6

Stability Robustness

Since no mathematical model can completely describe the exact behavior of a physical system, the consideration of model uncertainty in the analysis and design of feedback systems is an issue of unarguable theoretical and practical significance. In this respect, one of the fundamental problems is the analysis of the *stability robustness* of the control system, i.e., the property by which the closed-loop system remains stable under perturbations. This is a well-studied problem for FDLTI systems, where several useful tools, like H_∞ and μ methods, have proven successful.

The analysis of stability robustness for sampled-data systems is more difficult, again due to their time-varying characteristics, and has attracted the attention of a number of researchers in recent years. For example, Thompson et al. [1983] and Thompson et al. [1986] have used conic sector techniques to obtain sufficient conditions for robust stability. Similar results have been derived by Hara et al. [1991] using the L_2-induced norm and the Small-gain Theorem. More recently, Sivashankar and Khargonekar [1993] have shown that the L_2-induced norm actually gives both necessary and sufficient conditions for robust stability when the class of unstructured perturbations include periodic time-varying perturbations. However, as illustrated in Dullerud and Glover [1993], the L_2-induced norm may be a very conservative measure of robust stability under LTI perturbations, which are a more natural class of uncertainties to consider since the plant is normally assumed LTI. Indeed, under the assumption of stable LTI perturbations, Dullerud and Glover [1993] have shown that the necessary and sufficient condition for robust stability reduces to a μ type of test. This result has now been generalized to the case of unstable perturbations by Hagiwara and Araki [1995], who used Nyquist type of arguments and the frequency-domain framework suggested in Araki and Ito [1993] and Araki et al. [1993].

The approach followed in Dullerud and Glover [1993] is based on a state-space representation of the sampled-data system, and uses time-domain lifting techniques and a generalization of the $\mathcal{Z}$-transform to obtain a representation of the operators in frequency-domain. As pointed out by Yamamoto and Khargonekar [1996], this detour through state-space to describe input-output operators might complicate the analysis.

In this chapter, we show how these results can be obtained in a very intuitive and simple way — almost entirely by block-diagram manipulation — if the problem is set up directly in frequency-domain. In §6.1 we consider stable LTI multiplicative perturbations on the analog plant. Using the frequency-domain framework introduced in Chapter 5, we derive a μ-test that corresponds with the results of Dullerud and Glover. In the particular case of SISO systems, this test can be reduced to an ℓ_1-type condition involving the *fundamental complementary sensitivity function*, $T^0(s)$, introduced in §4.1. This has an important link with the results of Chapter 4, since it shows that peaks of T^0 will have direct deleterious effects on the stability robustness properties of the system.

It is interesting to note that under our framework, the problem is easily brought to the classical *basic perturbation model* of Figure 6.1 [see also Hagiwara and Araki, 1995]. Moreover — and perhaps unsurprisingly too — we shall see that the *interconnection matrix* G will be $\mathbf{T}_\omega$, the infinite matrix representation of the *sampled-data complementary sensitivity operator* introduced in §5.2. Note that this is in complete analogy with the corresponding LTI case, where the interconnection matrix is the complementary sensitivity function [e.g., Doyle et al., 1992].

Moreover, we shall see in §6.2 that this carries over to the problem of robust stability under a divisive perturbation model. Again in analogy with the LTI case, this time G is $\mathbf{S}_\omega$, the infinite matrix representation of the *sampled-data sensitivity operator*. The corresponding μ-test, though, will be only conjectured, since the sensitivity operator is *non-compact*, a fact that makes the analysis much more intricate than the multiplicative case. Nevertheless, a necessary condition for robust stability with the divisive perturbation model is easily obtained in the SISO case. This shows that peaks in the fundamental sensitivity function S^0 will necessarily reduce the stability margin of the hybrid system respect to this type of perturbations.

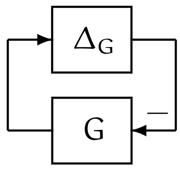

Figure 6.1: Basic perturbation model.

6.1 Multiplicative Perturbation

Consider the multivariable sampled-data system depicted in Figure 6.2. The perturbed plant is represented by the multiplicative uncertainty model

$$\tilde{P}(s) = (I + W(s)\Delta(s))\,P(s)\;, \tag{6.1}$$

where $\Delta(s)$ is a FDLTI perturbation given by a stable rational function satisfying $\|\Delta\|_\infty < 1$; we call such Δ and *admissible* perturbation. The weighting function $W(s)$ is assumed a fixed stable, minimum-phase rational function, and such that $F(s)W(s)P(s)$ is proper. This type of uncertainty model is useful to represent high frequency plant uncertainty Doyle and Stein [1981].

Assuming closed loop stability of the nominal hybrid system, i.e., for $\Delta(s) = 0$, we shall determine necessary and sufficient conditions for the perturbed system to remain stable under the class of admissible perturbations.

