
Conclusions

We have provided a frequency-domain framework to study sampled-data feedback control systems. This framework incorporates full information of the continuous-time response of the system, and has emphasized the description of the hybrid operators governing the steady-state response to output disturbances and measurement noise. Using this framework,

- (i) We have developed a theory of design limitations for SISO sampled-data systems. This theory allows the quantification of performance limitations that are inherent to open-loop properties of the plant and hold function. Briefly, we found that
 - Hybrid systems *inherit* the difficulty imposed upon analog feedback design by those plant properties such as NMP zeros, unstable poles, and time-delays. Furthermore, such difficulty is independent of the type of hold used.
 - Hybrid systems are subject to *extra* design limitations due to potential NMP zeros of the hold. In particular, if there is a hold zero close to a plant pole in the ORHP, sampling is “almost pathological”, and then system’s sensitivity, robustness, and response to disturbances will be poor.
 - Hybrid systems, unlike the analog case, are subject to limits upon the ability of high compensator gain to achieve disturbance rejection. This limits can be overcome in some cases by imposing additional constraints on the structure of the hold.
- (ii) We have derived MIMO closed-form expressions for the frequency-gains and L_2 -induced norms of hybrid sensitivity and complementary sensitivity operators. These expressions characterize the frequency-gain of both operators as the maximum eigenvalue of an associated finite-dimensional discrete transfer matrix. The induced norm is then computed by performing a search of maximum over a finite range of frequencies. The results admit straightforward implementation in a numerically reliable fashion.
- (iii) We have shown that certain robust stability tests for sampled-data systems

may be derived in simpler and more intuitive way with a pure frequency-domain approach than with alternative state-space based formulations.

- (iv) We have analyzed GSHFs and GSHF-based feedback control systems. From our results we conclude that control schemes relying on GSHF zero-placement capabilities cannot circumvent fundamental limitations imposed by analog NMP plant zeros. Furthermore, if the analog plant has a NMP zero within the desired system's closed-loop bandwidth, and GSHF zero-shifting is used to attempt removing the limitations imposed by this zero, then the following design tradeoffs arise:
- (a) If good nominal *discrete* performance is required, then necessarily the discrete response will be sensitive to uncertainty in the *analog* plant.
 - (b) If good nominal *discrete* performance *and* satisfactory intersample behavior is required, then necessarily the *analog* response will be sensitive to plant uncertainty, disturbances and sensor noise.

From the above discussion, it seems that only a marginal improvement in performance may be expected from using a GSHF instead of a ZOH. In any case, it should be noted that the potential advantages of GSHFs may altogether evaporate at the time of a practical implementation. Indeed, it is not obvious how to actually construct a LTI GSHF other than as an approximation by a PC GSHF, and even in this case, the realization will be considerably more demanding than that of the simpler ZOH.

A number of other issues remain as topics for future research. Perhaps an obvious first step would be the application of the analysis tools developed in this thesis to synthesis of discrete controllers. In this direction, the frequency-domain methods presented may prove useful, offering clear interpretations, and reliable numerical algorithms.

For example, a potential line of research is connected with the expressions derived in Chapter 5 for operator frequency-gains and L_2 -induced norms. In Corollary 5.2.6 we introduced the discrete function Φ_d , which was indicated as a measure of intersample activity, since it serves to quantify the difference in L_2 -induced norms between hybrid and discrete sensitivity operators. For example, Φ_d could be useful to perform hybrid H_∞ loop shaping, i.e., by considering the frequency-gain of the hybrid complementary sensitivity operator, and then "shaping" the responses of Φ_d and the discrete complementary sensitivity function. This function has some other intriguing interpretations that might be worthwhile analyzing further:

- (i) Φ_d may be seen as a "distance" between the spaces spanned by the plant, hold, and anti-aliasing filter in the lifted domain (cf. Remark 5.2.3). The minimization of this distance might be considered, for example, to draw alternative design guidelines for the anti-aliasing filter.
- (ii) Φ_d is linked to the degree of conservativeness of the L_2 -induced norm as a measure of hybrid stability robustness against LTI uncertainties [Hagiwara and Araki, 1995]. In relation to this, we might consider the problem of

mapping analog uncertainties to discrete, and devise a procedure to reduce a hybrid robust stability problem to a simpler discrete one. More concretely, suppose that Δ is some admissible uncertainty in the analog plant P ,

$$\tilde{P} = P(1 + \Delta).$$

Then, if F is the anti-aliasing filter, and H the hold, we may write the discretized perturbed plant as

$$(F\tilde{P}H)_d = (FPH)_d \left(1 + \frac{(FP\Delta H)_d}{(FPH)_d} \right).$$

Let us then define the discrete perturbation by $\Delta_d \triangleq (FP\Delta H)_d / (FPH)_d$. Thus, if $\|\Delta\|_\infty \leq \gamma$, then it is not difficult to see that

$$\|\Delta_d\|_\infty \leq \gamma \|\Phi_d\|_\infty,$$

which characterizes a class of uncertainties for an associated discrete-time robust stability problem. The conditions obtained from this discrete problem will be conservative, but this may be quantified from the analysis of Φ_d .

An extension of our theory of hybrid performance limitations to a multivariable setting is also a path worth pursuing in the future. This could be approached, for example, by combining our results with those obtained by Freudenberg and Looze [1988], and more recently by Gómez and Goodwin [1995], for analog multivariable linear systems.

In relation to the possible improvement in performance obtained from using GSHFs, it would be indeed interesting to compare the different optimal H_∞ solutions to the sampled-data control problem; as for example those given by Bamieh and Pearson [1992] and Sun et al. [1993]. Bamieh and Pearson solve the problem assuming that the hold is a ZOH, whereas Sun et al. do not make this assumption, and thus obtain a more general solution that involves a discrete controller and a GSHF. The loss of performance arising from the use of the ZOH may well be quantified by using our formulas for the L_2 -induced norms, which consider GSHFs, and are easily programmable.

