The Half-Duplex Gaussian Two-Way Relay Channel with Direct Links

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Abstract—We study the half-duplex Gaussian two-way relay channel with direct user-to-user links. In this setup, two users exchange data via a relay and via direct user-to-user links. Due to the half-duplex constraint, the channel can be in one of eight different states at any time (two of which are useless: no node transmitting and no node listening). Restricting to only four states, we propose a scheme that utilizes lattice codes to improve upon existing four-state schemes. Using all six states, we propose another scheme that utilizes lattice codes and coherent combining, and show that it can outperform existing schemes.

Index Terms—Two-way relay channel, bi-directional relaying, lattice codes, half duplex, Gaussian

I. INTRODUCTION

We study a class of relay-aided networks where two users exchange data through a relay. This network configuration—commonly found in the cellular mobile network, the satellite network, and the WiFi network—is referred to as the two-way relay channel or the bi-directional relaying channel. Common assumptions for the two-way relay channel are that the nodes operate in the half-duplex mode, (i) the nodes operate in the half-duplex mode, and (ii) the nodes operate in the half-duplex mode, and both users listen, and (2) the relay listens and both users transmit.

Under the half-duplex constraint, the channel operates in one of eight states at any time, depending on which nodes transmit and which listen. Using only four states, Kim et al. [6] designed a coding scheme where the relay completely decodes all the messages, bins the messages, and forwards the addition of the bin indices. Also using only four states, Ghasemi-Goojani and Behroozi [7] built on the idea of nested lattice codes (which was designed for the full-duplex two-way relay channel [1]) by proposing an intermediate lattice to achieve a better rate region. In this paper, we consider the case where (i) the nodes operate in the half-duplex mode, and (ii) there are user-to-user links.

In the two-way relay channel (see Figure 1), node 1 wishes to send a message, denoted by $M_1$, to node 2; node 2 wishes to send a message, $M_2$, to node 1. Node 3, who has no message to transmit, facilitates the message exchange. In this setup, nodes 1 and 2 are the users, and node 3 the relay. Let $X_i \in \mathbb{R}$ and $Y_i \in \mathbb{R}$ be the channel input and channel output, respectively, of node $i$, for $i \in \{1, 2, 3\}$.

We consider the half-duplex channel, where the channel state (a description of which node transmits and which node listens) is pre-determined and made known to all nodes a priori. Denote the channel state by a triplet $(s_1, s_2, s_3)$, where $s_i = 0$ if node $i$ listens, and $s_i = 1$ otherwise (i.e., if node $i$ transmits). It is convenient to represent the channel state by an integer $s = \sum_{i=1}^{3} s_i 2^{i-1}$. With this notation, the channel states for all $s \in \{0, 1, \ldots, 7\}$ are depicted in Figure 2.

Consider $n$ uses of the memoryless Gaussian two-way relay channel, where the $t$-th channel use is defined as follows:

\[ X_t \rightarrow R \rightarrow Y_t \]

\[ X_t \rightarrow Y_t \]

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\[ 1 \]There are fundamental differences between a full-duplex channel and its half-duplex counterpart. See the single-relay channel [5] for example.

\[ 2 \]Without the user-to-user links, there are only two useful states: (1) the relay transmits and both users listen, and (2) the relay listens and both users transmit.
Y_j[t] = \begin{cases} 
\sum_{i \in \{1,2,3\} \setminus \{j\}} h_{i,j}X_i[t] + Z_j[t], & \text{if } s_j[t] = 0 \\
0, & \text{otherwise,} 
\end{cases} 

(1)

for j \in \{1, 2, 3\} and t \in \{1, 2, \ldots, n\}. Here, h_{i,j} is the channel gain from node i to node j, which is constant for all channel uses. We do not impose the restriction that h_{i,j} = h_{j,i}. The noise term Z_j[t] is a zero-mean Gaussian random variable with variance \sigma_j^2, and is independent for each j and t. (s_1[t], s_2[t], s_3[t]) is the pre-determined channel state of the t-th channel use.

For simplicity, for any user i \in \{1, 2\}, we denote the other user by \tilde{i} \triangleq (i \mod 2) + 1. Let \nu_k be the number of channel uses (out of n) when the channel is in state k, i.e., \nu_k \triangleq |\{t \in \{1, 2, \ldots, n\} : s[t] = k\}|.