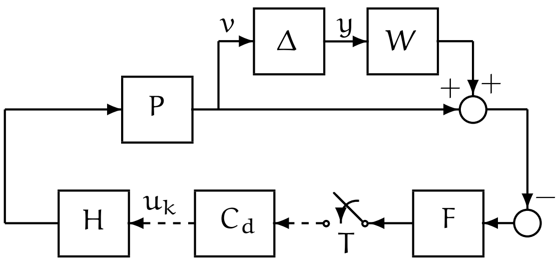

Figure 6.2: System with multiplicative uncertainty.

Suppose that we discretize the system of Figure 6.2 by opening the loop at the input and output of the discrete controller C_d. Then, we obtain the simplified discrete diagram of Figure 6.3, where $(F\tilde{P}H)_d$ is the discretized series of hold, perturbed plant and anti-aliasing filter. Applying Corollary 2.1.4 to $(F\tilde{P}H)_d$ yields the infinite sum representation

$$(F\tilde{P}H)_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(s)(I + W_k(s)\Delta_k(s))P_k(s)H_k(s)\ . \tag{6.2}$$

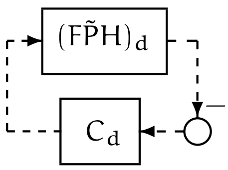

Figure 6.3: Discretized perturbed system.

Equation (6.2) displays the multi frequency structure induced by the sampling operation. This relation can be translated directly into the block diagram of Figure 6.4. Note in this picture that although the sampler is not represented explicitly, its action is structurally embedded in the block diagram as the parallel of an infinite number of direct paths where each harmonic component of the signals operates. We use this representation to derivate an expression where all the perturbations Δ_k are blocked together.

Take the k-harmonic direct path in Figure 6.4. Then, we can write

$$V_k(s) = \frac{1}{T}P_k(s)\,H_k(s)\,U_d(e^{sT}), \tag{6.3}$$

where U_d is the $\mathcal{Z}$-transform of the output of the controller. To ease notation, we shall drop the independent variables in the sequel of this derivation, understanding that all signals and transfer functions are functions of s, save for the discrete ones, like C_d and U_d, which are functions of e^{sT}. Now, we have that U_d is given

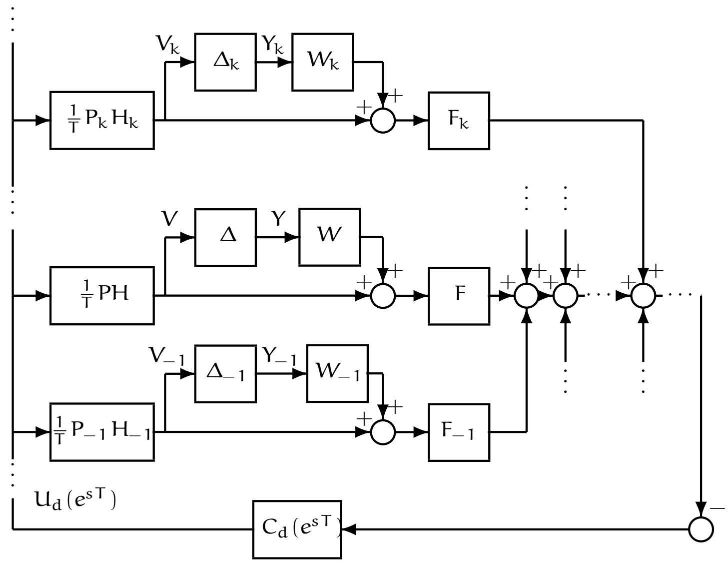

Figure 6.4: Harmonic structure of the perturbed system.

by

$$\begin{aligned} U_d &= -C_d \sum_{k=-\infty}^{\infty} F_k \left(W_k Y_k + \frac{1}{T} P_k H_k U_d \right) \\ &= -C_d \sum_k F_k W_k Y_k - C_d \left(\frac{1}{T} \sum_k F_k P_k H_k \right) U_d \ . \end{aligned} \tag{6.4}$$

Noting that by Corollary 2.1.4 $1/T \sum_k F_k P_k H_k$ is the nominal discretized plant $(FPH)_d$, from (6.4) we get

$$U_d = -S_d C_d \sum_k F_k W_k Y_k \ , \tag{6.5}$$

where

$$S_d(z) = [I + C_d(z)(FPH)_d(z))]^{-1} \tag{6.6}$$

is the nominal discrete Sensitivity Function. Now, replacing U_d from (6.5) and $Y_k = \Delta_k V_k$ into (6.3) yields

$$V_k = \frac{1}{T} P_k H_k S_d C_d \sum_m F_m W_m \Delta_m V_m \ . \tag{6.7}$$

In the lifted domain, (6.7) can be written as

$$(\mathbf{I} + \mathbf{T}_\omega \mathbf{W}_\omega \mathbf{\Delta}_\omega)\, \mathbf{v} = 0, \tag{6.8}$$