In a wider perspective for further work, one of the issues that comes to mind is to examine how pervasive these fundamental design limitations are. For some time, we have known that NMP zeros and unstable poles of the plant impose design constraints on analog systems. We have now shown that these limitations carry over to sampled-data systems. It seems that this would also follow to some extent to related control schemes such as periodic and multirate, which are subjects of current research. Do these limitations apply to *any* linear controller? Or even perhaps to *any controller whatsoever*?

At present, no answer to these general questions seems to be known, but it is expected that different analysis techniques would need to be applied.

A

Proofs of Some Results in the Chapters

A.1 Proofs for Chapter 2

In this section we prove Lemma 2.1.2. A proof for the strict conditions we stated may be found in Henrici [1977, Theorem 10.10a]¹; we shall give here a more compact version under an additional hypothesis.

We start with a few definitions and preliminary results. Given a function G (the Laplace transform of a function g) we introduce the following sequence of functions defined over the domain \mathcal{D}_G .

$$\Gamma_N(s) \triangleq \frac{1}{T} \sum_{n=-N}^N G(s + jn\omega_s), \quad \text{for } N = 0, 1, 2, \dots \quad (\text{A.1})$$

We shall assume the following, which is required for our proof of Lemma 2.1.2.

Assumption 6

The sequence $\{\Gamma_N\}_{N=0}^{\infty}$ is uniformly convergent in the strip \mathcal{D}_G . ◦

The convergence of the sequence $\{\Gamma_N\}_{N=0}^{\infty}$ established above delineates the conditions under which the RHS of (2.6) is mathematically meaningful.

Remark A.1.1 The uniform convergence of the series $\frac{1}{T} \sum_{n=-\infty}^{\infty} G(s + jn\omega_s)$ is also the condition required by the proof of Doetsch [1971]. We suspect that assuming that g is a function of BV should imply the uniform convergence of the series, and then allow a reasonably compact proof of Lemma 2.1.2 without resorting to the Poisson Summation Formula Henrici [1977]. However, we could not complete this proof by the time of writing this monograph. This assumption on g would then be a more restrictive condition, although somehow more insightful if one is interested in a time-domain characterization. That this condition is in fact sufficient to prove Lemma 2.1.2 follows from Henrici [1977]. ◇

¹Henrici refers to this result as the Polya Formula, and derives it as a corollary of the Poisson Summation Formula.

Many of the proofs for results related to Lemma 2.1.2 available in the literature rely on the introduction of the “function”

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT),$$

defined as an infinite series of impulses, or Dirac’s deltas Pierre and Kolb [1964], Carroll and W.L. McDaniel [1966], Phillips et al. [1966], Åström and Wittenmark [1990]. A Dirac’s delta is *not* well-defined as a function; it is in fact a *distribution*, and so special care must be taken regarding the sense in which certain mathematical manipulations are performed [cf. Zemanian, 1965].

Our approach dispenses with the use of δ_T , and instead resources to the *Dirichlet Kernel*, a classical tool in proving convergence of Fourier series. The Dirichlet Kernel is defined by

$$D_N(t) = \frac{\sin((2N + 1)t)}{\sin(t)},$$

where N is a positive integer. D_N is periodic and its integral on $[0, \pi/2]$ has a fixed value independent of N ,

$$\int_0^{\pi/2} D_N(t) dt = \frac{\pi}{2}.$$

A key property of the Dirichlet Kernel is related to the following *Dirichlet Integral* [e.g., Carslaw, 1950, § 94].

Lemma A.1.1 (Dirichlet Integral.)

If f is a function of BV on the interval $[0, \pi]$, then

$$\lim_{N \rightarrow \infty} \int_0^{\pi} f(t) D_N(t) dt = \frac{\pi}{2} [f(0^+) + f(\pi^-)]$$

◦

Note that D_N is very much like an approximation to δ_T , with many similar properties, but is well-defined as a function (see Figure A.1).

Proof of Lemma 2.1.2 Consider the finite series $\sum_{|n| \leq N} G(s + jn\omega_s)$ for some s in \mathcal{D}_G . Then, we have:

$$\begin{aligned} \sum_{|n| \leq N} G(s + jn\omega_s) &= \sum_{|n| \leq N} \int_0^{\infty} e^{-(s + jn\omega_s)t} g(t) dt \\ &= \sum_{|n| \leq N} \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} e^{-(s + jn\omega_s)t} g(t) dt \\ &= \sum_{|n| \leq N} \sum_{k=0}^{\infty} \int_0^T e^{-s(t+kT)} e^{-jn\omega_s t} g(t + kT) dt \\ &= \sum_{k=0}^{\infty} \int_0^T e^{-s(t+kT)} g(t + kT) \left(\sum_{|n| \leq N} e^{-jn\omega_s t} \right) dt \quad (\text{A.2}) \end{aligned}$$

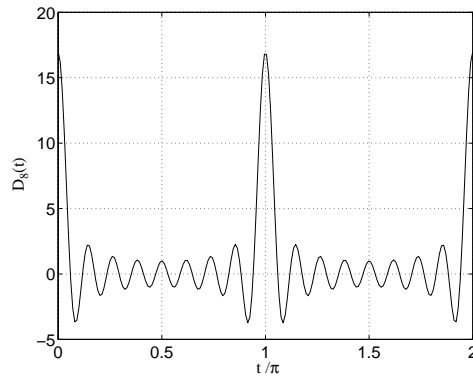


Figure A.1: Dirichlet Kernel for $N = 8$.