An (R_1, R_2, n) block code is defined as follows:

- A set of channel states: s = (s[1], s[2], \ldots, s[n]).
- Two message sets: \mathcal{M}_i \in \{1, 2, \ldots, 2^{nR_i}\}, for i \in \{1, 2\}.
- Three sets of encoding functions: X_i[t] = f_{i,t}(M_i, Y_i[1], Y_i[2], \ldots, Y_i[t-1], s_i[t]), for i \in \{1, 2, 3\} and t \in \{1, 2, \ldots, n\}, subject to the constraint that X_i[t] = 0 if s_i[t] = 0, and the following per-state\textsuperscript{3} transmitted power constraint:

\[ \frac{1}{\nu_k} \sum_{l \in \{1, 2, \ldots, \nu_k\} \text{s.t. } s[l] = k \text{ and } s_i[l] = 1} E[X_i[l]^2] \leq P_i, \]

(2)

for each node i \in \{1, 2, 3\} and each state k. We set M_0 = 0.

- Two decoding functions: \tilde{M}_i = g_i(M_i, Y_i[1], Y_i[2], \ldots, Y_i[n], s), for i \in \{1, 2\}.

Assuming that each message is uniformly distributed, the error probability is defined as \Pr(M_1 \neq M_i \text{ or } M_2 \neq M_i). A rate pair (R_1, R_2) is said to be achievable if given any \epsilon > 0, there exists an (R_1, R_2, n) code such that \Pr \leq \epsilon.

To simplify expressions in our analyses, we define pair-wise signal-to-noise ratio (SNR) \lambda_{i,j} \triangleq h_{i,j}^2P_i/\sigma_j^2 for all node pairs (i, j), and \tau_k \triangleq \nu_k/n for all states k.

Note that Y_i[t] = 0 for all i \in \{1, 2, 3\} in states 0 and 7. So, it suffices to consider only states 1 to 6, and set t_0 = t_7 = 0.

III. ACHIEVABLE REGIONS USING ONLY FOUR STATES

In this section, we study two existing schemes that use only four channel states (i.e., states 1–4). We identify their shortcomings, and propose a new scheme that utilizes index coding and simultaneous decoding.

A. The Hybrid Broadcast Scheme using Decode-Forward and Binning

Kim et al. [6] proposed the hybrid broadcast (HBC) scheme, which uses functional-decode-forward and binning, described briefly as follows:

- In state i \in \{1, 2\}, user i broadcast its message \mathcal{M}_i to the relay and user \tilde{i}.
- In state 3, both users simultaneously send \{M_i\} again to the relay, using another independently generated codebook.
- The relay decodes \mathcal{M}_1 and \mathcal{M}_2 over states 1–3.
- In state 4, the relay randomly assigns the message \mathcal{M}_i into 2^{nR_i} bins. Let the bin indices be b_i(M_i). The relay broadcasts s_1(M_1) + s_2(M_2) mod 2^{n\max\{R_1, R_2\}}.
- User \tilde{i} \in \{1, 2\} decodes b_i(M_i) over state 4, and then \mathcal{M}_i over state \tilde{i} with the help of b_i(M_i).

Lemma 1 (The HBC Scheme [6]): The HBC scheme achieves a rate region \mathcal{R}_{HBC} \subset \mathbb{R}_+^2, consisting of all non-negative rate pairs (R_1, R_2), each satisfying

\[ R_1 < t_1 \left( \frac{t_1}{2} \log(1 + \lambda_{1,3}) + \frac{t_3}{2} \log(1 + \lambda_{1,3}) \right) \leq A_i, \]

(3)

\[ R_1 < t_1 \left( \frac{t_1}{2} \log(1 + \lambda_{1,3}) + \frac{t_3}{2} \log(1 + \lambda_{1,3}) \right) \leq B_i, \]

(4)

\[ R_1 + R_2 < A_1 + A_2 + \frac{t_2}{2} \log(1 + \lambda_{1,3} + \lambda_{2,3}) \leq D_i, \]

(5)

for all i \in \{1, 2\}.

B. The Ghasemi-Behroozi Scheme using Lattice Codes

Recently, Ghasemi-Goojani and Behroozi [7] proposed a coding scheme, which extends the lattice-based coding scheme for the full-duplex two-way relay channel [1] using their proposed intermediate lattice. They claimed that their scheme achieves a larger region than the HBC scheme.

The scheme, referred to as the G-S scheme, is as follows:

- Form four nested lattices \mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_3 \subset \mathcal{L}_c \subset \mathbb{R}^{n_3}, where \mathcal{L}_0 is the intermediate lattice.
- User i \in \{1, 2\} maps a message \mathcal{M}_i to a lattice point \mathcal{V}_i \in \{\mathcal{L}_c \cap \mathcal{V}_i\}, where \mathcal{V}_i is the Voronoi region of \mathcal{L}_i [10]. This lattice point is decomposed into two sub-messages: (i) \mathcal{V}_{a_i} \in \{\mathcal{L}_c \cap \mathcal{V}_i\} of rate \mathcal{R}_{a_i}, and (ii) \mathcal{V}_{b_i} \in \{\mathcal{L}_b \cap \mathcal{V}_i\} of rate \mathcal{R}_{b_i}, where \mathcal{R}_{a_i} + \mathcal{R}_{b_i} = \mathcal{R}_i.
- In state i \in \{1, 2\}, user i sends \mathcal{V}_{a_i} and \mathcal{V}_{b_i} using the superposition of two random Gaussian codewords.
• The relay recovers only $V_{ai}$ over state $i$, $i \in \{1, 2\}$.
• In state 3, the two users simultaneously send $\{V_{bi}\}$.
• Knowing $V_{a1}$ and $V_{a2}$, the relay decodes the lattice-modulo sum $[10]$ of $V_{a2}$ and $V_{b2}$, denoted as $V_{b-sum}$, over states 1–3.
• In state 4, the relay broadcasts the lattice-modulo sum of $V_{b-sum}$, $V_{a1}$, and $V_{a2}$ using a random Gaussian codeword.
• User $i \in \{1, 2\}$ decodes $V_i$ over states 1 and 4. From $V_i$, user $i$ obtains $M_i$.

**Lemma 2 (The G-B Scheme [7]):** The G-B scheme achieves a rate region $R_{G-B} \in \mathbb{R}^2$, consisting of all non-negative rate pairs $(R_1, R_2)$, each satisfying

$$R_i < A_i + \left[ \frac{t_3}{2} \log(\lambda_{i,3}) \right]^+, \quad (6)$$

$$R_i < C_i + D_i, \quad (7)$$

for all $i \in \{1, 2\}$, where $[x]^+ \equiv \max\{0, x\}$.

Note that while the G-B scheme got rid of the sum-rate constraint (5) in the HBC scheme (this is because using lattice codes, the relay is not required to decode both messages $M_1$ and $M_2$), its constraint (6) is tighter than (3) in the HBC, as $B_i' < B_i$.

**Issue:** It is not clear if the rate bound (6) can be obtained with the aforementioned coding scheme. The authors have shown that considering state $i \in \{1, 2\}$, the relay can decode $V_{ai}$ and $V_{bi}$ (using successive decoding) if

$$R_{ai} \leq \frac{1}{2} \log \left( 1 + \frac{\alpha \lambda_{i,3}}{1 + (1 - \alpha) \lambda_{i,3}} \right), \quad (8)$$

$$R_{bi} \leq \frac{1}{2} \log(1 + (1 - \alpha) \lambda_{i,3}), \quad (9)$$

for some $0 \leq \alpha \leq 1$. Considering state 3, having decoded $V_{a1}$ and $V_{a2}$, the relay can decode $V_{b-sum}$ if

$$R_{bi} \leq \left[ \frac{1}{2} \log(\lambda_{i,3}) \right]^+, \quad (10)$$

for $i \in \{1, 2\}$. The authors then argued that by using time sharing between states $i$ and 3, (9) and (10) give

$$R_{bi} \leq \frac{t_i}{2} \log(1 + (1 - \alpha) \lambda_{i,3}) + R_i', \quad (11)$$

which is then combined with (8) to give (6).

An issue with this argument is that using time sharing, different messages (say $V_{b1}'$ and $V_{b2}'$) are transmitted in different states, and this necessarily imposes additional rate constraints on individual components. These components also impose more rate constraints for them to be decoded at user $i$.

On the other hand, if the same message $V_{bi}$ is transmitted in states $i$ and 3 (which is intended by the authors), one cannot invoke time sharing, but one can use simultaneous decoding. However, there is still an issue of how the relay can simultaneously decode the required lattice-modulo sum $V_{b-sum}$ from two different codewords, namely a Gaussian codeword in state $i$ (where only $V_{bi}$ has been transmitted) and a lattice codeword in state 3 (where both $V_{b1}$ and $V_{b2}$ have been transmitted), if (11) is satisfied.

**C. Our Proposed Scheme 1**

In addition to the aforementioned issues related to the G-B scheme, we also note that using the intermediate lattice $\Lambda_b$ incurs some rate loss in $B_i'$ (c.f. Nam et al. [1]).

We propose the following scheme, which uses nested lattice codes without needing the intermediate lattice, and simultaneous decoding:

• Create nested lattices as per the full-duplex case [1], $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_c \in \mathbb{R}^n$.
• Each user $i$ splits its message into two parts: $M_{ai}$ of rate $R_{ai}$, and $M_{bi}$ of rate $R_{bi}$. It then maps its second sub-message to a lattice point $M_{bi} \rightarrow V_{bi} \in \{\Lambda_c \cap V_i\}$.
• In state $i \in \{1, 2\}$, user $i$ sends $M_{ai}$ using a random Gaussian codeword, and the relay decodes $M_{ai}$.
• In state 3, the users send lattice codewords simultaneously, and the relay decodes the lattice-modulo sum $V_{b-sum}$ (the lattice-modulo sum of $V_{b1}$ and $V_{b2}$).
• In state 4, the relay encodes $V_{b-sum}$ and $(M_{a1} + M_{a2}) \mod 2^n \max\{R_{a1}, R_{a2}\}$ together using a random Gaussian codeword.
• User $i$ decodes $V_{b-sum}$ and $M_{ai}$ simultaneously over states 1 and 4.