where $\mathbf{I}$ is the infinite identity matrix (the identity operator in ℓ_2) and $\mathbf{T}_\omega$ is the infinite matrix representation of the complementary sensitivity operator, defined in (5.7). $\mathbf{W}_\omega$ and $\mathbf{\Delta}_\omega$ are infinite-dimensional block diagonal matrices,

$$\mathbf{W}_\omega \triangleq \text{diag}[\ldots, W_k(j\omega), W_{k-1}(j\omega), \ldots],$$

and

$$\mathbf{\Delta}_\omega \triangleq \text{diag}[\cdots, \Delta_k(j\omega), \Delta_{k-1}(j\omega), \cdots],$$

while $\mathbf{v}$ is the lifted vector

$$\mathbf{v}(\omega) \triangleq \begin{bmatrix} \vdots \\ V_1(\omega) \\ V_0(\omega) \\ V_{-1}(\omega) \\ \vdots \end{bmatrix} . \tag{6.9}$$

Equation (6.8) collects system knowns and perturbations in two separated blocks, as in the basic perturbation model of Figure 6.5. Thus, we can see clearly in the form of $\mathbf{\Delta}_\omega$ how the original time-varying problem with unstructured analog perturbations conduces to a time-invariant, infinite-dimensional, problem with a very structured class of perturbations. From this setup it is standard to derive the conditions for the internal stability of the loop of Figure 6.5 as a μ-test.

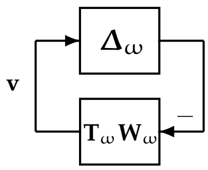

Figure 6.5: Basic perturbation model for multiplicative uncertainty.

Before proceeding, we need to recall a few definitions relative to the structured singular value μ required to state the results; we refer for example to Packard and Doyle [1993] for more details. The structured singular value of a given $n \times n$ complex matrix M is a nonnegative real number defined with respect to a set $\mathbf{\Delta}$ of perturbation matrices Δ in $\mathbb{C}^{n\times n}$ of prescribed structure. Denote by $\bar{\sigma}\{\Delta\}$ the maximum singular value of Δ. Then we define $\mu_{\mathbf{\Delta}}(M)$ as

$$\mu_{\mathbf{\Delta}}(M) \triangleq \frac{1}{\min\limits_{\Delta\in\mathbf{\Delta}}\{\bar{\sigma}\{\Delta\} : \det(I - M\Delta) = 0\}},$$

unless no $\Delta \in \mathbf{\Delta}$ makes $(I - M\Delta)$ singular, in which case $\mu_{\mathbf{\Delta}}(M) \triangleq 0$. The perturbation set $\mathbf{\Delta}$ is defined as the set ol perturbations Δ of the form

$$\Delta = \text{diag}[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_S I_{r_S}, \Delta_{S+1}, \ldots, \Delta_{S+F}], \tag{6.10}$$

where $\delta_i \in \mathbb{C}$, $\Delta_{S+j} \in \mathbb{C}^{m_j \times m_j}$, for $i = 1, 2, \ldots, S$, and $j = 1, 2, \ldots, F$. With I_{r_i} we denote $\mathbb{C}^{r_i \times r_i}$ identity matrices. Note that for dimensional consistency it is necessary that $\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$.

With these definitions, we can now state necessary and sufficient conditions for robust stability of the hybrid system of Figure 6.2, adapted from the result obtained by Dullerud and Glover [1993]. The result reduces to a μ-problem on the infinite dimensional matrices of Figure 6.5, and it is expressed as a sequence of all the finite dimensional μ-problems obtained by truncating the original matrices. Denote by $[\mathbf{T}_\omega]^n$, $[\mathbf{W}_\omega]^n$ and $[\boldsymbol{\Delta}_\omega]^n$ the corresponding truncations keeping all harmonics between $-n$ and n, for some positive integer n. For each ω in Ω_N, $[\boldsymbol{\Delta}_\omega]^n$ has a block diagonal structure, where each block $\Delta_k(j\omega)$ is as in (6.10). Let $\boldsymbol{\Delta}^n$ denote the set of all these finite dimensional block diagonal matrix perturbations, $\boldsymbol{\Delta}^n \triangleq \{\text{diag}[\Delta_n, \ldots, \Delta_{-n}] : \Delta_i \in \boldsymbol{\Delta}\}$. Then, we have the following proposition.