Note that the summation inside the integral in (A.2) is precisely the Dirichlet Kernel introduced before, since

$$\sum_{|n| \leq N} e^{-jn\omega_s t} = D_N(\omega_s t/2).$$

Hence, for each k we obtain a Dirichlet Integral on $e^{-s(t+kT)} g(t+kT)$,

$$\sum_{|n| \leq N} G(s + jn\omega_s) = \sum_{k=0}^{\infty} \int_0^T e^{-s(t+kT)} g(t+kT) D_N(\omega_s t/2) dt. \quad (A.3)$$

Take limits on both sides of (A.3) and, since the series on the LHS is uniformly convergent, we can interchange limit and summation, which yields

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{|n| \leq N} G(s + jn\omega_s) &= \sum_{k=0}^{\infty} \lim_{N \rightarrow \infty} \int_0^T e^{-s(t+kT)} g(t+kT) D_N(\omega_s t/2) dt. \\ &= \frac{T}{2} \sum_{k=0}^{\infty} (g(kT^+) + g((k+1)T^-)) e^{-skT}. \end{aligned} \quad (A.4)$$

Adding and subtracting $\sum_{k=0}^{\infty} g(kT^+) e^{-skT}/2$, we obtain $T F_d(e^{sT})$ on the RHS of (A.4). Finally, noting that $g(0^-) = 0$, expression (2.6) follows, completing the proof. \square

A.2 Proofs for Chapter 3

This section provides the proof of Lemma 3.2.2, on the asymptotic location of the zeros of a FDLTI GSHP.

Proof of Lemma 3.2.2 We first prove, by contradiction, that $n, m < N - 1$, where $L \in \mathbb{R}^{N \times N}$. Suppose $n \geq N - 1$. Then $KL^i M = 0$ for $i = 1, 2, \dots, N - 1$.

By the Cayley-Hamilton Theorem Chen [1984], $KL^iM = 0$ for all i . However, since $d^i h/dt^i|_{t=T} = KL^iM$, this implies that h is identically zero [Chen, 1984, Appendix B]. This argument also shows that the coefficient of e^{-sT} in (3.12) is nonzero. A similar argument shows that $m < N - 1$.

Consider next the ratio

$$F(s) = \frac{K(sI + L)^{-1}Me^{-sT}}{K(sI + L)^{-1}e^{LT}M}.$$

Note that F is analytic in a neighborhood of infinity and has an essential singularity at infinity (the latter is due to the presence of e^{-sT} with a nonzero coefficient). It follows from the Great Picard Theorem [Conway, 1973, p. 302] that in each neighborhood of infinity F assumes each complex number with one possible exception, infinitely many times. Because of the term e^{-sT} , this exceptional value must equal zero. Hence there exists a sequence $\{\gamma_\ell\}_{\ell=1}^\infty$ converging to infinity such that $F(\gamma_\ell) = 1$, and thus $H(\gamma_\ell) = 0$ for all ℓ .

We now show that if $n = m$, then the γ_ℓ 's necessarily converge to the values given in (3.19). To do this note that for each integer k ,

$$(sI + L)^{-1} = \frac{1}{s} \left(\left(\frac{-L}{s} \right)^{k+1} \left(I + \frac{L}{s} \right)^{-1} + \sum_{i=0}^k \left(\frac{-L}{s} \right)^i \right)$$

Using this identity and the definitions of m and n yields

$$H(s) = \frac{1}{s^{m+1}} (Q(s) - R(s) e^{-sT} s^{m-n}),$$

where

$$Q(s) = K(-L)^m e^{LT}M + \frac{1}{s} K(-L)^{m+1} \left(I + \frac{L}{s} \right)^{-1} e^{LT}M,$$

and

$$R(s) = K(-L)^n M + \frac{1}{s} K(-L)^{n+1} \left(I + \frac{L}{s} \right)^{-1} M.$$

For γ_ℓ a zero of H , we have $Q(\gamma_\ell) = \gamma_\ell^{m-n} R(\gamma_\ell) e^{\gamma_\ell T}$. Note that for ℓ sufficiently large, $Q(\gamma_\ell)$ and $R(\gamma_\ell)$ are both nonzero and constant. Taking logarithms and rearranging shows that there exists k such that

$$\gamma_\ell = -\frac{1}{T} \log \frac{Q(\gamma_\ell)}{R(\gamma_\ell)} + \frac{m-n}{T} \log \gamma_\ell + jk\omega_s.$$

If $n = m$, then taking limits yields

$$\gamma_\ell \rightarrow -\frac{1}{T} \log \eta + jk\omega_s.$$

Noting that zeros must occur in conjugate pairs yields (3.19). □

A.3 Proofs for Chapter 4

This section provides an sketch of a proof for Lemma 4.1.2 on the steady-state frequency response of the hybrid system to input disturbances and noise. We also give here a proof for the complementary sensitivity integral constraint of Theorem 4.4.11.

A.3.1 Proof of Lemma 4.1.2

We consider only the disturbance response, calculations for the noise response being entirely analogous. To evaluate the steady-state response to $d(t) = e^{j\omega t}$, we must first evaluate the inverse Laplace transform of Y^d , and then discard all terms due to those poles lying in \mathbb{C}^- . Inverting the Laplace transform requires that we evaluate the Bromwich integral Levinson and Redheffer [1970]

$$y^d(t) = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} e^{st} Y^d(s) ds, \quad (\text{A.5})$$

where $\gamma > 0$. This integral may be evaluated using the residue theorem.

It follows from (4.1) that Y^d has poles due to the disturbance located along the imaginary axis at $s = j(\omega + k\omega_s)$, $k = 0, \pm 1, \pm 2, \dots$. By the assumption of closed loop stability all other poles of Y^d lie in the \mathbb{C}^- . Using (4.1), it may be shown that these poles have the following properties:

- (i) they all lie to the right of some vertical line $\text{Re}[s] = c < 0$,
- (ii) there are finitely many poles due to P and no poles due to H , and
- (iii) there are finitely many sequences of poles due to $C_d(e^{sT})$, $S_d(e^{sT})$, and $F(s + jk\omega_s)$, $k = 0, \pm 1, \pm 2, \dots$ lying on vertical lines and spaced at intervals equal to ω_s .