Proposition 6.1.1 (Dullerud and Glover [1993])
For all Δ such that $\|\Delta\|_\infty < 1$ the system of Figure 6.2 is internally stable if and only if for each integer $n > 0$ the following inequality is satisfied

$$\max_{\omega \in \Omega_N} \mu_{\boldsymbol{\Delta}^n}([\mathbf{T}_\omega]^n [\mathbf{W}_\omega]^n) \leq 1 \ . \tag{6.11}$$

◦

As mentioned before, although we started with unstructured perturbations on the analog plant, they are mapped into a very structured type of perturbations in the lifted space $L_2(\ell_2; \Omega_N)$. In general, assuming also $\boldsymbol{\Delta}_\omega$ unstructured Sivashankar and Khargonekar [1993] will lead to a small-gain type of test in terms of the L_2-induced norm

$$\|\mathbf{T}_\omega \mathbf{W}_\omega\| \leq 1.$$

This small-gain condition is only sufficient for LTI perturbations, and it may be quite conservative. This has been analyzed by means of example by Dullerud and Glover [1993]. Other interesting related remarks are given in Hagiwara and Araki [1995].

A necessary condition for robust stability may be stated in terms of the fundamental complementary sensitivity function of Chapter 4.

Theorem 6.1.2 (Necessary Condition for Robust Stability)
A necessary condition for the the system of Figure 6.2 to remain stable for all Δ such that $\|\Delta\|_\infty < 1$ is that

$$\|T^0(j\omega) W(j\omega)\|_\infty \leq 1. \tag{6.12}$$

Proof: It is necessary for closed loop stability that

$$\tilde{S}_d(z) = [I + C_d(z)(F\tilde{P}H)_d(z)]^{-1} \tag{6.13}$$

have no poles in $\mathbb{D}^C$. Rearranging yields

$$\tilde{S}_d(z) = [I + S_d(z)C_d(z)(FW\Delta PH)_d(z)]^{-1}S_d(z). \tag{6.14}$$

Since the nominal system is stable, then $\tilde{S}_d$ will have no poles in $\mathbb{D}^C$ if and only if

$$\det[I + S_d(e^{j\omega T})C_d(e^{j\omega T})(FW\Delta PH)_d(e^{j\omega T})] \neq 0 \quad \text{for all } \omega. \tag{6.15}$$

The proof proceeds by contradiction, following that in Chen and Desoer [1982, Theorem 2]. Denote $Q(j\omega) \triangleq T^0(j\omega)W(j\omega)$, and suppose that (6.12) is violated. Then there exists a frequency ω_1 such that $\sigma_1 \triangleq \bar{\sigma}\{Q(j\omega_1)\} > 1$, where $\bar{\sigma}\{\cdot\}$, recall, denotes the maximum singular value. Performing a singular value decomposition of $Q(j\omega_1)$ yields

$$Q(j\omega_1) = U\,\mathrm{diag}[\sigma_1 \ldots]\,V^*,$$

where $U \triangleq \{u_{ij}\}$ and $V \triangleq \{v_{ij}\}$ are unitary matrices. Now assume for the moment that there exists an admissible $\check{\Delta}$ that also satisfies

$$\begin{aligned}\check{\Delta}(j\omega_1) &= \begin{bmatrix} v_{11} \\ \vdots \\ v_{n1} \end{bmatrix} (-\sigma_1)^{-1} \begin{bmatrix} u_{11}^* & \ldots & u_{n1}^* \end{bmatrix} \\ &= V\,\mathrm{diag}[(-\sigma_1)^{-1}, 0, \ldots, 0]\,U^*, \end{aligned} \tag{6.16}$$

and

$$\check{\Delta}(j(\omega_1 + k\omega_s)) = 0 \quad \text{for } k = \pm 1, \pm 2, \ldots, \text{ and } k \neq -2\omega_1/\omega_s. \tag{6.17}$$

The assumptions on W, and Δ, imply that Corollary 2.1.4 may be used to calculate $(FW\check{\Delta}PH)_d$. Using (6.16) and (6.17) yields

$$(FW\check{\Delta}PH)_d(e^{j\omega_1 T}) = -\frac{1}{T}F(j\omega_1)W(j\omega_1)V\,\mathrm{diag}[-\frac{1}{\sigma_1}, 0, \ldots, 0]\,U^*P(j\omega_1)H(j\omega_1), \tag{6.18}$$

and therefore[1]

$$\begin{aligned} &\det[I + S_d(e^{j\omega_1 T})C_d(e^{j\omega_1 T})(FW\check{\Delta}PH)_d(e^{j\omega_1 T})] \\ &\qquad = \det[I + S_d C_d \frac{1}{T} FWV\,\mathrm{diag}[(-\sigma_1)^{-1}, 0, \ldots, 0]\,U^*PH] \\ &\qquad = \det[I + V\,\mathrm{diag}[(-\sigma_1)^{-1}, 0, \ldots, 0]\,U^*\frac{1}{T}PHS_dC_dFW] \\ &\qquad = \det[I + V\,\mathrm{diag}[(-\sigma_1)^{-1}, 0, \ldots, 0]\,U^*Q(j\omega_1)] \\ &\qquad = [I + V\,\mathrm{diag}[-1, 0, \ldots, 0]\,V^*] \\ &\qquad = \det[V]\ \det[\mathrm{diag}[0, 1, 1, \ldots, 1]]\ \det[V^*] \\ &\qquad = 0. \end{aligned}$$

[1]We suppress dependence on the transform variable when convenient, meaning will be clear from context.

Hence, (6.15) fails and so the perturbed system is unstable.

It remains to show that $\check{\Delta}$ satisfying the required properties exists. We do this following a construction in Chen and Desoer [1982]. Consider

$$\check{\Delta}(s) \triangleq \begin{bmatrix} \alpha_1(s) \\ \vdots \\ \alpha_n(s) \end{bmatrix} \left(-\frac{1}{\sigma_1}\right) f_q(s)^{k'} z(s) \left[\beta_1(s), \dots, \beta_n(s)\right]$$

where k' is a natural number, and

$$\begin{aligned} f_q(s) &\triangleq \frac{\omega_1 s}{q(s^2+\omega_1^2)+\omega_1 s}, \quad q > 0 \\ \alpha_i(s) &\triangleq \frac{s}{\omega_1}\mathbb{I}m\{v_{i1}\} + \mathbb{R}e\{v_{i1}\} \\ \beta_i(s) &\triangleq -\frac{s}{\omega_1}\mathbb{I}m\{u_{i1}\} + \mathbb{R}e\{u_{i1}\} \\ z(s) &\triangleq \frac{H_{ZOH}(s-j\omega_1)H_{ZOH}(s+j\omega_1)}{T|H_{ZOH}(j2\omega_1)|}\eta(s), \\ \eta(s) &\triangleq \left(-\frac{s}{\omega_1}\sin(\sphericalangle H_{ZOH}(j2\omega_1)) + \cos(\sphericalangle H_{ZOH}(j2\omega_1))\right), \end{aligned}$$

where $H_{ZOH}(s)$ is the frequency response function of the ZOH, and $\sphericalangle$ denotes the phase of a complex number. It is then straightforward to verify that

(i) $\check{\Delta}(j\omega_1)$ satisfies (6.16) and (6.17), and

(ii) by choosing both k' and q large enough, $\check{\Delta}$ is exponentially stable and, for all $\omega \neq \pm\omega_1$, $\lim_{\omega\to\infty} \bar{\sigma}\{\check{\Delta}(j\omega)\} \to 0$, i.e., $\|\check{\Delta}\|_\infty < 1$ is satisfied.

□

In relation to the results of Chapter 4, for SISO systems follows that if $|T^0(j\omega)|$ is very large at any frequency, then the hybrid system will exhibit poor robustness to uncertainty in the analog plant at that frequency.

The necessary and sufficient condition of Proposition 6.1.1 yields an explicit expression that also involves T^0 in the SISO case. We state this in the following corollary.

Corollary 6.1.3 (Robust Stability Test — SISO case)
If the system of Figure 6.2 is SISO, then, for all Δ satisfying $\|\Delta\|_\infty < 1$, the system is internally stable if and only if

$$\sum_{k=-\infty}^{\infty} |T^0(j(\omega + k\omega_s))\, W(j(\omega + k\omega_s))| \leq 1 \quad \textit{for all } \omega \textit{ in } \Omega_N \tag{6.19}$$

Proof: By Proposition 6.1.1, the system will be robustly stable if and only if all truncated systems satisfy μ-condition (6.11). Fix an integer $n > 0$ and ω in Ω_N.

The standard approach to evaluate $\mu_{\mathbf{\Delta}^n}([\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n)$ is through the computation of upper and lower bounds. Define the sets

$$\mathcal{Q} \triangleq \{\mathbf{Q} \in \mathbf{\Delta}^n : \mathbf{Q}^*\mathbf{Q} = I\} \tag{6.20}$$
$$\mathcal{D} \triangleq \{\mathbf{D} \in \mathbf{\Delta}^n : \mathbf{D} = \mathbf{D}^* > 0 \text{ and } \mathbf{D}\mathbf{\Delta}^n_\omega = \mathbf{\Delta}^n_\omega\mathbf{D} \text{ for all } \mathbf{\Delta}^n_\omega \in \mathbf{\Delta}^n\} \;. \tag{6.21}$$

Then we have the following inequalities Packard and Doyle [1993]

$$\max_{\mathbf{Q}\in\mathcal{Q}} \rho(\mathbf{Q}[\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n) \le \mu_{\mathbf{\Delta}^n}([\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n) \le \inf_{\mathbf{D}\in\mathcal{D}} \bar{\sigma}\{\mathbf{D}[\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n\mathbf{D}^{-1}\} \,. \tag{6.22}$$