Next, it is straightforward to verify that the residues of $e^{st} Y^d$ at the $j\omega$ -axis poles are given by

$$\lim_{s \rightarrow j(\omega + k\omega_s)} (s - j(\omega + k\omega_s)) e^{st} Y^d(s) = \begin{cases} S^0(j\omega) e^{j\omega t} & \text{if } k = 0 \\ -T_k(j\omega) e^{j(\omega + k\omega_s)t} & \text{if } k \neq 0 \end{cases}. \quad (\text{A.6})$$

We need not calculate explicitly the residues at the other poles; as we shall show, they do not contribute to the steady-state response.

Consider the contours of integration \mathcal{C}_n , $n = 1, 2, 3, \dots$ depicted in Figure A.2, and chosen so that (i) \mathcal{C}_1 encloses only that $j\omega$ -axis pole lying in Ω_N , (ii) the horizontal line $\text{Im}[s] = R_1$ does not contain any OLHP poles of Y^d , and (iii) $R_{n+1} = R_n + \omega_s$.

Figure A.2 and subsequent calculations are appropriate for the case that ω is in Ω_N (modifications to the general case are straightforward). Our construction

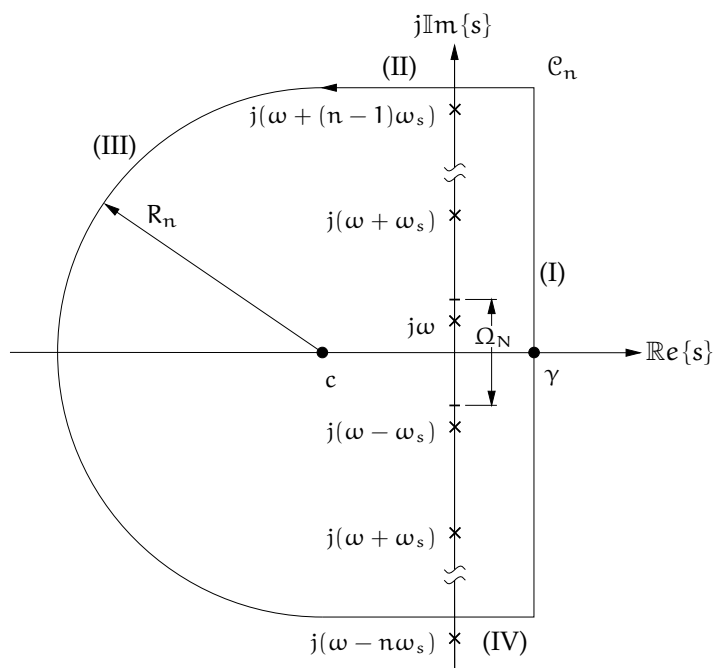


Figure A.2: Contours of integration.

of the contour of integration guarantees that for n sufficiently large no poles of Y^d will lie on C_N . Hence the residue theorem may be applied to yield

$$\begin{aligned} & \frac{1}{2\pi j} \left\{ \int_I e^{st} Y^d(s) ds + \int_{II} e^{st} Y^d(s) ds + \int_{III} e^{st} Y^d(s) ds + \int_{IV} e^{st} Y^d(s) ds \right\} \\ & = S^0(j\omega) e^{j\omega t} - \sum_{\substack{k=-N \\ k \neq 0}}^N T_k(j\omega) e^{j(\omega+k\omega_s)t} + \Psi(t), \end{aligned} \quad (A.7)$$

where $\Psi(t)$ denotes the contribution of the poles in C^- .

We now sketch a proof that as $t \rightarrow \infty$, $\Psi(t) \rightarrow 0$. First, it is clear that the contribution to Ψ from each pole of P converges to zero. Consider next the contribution of one of the finitely many sequences of poles described in (iii) above. Let this sequence be denoted $\rho_k \triangleq \rho + jk\omega_s$, $k = 0, \pm 1, \pm 2, \dots$, and $\Re\{\rho\} < 0$. We shall assume that ρ is real for notational simplicity, and shall also assume for simplicity that each ρ_k is a simple pole. Then, for any fixed value of t , the contribution to Ψ from this sequence of poles is given by

$$y^p(t) \triangleq e^{\rho t} \lim_{K \rightarrow \infty} \sum_{k=-K}^K \text{Res}(\rho_k) e^{jk\omega_s t}, \quad (A.8)$$

where $\text{Res}(\rho_k) = \lim_{s \rightarrow \rho_k} (s - \rho_k) Y^d(s)$. By the Riesz-Fischer Theorem [Riesz and Sz.-Nagy, 1990, p.70], if it may be shown that the sequence $\{\text{Res}(\rho_k)\}$ is square-summable, then the series in (A.8) will converge to a bounded periodic function of t . Since $\rho < 0$, it thus follows that $y^p(t) \rightarrow 0$ as $t \rightarrow \infty$. Since there are only finitely many sequences of the form (A.8), we then have that $\Psi(t) \rightarrow 0$.

We now show that the sequence $\{\text{Res}(\rho_k)\}$ is square-summable. From the (4.1), we have that

$$Y^d(s) = D(s) - P(s)H(s)C_d(e^{sT})S_d(e^{sT})V_d(e^{sT}), \quad (A.9)$$

where $V_d(e^{sT})$ is given by (cf. the proof of Lemma 4.1.1)

$$V(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F_k(s) D_k(s).$$

Hence

$$\text{Res}(\rho_k) = -P(\rho_k)H(\rho_k) \lim_{s \rightarrow \rho_k} (s - \rho_k) C_d(e^{sT}) S_d(e^{sT}) V_d(e^{sT}). \quad (A.10)$$

Because $C_d(e^{sT})$, $S_d(e^{sT})$, and $V_d(e^{sT})$ are each periodic in s along vertical lines, it may be shown that the limit on the right hand side of (A.10) is independent of k . Denote the common value of this limit by L_ρ . Then (A.8) becomes

$$y^p(t) = e^{\rho t} L_\rho \lim_{K \rightarrow \infty} \sum_{k=-K}^K P(\rho_k) H(\rho_k) e^{jk\omega_s t}. \quad (A.11)$$