Note that the structure of the uncertainty in this case is diagonal,

$$\mathbf{\Delta}^n_\omega = \mathrm{diag}[\delta_n(j\omega), \ldots, \delta_{-n}(j\omega)],$$

with $\delta_i(j\omega)$ in $\mathbb{C}$. As the truncated $[\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n$ is rank-one, we can work out in closed form the values of $\rho(\mathbf{Q}[\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n)$ and $\bar{\sigma}\{\mathbf{D}[\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n\mathbf{D}^{-1}\}$ in (6.22). We show that there exist matrices $\mathbf{Q}_0$ and $\mathbf{D}_0$ such that upper and lower bounds in (6.22) coincide, yielding the expression for $\mu_{\mathbf{\Delta}^n}([\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n)$. To lighten notation we write in the remaining $\mathbf{TW}$ for $[\mathbf{T}_\omega]^n[\mathbf{W}_\omega]^n$.

We compute first the lower bound, i.e., the spectral radius $\rho(\mathbf{QTW})$. Since in the SISO case the complementary sensitivity operator is rank-one, so is $\mathbf{TW}$, and its matrix may then be written as a dyad, i.e., in an outer product form, $\mathbf{TW} = \mathbf{g}\,\mathbf{w}^*$, where the vectors

$$\mathbf{g} = \frac{1}{T} S_d\, C_d \begin{bmatrix} P_n\, H_n \\ P_{n-1}\, H_{n-1} \\ \vdots \\ P_{-n}\, H_{-n} \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} F_n^* W_n^* \\ F_{n-1}^* W_{n-1}^* \\ \vdots \\ F_{-n}^* W_{-n}^* \end{bmatrix} \,.$$

Then, $\mathbf{QTW}$ is also a rank-one matrix, and its only eigenvalue is $\lambda = \mathbf{w}^*\mathbf{Q}\mathbf{g}$, so $\rho(\mathbf{QTW}) = |\mathbf{w}^*\mathbf{Q}\mathbf{g}|$.

Consider the particular matrix $\mathbf{Q}_0 = \mathrm{diag}[Q_n, Q_{n-1}, \ldots, Q_{-n}]$, with

$$Q_i \triangleq \begin{cases} \dfrac{P_i^*\, H_i^*\, C_d^*\, S_d^*\, F_i^* W_i^*}{|P_i\, H_i\, C_d\, S_d\, F_i W_i|} & \text{if } P_i\, H_i\, C_d\, S_d\, F_i\, W_i \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

Then

$$\rho(\mathbf{Q}_0\mathbf{TW}) = \sum_{i=-n}^{n} |P_i\, H_i\, C_d\, S_d\, F_i|\, W_i, \tag{6.23}$$

and $\mathbf{Q}_0$ is certainly in $\mathcal{Q}$.

We now consider the upper bound $\bar{\sigma}\{\mathbf{DTWD}^{-1}\}$. The 2-norm of a rank-one matrix $\mathbf{TW} = \mathbf{g}\,\mathbf{w}^*$ is given by $\bar{\sigma}\{\mathbf{TW}\} = \rho(\mathbf{TW}^*\mathbf{TW})^{1/2} = \bar{\sigma}\{\mathbf{g}\}\,\bar{\sigma}\{\mathbf{w}\}$. Consider $\bar{\sigma}\{\mathbf{D}_0\mathbf{TWD}_0^{-1}\}$ with $\mathbf{D}_0 = \mathrm{diag}[D_n, D_{n-1}, \ldots, D_{-n}]$ and let

$$D_i \triangleq \begin{cases} \left|\dfrac{TF_i W_i}{P_i H_i C_d S_d}\right|^{1/2} & \text{if } P_i H_i C_d S_d F_i W_i \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore,

$$\bar{\sigma}\{\mathbf{D}_0\mathbf{TWD}_0^{-1}\} = \sum_{i=-n}^{n} |P_i\, H_i\, C_d\, S_d\, F_i\, W_i| \,, \tag{6.24}$$

with $\mathbf{D}_0$ in $\mathcal{D}$.

From (6.22), (6.23), and (6.24), we conclude that

$$\mu_{\Delta^n}(\mathbf{QTW}) = \sum_{i=-n}^{n} |P_i\, H_i\, C_d\, S_d\, F_i\, W_i| \,. \tag{6.25}$$

Note that (6.25) is valid for an arbitrary integer $n > 0$ and ω in Ω_N.

The proof is completed by recalling that

$$T^0(s) = \frac{1}{T} P(s)\, H(s)\, C_d(e^{sT})\, S_d(e^{sT}) F(s)$$

and using Proposition 6.1.1. □

Again, as for Theorem 6.1.2, we see from this result that a large value of T^0 at any frequency reduces the stability robustness properties of the system at that frequency. Notice that in this case the condition is an ℓ_1-type condition on the lifted vector representing T^0, in contrast to that of Theorem 6.1.2, which is an ℓ_∞-type condition. Hence, for the SISO case, the condition of Theorem 6.1.2 is straightforwardly implied by condition (6.19), since $\ell_1 \subset \ell_\infty$.