By Assumption 3, $|P(\rho_k)|$ converges to a finite constant as $k \rightarrow \infty$. Next, using the definition of H (2.4) and integration by parts write for s in \mathbb{C}^-

$$\begin{aligned} |sH(s)| &= \left| h(0^+) - e^{-sT}h(T^-) + \int_0^T e^{-st}\dot{h}(t) dt \right| \\ &\leq |h(0^+)| + |h(T^-)|e^{-\Re\{s\}T} + e^{-\Re\{s\}T} \int_0^T |\dot{h}(t)| dt. \end{aligned} \quad (\text{A.12})$$

Since h is of BV by Assumption 1, \dot{h} is integrable on $[0, T]$, and then from (A.12)

$$|H(\rho_k)| \leq \frac{c_1 + c_2 e^{-\rho T}}{|\rho_k|}.$$

It follows that the sequence $\{P(\rho_k)H(\rho_k)\}$ is square summable, thus completing the proof that $\Psi(t) \rightarrow 0$.

The desired result (4.9) will hold if it may be shown that the last three integrals in (A.7) converge to zero as $N \rightarrow \infty$. We now show that the integral (II) converges to zero; similar calculations apply to (IV). Consider values of s such that $s = x + jR_n$, $c \leq x \leq \gamma$, and R_n is sufficiently large that $R_n > \omega$ and that \mathcal{C}_n encloses all poles of P . It may be shown that there exists constants M and M_p , independent of n , such that $|C_d(e^{sT})S_d(e^{sT})V_d(e^{sT})| \leq M$ and $|P(s)| \leq M_p$ for all such s . Furthermore, it is not difficult to see from similar arguments as those in (A.12) that for $t \geq T$ there exists a constant M_γ with

$$|se^{st}H(s)| \leq M_\gamma, \quad \text{for all } s \text{ with } \Re\{s\} \leq \gamma. \quad (\text{A.13})$$

Using these bounds in (A.8) yields

$$|e^{st}Y^d(s)| \leq e^{\gamma t}(R_n - \omega)^{-1} + MM_p M_\gamma (R_n - \omega)^{-1} \quad (\text{A.14})$$

Using this bound in (II) yields that the integral converges to zero as $R_n \rightarrow \infty$.

It remains to show that the integral (III) converges to zero. This follows by (i) parameterizing (III) by $s = c + R_n e^{j\theta}$, with $\pi/2 \leq \theta \leq 3\pi/2$, and defining $\sigma = s - c$; contour (III) is then a semicircle φ_n centered at the origin of the σ -plane and extended into the left half plane; (ii) showing that Y^d is bounded on φ_n ; and (iii) using Jordan's Lemma [Levinson and Redheffer, 1970, p. 199] to obtain a bound on the integral

$$\int_{\varphi_n} |e^{\sigma T} d\sigma|.$$

Hence, for $t > T$, integral (III) converges to zero as $R_n \rightarrow \infty$. \square

A.3.2 Proof of Theorem 4.4.11

This result is analogous to the integral constraint we proved for holds in Chapter 3, Proposition 3.3.3. The proof is in a similar pattern. We need the following preliminary result.

Lemma A.3.1

Suppose that G is an analytic function bounded in the CRHP; suppose that $G(0) = 1$. Then

$$\lim_{x \rightarrow 0} \int_0^{\infty} \frac{\log |G(j\omega)|}{x^2 + \omega^2} d\omega = \int_0^{\infty} \frac{\log |G(j\omega)|}{\omega^2} d\omega \quad (\text{A.15})$$

Proof: The result follows from the Lebesgue Dominated Convergence Theorem [e.g., Riesz and Sz.-Nagy, 1990, p. 37]. To apply this result, it suffices to

- note that $|\log |G(j\omega)|/(x^2 + \omega^2)| \leq |\log |G(j\omega)|/\omega^2|$ for all x and ω , and
- show that the integral on the right hand side of (A.15) is finite.

The latter follows by noting that

- (i) $|\log |G(j\omega)||$ is bounded on the $j\omega$ -axis except at zeros of $G(j\omega)$,
- (ii) these zeros, including a possible zero at infinity, are removable singularities Levinson and Redheffer [1970] and thus do not cause the integral to become unbounded, and
- (iii) the integral approaches a finite limit as $\omega \rightarrow 0$.

Statement (iii) follows by using L'Hospital's Rule to show that

$$\lim_{\omega \rightarrow 0} \frac{\log G(j\omega)}{\omega^2} = \frac{G'(0)^2 - G''(0)}{2},$$

where $G'(0) = dG(s)/ds|_{s=0}$ and $G''(0) = d^2G(s)/ds^2|_{s=0}$. □

Proof of Theorem 4.4.11 We begin by applying the Poisson integral to the fundamental complementary sensitivity function for an arbitrary real $x > 0$. Subtracting $\log |T^0(0)|$ from both sides yields

$$\begin{aligned} \frac{2}{\pi} \int_0^{\infty} \log \left| \frac{T^0(j\omega)}{T^0(0)} \right| \frac{x}{x^2 + \omega^2} d\omega &= x(\tau_p + \tau_H + N_c T) + \log |B_z^{-1}(x)| \\ &+ \log |B_\gamma^{-1}(x)| + \sum_{k=1}^{N_p} \log |B_{p_k}^{-1}(x)| \quad (\text{A.16}) \\ &+ \sum_{k=1}^{N_a} \log |B_{a_k}^{-1}(x)| + \log \left| \frac{T^0(x)}{T^0(0)} \right|, \end{aligned}$$

where the terms on the right hand side are as defined in Subsection 4.4.1. Dividing both sides by x , taking the limit as $x \rightarrow 0$, and applying Lemma A.3.1

yields

$$\begin{aligned}
\frac{2}{\pi} \int_0^\infty \log \left| \frac{T^0(j\omega)}{T^0(0)} \right| \frac{1}{x^2 + \omega^2} d\omega &= \tau_P + \tau_H + N_c T + \lim_{x \rightarrow 0} \frac{1}{x} \log |B_\zeta^{-1}(x)| \\
&+ \lim_{x \rightarrow 0} \frac{1}{x} \log |B_\gamma^{-1}(x)| + \lim_{x \rightarrow \omega} \frac{1}{x} \sum_{k=1}^{N_p} \log |B_{p_k}^{-1}(x)| \quad (\text{A.17}) \\
&+ \lim_{x \rightarrow 0} \frac{1}{x} \sum_{k=1}^{N_a} \log |B_{a_k}^{-1}(x)| + \lim_{x \rightarrow 0} \frac{1}{x} \log \left| \frac{T^0(x)}{T^0(0)} \right|.
\end{aligned}$$

We now use L'Hospital's rule and the fact that the zeros and poles (4.58)-(4.63) must occur in complex conjugate pairs to evaluate the various limits on the right hand side of (A.17):

(i)

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{1}{x} \log |B_\zeta^{-1}(x)| &= \lim_{x \rightarrow 0} \frac{1}{x} \sum_{k=1}^{N_\zeta} \log \left| \frac{\bar{\zeta}_k + x}{\zeta_k - x} \right| \\
&= \sum_{k=1}^{N_p} \lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{1}{x} \log \left[\frac{\bar{\zeta}_k + x}{\zeta_k - x} \right] \right) \\
&= \sum_{k=1}^{N_p} \frac{2\text{Re}(\zeta_k)}{|\zeta_k|^2} \\
&= 2 \sum_{k=1}^{N_p} \frac{1}{\zeta_k}. \quad (\text{A.18})
\end{aligned}$$

(ii) A calculation similar to (i) applies to the fifth term in the RHS of (A.17) if it may be shown that the possibly infinite sum $\sum_{k=1}^{N_p} 1/\gamma_k$ converges. Convergence of this series follows from: (i) the fact that $T^0(0) \neq 0 \Rightarrow H(0) \neq 0$ and (ii) applying arguments based on properties of zeros of functions analytic in the CRHP (cf. p. 132 of Hoffman [1962]).

(iii)

$$\lim_{x \rightarrow 0} \frac{1}{x} \sum_{k=1}^{N_a} \log |B_{a_k}^{-1}(x)| = \lim_{x \rightarrow 0} \frac{1}{x} \sum_{k=1}^{N_a} \log \prod_{\ell=-\infty}^{\infty} \left| \frac{\bar{a}_{k\ell} + x}{\zeta_\ell - x} \right|. \quad (\text{A.19})$$

It follows from p. 175 of Conway [1973] that

$$\prod_{\ell=-\infty}^{\infty} \left| \frac{\bar{a}_{k\ell} + x}{a_{k\ell} - x} \right| = \left| \frac{\sinh((\bar{a}_k + x)\frac{T}{2})}{\sinh((a_k - x)\frac{T}{2})} \right| \quad (\text{A.20})$$

Substituting (A.20) into (A.19) and applying L'Hospital's rule yields the desired result.

- (iv) A calculation similar to (iii) applies to the sixth term on the RHS of (A.17), keeping in mind that the factor $\ell = 0$ is not present in the infinite product that corresponds to (A.20).
- (v) Applying L'Hospital's rule yields

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{x} \log \left| \frac{T^0(x)}{T^0(0)} \right| &= \lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{1}{2} \log \left(\frac{T^0(x)}{T^0(0)} \right)^2 \right) \\ &= \frac{T'^0(0)}{T^0(0)} \end{aligned}$$

□

A.4 Proofs for Chapter 5

In this section we prove that the frequency-domain lifting transformation defined in Chapter 5 is an isometric isomorphism between the spaces $L_2(-\infty, \infty)$ and $L_2(\ell_2; \Omega_N)$.

Proof of Lemma 5.1.1 Let $Y(j\omega)$ be in L_2 . Then we have that

$$\|Y\|^2 = \int_{-\infty}^{\infty} |Y(j\omega)|^2 d\omega \tag{A.21}$$

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\omega_N}^{(2k+1)\omega_N} |Y(j\omega)|^2 d\omega \\ &= \sum_{-\infty}^{\infty} \int_{-\omega_N}^{\omega_N} |Y(j(\omega + k\omega_s))|^2 d\omega. \end{aligned} \tag{A.22}$$

As $\|Y\|^2$ is finite by assumption, the series $\sum \int |Y_k(j\omega)|^2 d\omega$ is convergent. Then, by Levi's Theorem Riesz and Sz.-Nagy [1990], we can interchange summation and integration in (A.22), and using (5.1), we have that

$$\sum_{-\infty}^{\infty} \int_{-\omega_N}^{\omega_N} |Y_k(j\omega)|^2 d\omega = \int_{-\omega_N}^{\omega_N} \sum_{-\infty}^{\infty} |Y_k(j\omega)|^2 d\omega \tag{A.23}$$

$$= \|\mathbf{y}\|^2 \tag{A.24}$$

From (A.21)-(A.24) it follows that there is an isometry between $L_2(-\infty, \infty)$ and $L_2(\Omega_N; \ell_2)$. To see that the isometry is isomorphic, we have to show that it is *onto*, that is, *each* function in $L_2(\Omega_N; \ell_2)$ is the image of a function in $L_2(-\infty, \infty)$. Actually, it suffices to show that this is the case for each element in a basis for $L_2(\Omega_N; \ell_2)$, and so we shall do next.