Remark 6.1.1 This result may also be obtained dispensing with the μ-framework, in a similar way to Theorem 6.1.2. An outline of this alternative proof is provided in Appendix A, §A.5. ◇

6.2 Divisive Perturbation

We now consider the stability robustness properties of the sampled-data system of Figure 6.6, i.e., with a divisive type of uncertainty model. We assume that Δ and W satisfy the conditions stated in §6.1. The perturbed plant is represented by

$$\tilde{P}(s) = (I + W(s)\Delta(s))^{-1}\, P(s) \,. \tag{6.26}$$

The derivation of necessary and sufficient conditions for robust stability of the hybrid system with this class of perturbations is considerably more difficult than the multiplicative case, and remains as a challenging open problem. In this section we show that the problem can be also represented by a basic perturbation model, where the infinite dimensional matrix $\mathbf{S}_\omega$ appears in the interconnection matrix. A small-gain type sufficient condition follows immediately from this representation. We also provide a necessary condition for the SISO case, that imposes a bound on the values of the fundamental sensitivity function of Chapter 4 on the $j\omega$-axis.

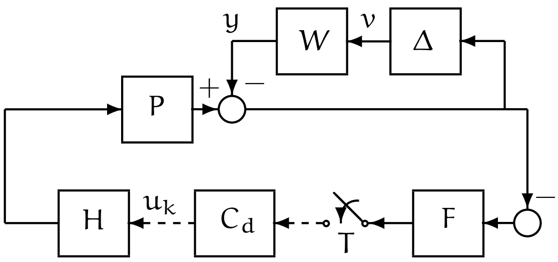

Figure 6.6: System with divisive uncertainty.

Again, we have for Δ and W the same assumptions made for the multiplicative uncertainty in the previous subsection. This time, the discretized perturbed plant is given by

$$(F\tilde{P}H)_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(s)(I + W_k(s)\Delta_k(s))^{-1}P_k(s)H_k(s)\;. \tag{6.27}$$

Following similar steps to those for the multiplicative case, we obtain the block diagram of Figure 6.7 displaying the harmonic structure of the system arising from the sampling process.

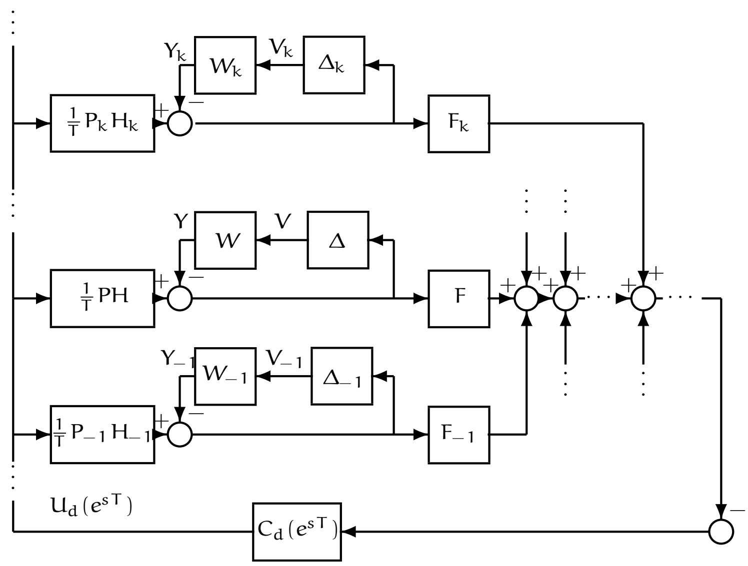

Figure 6.7: Harmonic structure of the perturbed system with divisive uncertainty.

Based on this picture, we compute an expression analogous to (6.7) for the

k-harmonic of the input to the uncertainty block, V, getting this time

$$V_k = -W_k \Delta_k V_k + \frac{1}{T} P_k H_k S_d C_d \sum_m F_m W_m \Delta_m V_m \ . \tag{6.28}$$

The lifted version of (6.28) is then

$$(\mathbf{I} + \mathbf{S}_\omega \mathbf{W}_\omega \boldsymbol{\Delta}_\omega)\, \mathbf{v} = 0 \ , \tag{6.29}$$

with the same notation used in the preceding subsection. As could have been intuitively expected, now the *sensitivity operator* appears in the formula through its infinite matrix representation $\mathbf{S}_\omega$, defined in (5.8). As before, the underlying μ-problem is evident from (6.29), also represented in the basic perturbation model of Figure 6.8.