Let $\{\gamma_k\}_{k=-\infty}^{\infty}$ be an orthonormal basis for ℓ_2 , and $\{\psi_k(\omega)\}_{k=-\infty}^{\infty}$ an orthonormal basis for $L_2(\Omega_N)$. It is not difficult to prove that the double sequence

$$\{\psi_n(\omega) \gamma_m\}_{n,m=-\infty}^{\infty}$$

is an orthonormal base on $L_2(\Omega_N; \ell_2)$. Now, take for example $\psi_n(\omega) \gamma_m$, for fixed integers n, m . This element of $L_2(\Omega_N; \ell_2)$ is precisely

$$\begin{bmatrix} \vdots \\ 0 \\ \psi_n(\omega) \\ 0 \\ \vdots \end{bmatrix} \begin{matrix} \vdots \\ m+1 \\ m \\ m-1 \\ \vdots \end{matrix}, \quad (\text{A.25})$$

which corresponds to the function

$$\psi(\omega) = \begin{cases} \psi_n(\omega) & \text{if } \omega \in [-\omega_N + m\omega_s, \omega_N + m\omega_s] \\ 0 & \text{otherwise} \end{cases}. \quad (\text{A.26})$$

But $\psi(\omega)$ is obviously in $L_2(-\infty, \infty)$, since it is a function of finite support and integrable there. Therefore, every element in $L_2(\Omega_N; \ell_2)$ is the image of an element in $L_2(-\infty, \infty)$ and the proof is completed. \square

A.5 Proofs for Chapter 6

In this section we include the sketch of an alternative proof for Corollary 6.1.3 that dispenses with the μ -framework. The arguments are similar to those in Theorem 6.1.2.

Proof of Corollary 6.1.3 We start by noting that the perturbed discrete sensitivity function can be written as

$$\begin{aligned} \tilde{S}_d &= \frac{S_d}{1 + (\text{FPW}\Delta H)_d S_d C_d} \\ &= \frac{S_d}{1 + \sum_{n=-\infty}^{\infty} T_k^0 W_k \Delta_k} \end{aligned} \quad (\text{A.27})$$

We prove both implications in Corollary 6.1.3 by contrapositive arguments.

(\Leftarrow) Suppose that there exist an admissible Δ such that \tilde{S}_d is unstable. Then, by continuity arguments there also exist some admissible Δ' such that \tilde{S}_d is *marginally* stable, i.e., it has a pole at $s = j\omega_0$, for some ω_0 . From (A.27) it follows that

$$\sum_{n=-\infty}^{\infty} T_k^0(\omega_0) W_k(\omega_0) \Delta'_k(\omega_0) = -1.$$

Hence,

$$\sum_{n=-\infty}^{\infty} |T_k^0(\omega_0) W_k(\omega_0) \Delta'_k(\omega_0)| \geq 1,$$

but as $\|\Delta'\|_\infty < 1$, then

$$\sum_{n=-\infty}^{\infty} |T_k^0(\omega_0)W_k(\omega_0)| > 1.$$

(\Rightarrow) Suppose that there exists ω_0 in Ω_N such that

$$\sum_{n=-\infty}^{\infty} |T_k^0(\omega_0)W_k(\omega_0)| = \alpha > 1.$$

Then, it is possible to find an admissible perturbation Δ that interpolates

$$\Delta(j(\omega_0 + k\omega_s)) = -1/\alpha e^{-j\theta_k},$$

where $\theta_k \triangleq \angle T_k^0(\omega_0)W_k(\omega_0)$. Then \tilde{S}_d has a pole at $z = e^{j\omega_0 T}$ and so the perturbed system is not asymptotically stable. □

B

Order and Type of an Entire Function

This appendix provides a brief description of the concepts of *order* and *type* of entire functions; for further reference see Markushevich [1965]. We recall that an entire function, F , is a function defined and analytic for all finite values of the complex variable s . An entire function that is not a polynomial is called an entire *transcendental* function. For such a function F , define the *maximum modulus* as

$$M(r) = \max_{|s|=r} |F(s)|.$$

It can be seen [e.g., Markushevich, 1965] that, since F is analytic everywhere, $M(r)$ is a strictly increasing function, and, moreover, $\lim_{r \rightarrow \infty} M(r) = \infty$. An entire function is said to be of *finite order* if there exists a positive number μ such that as $|s| = r \rightarrow \infty$, we have that¹

$$F(s) = O(e^{r^\mu}). \quad (\text{B.1})$$

Clearly, if (B.1) is satisfied for some μ , it will also be satisfied for any $\mu' > \mu$. The infimum of the numbers satisfying (B.1) is defined as the *order*, ρ , of the entire function F . We shall be interested in entire functions of *exponential type*, i.e., of finite order 1 for which there exists a positive constant K such that as $|s| = r \rightarrow \infty$,

$$F(s) = O(e^{Kr}). \quad (\text{B.2})$$

The lower bound σ of numbers K for which (B.2) is true is called the *type* of the entire function. We say then that F is of exponential type σ .

¹Here we use the notation $F(s) = O(e^{r^\mu})$, which means that $M(r) < ke^{r^\mu}$ for some constant k when r is near to some given limit.

C

Discrete Sensitivity Integrals

Discrete sensitivity functions satisfy analytic constraints in the form of Bode and Poisson integral relations analogous to those satisfied by their continuous-time counterparts. The results in this section are adapted from Sung and Hara [1988], to which we refer for further details.

Let $d_i, i = 1, \dots, N_d$ denote the poles of $(\text{FPH})_d C_d$ lying in $\overline{\mathbb{D}}^c$. Then we have the following.

Proposition C.1.1 (Bode Discrete Sensitivity Integral)

Assume that S_d is stable and that $(\text{FPH})_d C_d$ is strictly proper. Then

$$\int_0^{\omega_N} \log |S_d(e^{j\omega T})| d\omega = \omega_N \sum_{i=1}^{N_d} \log |d_i|. \quad (\text{C.1})$$

◦

For a fixed sampling period, this integral implies a non-trivial sensitivity trade-off even if no bandwidth constraint is imposed. The next corollary is a straightforward consequence of Proposition C.1.1.