A sufficient condition for stability of the perturbed system is evident from (6.29). Indeed, if the following inequality is satisfied,

$$\|\mathbf{S}_\omega \mathbf{W}_\omega\| \leq 1,$$

then the operator $(\mathbf{I} + \mathbf{S}_\omega \mathbf{W}_\omega \boldsymbol{\Delta}_\omega)$ is non-singular, which implies internal stability of the basic perturbation model.

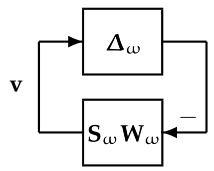

Figure 6.8: Basic perturbation model for divisive uncertainty.

Remark 6.2.1 (Robust Stability Test under Divisive Perturbations) We conjecture that a result analogous to Proposition 6.11 will be valid in this case also, i.e., the system will be robust stable under divisive perturbation if and only if all the truncated μ-problems corresponding to Figure 6.8 satisfy a stability condition. In other words, if and only if for all admissible LTI perturbations and each integer $n > 0$

$$\max_{\omega \in \Omega_N} \mu_{\boldsymbol{\Delta}^n}(\mathbf{S}^n_\omega \mathbf{W}^n_\omega) \leq 1 \ . \tag{6.30}$$

A proof for this result is not obvious to us at present, and it remains as a topic for future research. We can foresee a greater difficulty in this case since the sensitivity operator is *non-compact*, and non-compact operators are not necessarily *approximable* by a sequence of finite-rank operators. Therefore, special care should be taken to show that the infinite sequence of μ-problems in (6.30) indeed converges when $n \to \infty$.

Nevertheless, a hint that a proof for this conjecture could be possible is perhaps suggested by the same fact that allowed us to compute a “closed form” for the frequency-gain of this operator in Chapter 5, i.e., the sensitivity operator $\mathcal{S}$ is not an arbitrary non-compact operator, since it can be written as

$$\mathcal{S} = \mathcal{I} - \mathcal{T},$$

where $\mathcal{T}$ is always finite-rank. Moreover, notice that a condition like (6.30) would in principle be valid if we impose additional restrictions on the weighting function W, e.g., if it is assumed stable and strictly proper. In this case the corresponding infinite matrix $\mathbf{W}_\omega$ represents a compact operator, which also makes the product $\mathbf{S}_\omega \mathbf{W}_\omega$ compact. $\diamond$

A necessary condition for robust stability is easily obtained in the SISO case. In parallel with the result of Theorem 6.1.2, this condition involves the fundamental sensitivity function S^0, as we see next.

Lemma 6.2.1
A necessary condition for the the system of Figure 6.6 to remain stable for all Δ such that $\|\Delta\|_\infty < 1$ is that

$$\|S^0(j\omega)W(j\omega)\|_\infty \leq 1. \tag{6.31}$$

Proof: The proof follows the same lines of that of Theorem 6.1.2 after noting that we can alternatively write the perturbed discrete sensitivity function as

$$\begin{aligned}\tilde{S}_d &= \left[1 + C_d(F\tilde{P}H)_d\right]^{-1} \\ &= \left[1 + C_d(FPH)_d - C_d\left(\frac{F\Delta WPH}{1+W\Delta}_d\right)\right]^{-1} \\ &= \left[1 - S_dC_d\left(\frac{F\Delta WPH}{1+W\Delta}_d\right)\right]^{-1} S_d. \end{aligned} \tag{6.32}$$

That the nonsingularity of the term between brackets in (6.32) implies (6.31) may be shown by a contrapositive argument similar to that for the proof of Theorem 6.1.2, and is omitted here to avoid repetition. $\square$

In connection with the results of Chapter 4, this lemma shows that if $|S^0(j\omega)|$ is large at any frequency, then the system will have poor robustness to divisive uncertainties in the analog plant at that frequency.

6.3 Summary

In this chapter we have considered the stability robustness of a hybrid system to unstructured LTI perturbations of the analog plant.

Using the frequency-domain lifting introduced in Chapter 5, we have derived a robust stability test in the form of a structured singular value for the case of multiplicative perturbations. The expression obtained was first given by Dullerud and Glover [1993] based on time-domain lifting techniques. Our procedures, though, are considerably simplified by the use of the frequency-domain lifting technique.

For the case of divisive perturbations, our framework allows the problem to be easily recasted as a basic perturbation model, from which a small-gain type sufficient condition for robust stability is directly obtained. The derivation of necessary and sufficient conditions for this type of perturbation model is a much harder problem than that of multiplicative perturbations, and is left as subject of ulterior research.

For both types of perturbation models, we have drawn important connections with the discussion of Chapter 4 by obtaining necessary conditions for robust stability of the hybrid system in terms of the fundamental sensitivity and complementary sensitivity functions S^0 and T^0. A key conclusion of these results is that large peaks in either S^0 or T^0 will necessarily degrade the robustness stability properties of the hybrid system respect to uncertainty in the analog plant.