Corollary C.1.2

Assume the conditions of Proposition C.1.1. Suppose in addition that

$$|S_d(e^{j\omega T})| \leq \beta \quad \text{for } \omega \text{ in } [0, \omega_0], \text{ with } \omega_0 < \omega_N. \quad (\text{C.2})$$

Then necessarily

$$\sup_{\omega_0 < \omega < \omega_N} |S_d(e^{j\omega T})| \geq \left(\frac{1}{\beta}\right)^{\frac{\omega_0}{\omega_N - \omega_0}} \left| \prod_{i=1}^{N_d} d_i \right|^{\frac{\omega_N}{\omega_N - \omega_0}}. \quad (\text{C.3})$$

◦

Denote by $\rho_i, i = 1, \dots, N_\rho$ the poles of $(\text{FPH})_d$ in $\overline{\mathbb{D}}^c$, and denote by B_ρ the associated Blaschke product

$$B_d(z) \triangleq \prod_{i=1}^{N_d} \frac{z - d_i}{1 - \bar{d}_i z}.$$

Define also the Poisson kernel $\Psi_d(re^{j\theta}, \omega)$,

$$\Psi_d(re^{j\theta}, \omega) \triangleq \frac{\frac{T}{2}(r^2 - 1)}{1 - 2r \cos(\omega T + \theta) + r^2} + \frac{\frac{T}{2}(r^2 - 1)}{1 - 2r \cos(\omega T - \theta) + r^2}. \quad (\text{C.4})$$

Then S_d satisfy the following Poisson integral relation.

Proposition C.1.3 (Poisson Discrete Sensitivity Integral)

Assume that S_d is stable. Let $\nu = re^{j\theta}$ lie in \mathbb{D}^c . Then

$$\int_0^{\omega_N} \log |S_d(e^{j\omega T})| \Psi_d(\nu, \omega) d\omega \geq \pi \log |B_\rho^{-1}(\nu)| + \pi \log |S_d(\nu)|. \quad (\text{C.5})$$

◦

Note that equality may be achieved in (C.5) by incorporating terms due to unstable poles of the compensator into the Blaschke product B_ρ .

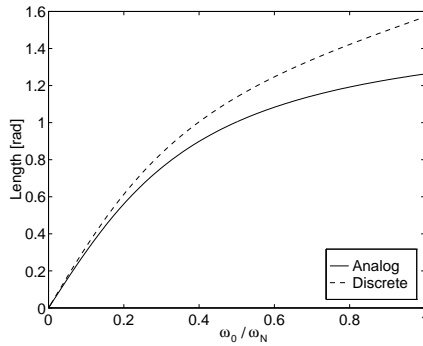


Figure C.1: Weighted lengths of intervals.

We shall require the weighted length of an interval by the Poisson kernel (C.4) (cf. the corresponding for the Poisson kernel for the half plane, in Chapter 3, (3.35)). Consider an interval $\Omega = [0, \omega_0)$, where $\omega_0 \leq \omega_N$, and a point $\xi = x + jy$ in the open right half plane. The image of the interval Ω under the mapping $z = e^{sT}$ is an arc, $\Omega_d = (1, e^{j\omega_0 T})$, of the unit circle, and the image of ξ is a point $e^{\xi T}$ in \mathbb{D}^c . Define the length of Ω_d , as weighted by $e^{\xi T}$, to be

$$\Theta_d(\xi, \Omega) \triangleq \int_0^{\omega_0} \Psi_d(e^{\xi T}, \omega) d\omega. \quad (\text{C.6})$$

In the case that ξ is real, we then have that

$$\Theta_d(\xi, \Omega) = -\angle \prod_{k=-\infty}^{\infty} \frac{\xi - j(\omega_0 - k\omega_s)}{\xi + j(\omega_0 + k\omega_s)} \quad (\text{C.7})$$

$$= -\angle \frac{\sinh\left(\frac{(\xi - j\omega_0)T}{2}\right)}{\sinh\left(\frac{(\xi + j\omega_0)T}{2}\right)}; \quad (\text{C.8})$$

i.e., the weighted length of the interval Ω equals the negative of the *sum* of the phase lags contributed by the Blaschke product $(\xi - s)/(\xi + s)$ at each of the points $\omega_0 + k\omega_s$, $k = 0, \pm 1, \pm 2, \dots$, that are mapped to the upper end point of the interval. It is straightforward to verify that the length of the discrete arc Ω_d weighted by the point $e^{\xi T}$ is *greater* than that of the corresponding analog interval $\Omega = [0, \omega_0)$ as weighted by the point ξ (cf. (3.36)). As an example, see Figure C.1, which contains plots of $\Theta(\xi, \Omega)$ and $\Theta_d(\xi, \Omega)$ for the point $\xi = 1/T$ and values of ω_0 ranging from 0 to ω_N . Similar remarks apply to the case of a complex ξ .

The following result is derived immediately from Proposition C.1.3.

Corollary C.1.4

Suppose that

$$|S_d(e^{j\omega T})| \leq \alpha, \quad \text{for all } \omega \text{ in } \Omega = [0, \omega_0), \quad (\text{C.9})$$

where $\omega_0 \leq \omega_N$, and let $\nu = e^{\xi T}$, where ξ lies in \mathbb{C}^+ . Then

$$\sup_{\omega \in [\omega_0, \omega_N)} |S_d(e^{j\omega T})| \geq (1/\alpha)^{\frac{\Theta_d(\xi, \Omega)}{\pi - \Theta_d(\xi, \Omega)}} |B_p^{-1}(\nu)|^{\frac{\pi}{\pi - \Theta_d(\xi, \Omega)}} |S_d(\nu)|^{\frac{\pi}{\pi - \Theta_d(\xi, \Omega)}} \quad (\text{C.10})$$

◦

If ν is a NMP zero of the discretized plant, then $S_d(\nu) = 1$ and $|S_d(e^{j\omega T})|$ is guaranteed to have a peak greater than one. Since $\Theta_d(\xi, \omega_0) \geq \Theta(\xi, \omega_0)$ it follows that the infimum of this peak is *guaranteed* to be greater than that given by (C.8) in the analog case.