Over state 3, the relay can decode $V_{b-sum}$ if [1, Sec. IV.A]

$$R_{bi} < \left[ \frac{t_3}{2} \log \left( \frac{\lambda_{i,3}}{\lambda_{i,3} + \lambda_{i,3}} + \lambda_{i,3} \right) \right]^+, \quad (12)$$

$$\subseteq B_i'$$

for both $i \in \{1, 2\}$; over state $i \in \{1, 2\}$, the relay can decode $M_{ai}$ if [11, Thm. 9.1.1]

$$R_{ai} < A_i, \quad (13)$$

where $A_i$ has been defined in (3).

Note that without using an intermediate lattice, we regain the missing term in red (compare (12) with (6)), as $B_i' > B_i$.

Using simultaneous decoding (see, e.g., Asadi et al. [12]) over states 1 and 4, we can show that user $i \in \{1, 2\}$ (knowing its own message $M_{ai}$) can decode $V_{b-sum}$ and $M_{ai}$ if

$$R_{ai} < C_i + D_i, \quad (14)$$

$$R_{bi} < D_i, \quad (15)$$

$$R_{ai} + R_{bi} < C_i + D_i. \quad (16)$$

Given (16), (14) is redundant.

From $M_{ai}$ and $V_{b-sum}$, user $i \in \{1, 2\}$ can obtain its required $M_i$. Combining (12), (13), (15), (16) using Fourier-Motzkin elimination, we have the following:

**Theorem 1 (Proposed scheme 1):** For the half-duplex Gaussian two-way relay channel, utilizing only states 1–4, a rate region $R_1 \in \mathbb{R}^2$, consisting all non-negative rate pairs $(R_1, R_2)$, each satisfying the following, is achievable:

$$R_i < A_i + B_i', \quad (17)$$

$$R_i < A_i + D_i, \quad (18)$$

$$R_i < C_i + D_i, \quad (19)$$

\*Note that the definition of $V_{b-sum}$ here is different from that in the G-B scheme, which uses an intermediate lattice.
for all $i \in \{1, 2\}$.

Here, (19) is common among all three schemes. (17) is looser than (6) in the G-B scheme (due to not using the intermediate lattice). There is an additional constraint (18) (due to rate splitting), but there is no sum rate constraint (as the relay does not decode both messages).

D. Comparison

Figure 3 compares the rate regions achievable by these three schemes for a specific channel configuration. It can be seen that our proposed scheme achieves a strictly larger rate region than the other two for this set of channel parameters. Also, it shows that $\mathcal{R}_{G,B}$ is not always strictly larger than $\mathcal{R}_{HBC}$.

IV. Achievable Regions Using All Six States

Instead of using only four states as in the previous section, we may be able to enlarge the achievable rate region by using all six states (states 1–6).

A. The Gong-Yue-Wang Scheme

A coding scheme using all six states has been proposed by Gong, Yue, and Wang [8]. The coding schemes are as follows:

- Each user $i$ splits its message into three parts, $M_i = (M_{ai}, M_{bi}, M_{ci})$ with rates $(R_{ai}, R_{bi}, R_{ci})$ respectively, where $R_{ai} + R_{bi} + R_{ci} = R_i$.
- In state $i \in \{1, 2\}$, user $i$ sends $M_{ai}$, and the relay decodes $M_{ai}$.
- In state 3, both users send $\{M_{bi}\}$ simultaneously, and the relay decodes both $M_{bi}$ and $M_{b2}$.
- In state 4, the relay sends $(M_{a1}, M_{b1}) \oplus (M_{a2}, M_{b2})$ (the messages are converted into binary bits and then XORed).
- In state $(4+i)$, for $i \in \{1, 2\}$, user $i$ sends $M_{ci}$, while the relay sends $(M_{a1}, M_{b1})$.

The scheme, referred to as the G-Y-W scheme achieves the following rate region:

3We follow the state definition by Kim et al. [6] and Ghasemi-Goojani and Behroozi [7], which is different from that by Gong et al. [8].

Lemma 3 (The G-Y-W Scheme [8]): The G-Y-W scheme achieves a rate region $\mathcal{R}_{G-Y-W} \subseteq \mathbb{R}^2$, consisting of all non-negative rate tuples $(R_1, R_2)$, each satisfying the following:

\begin{align*}
R_i &< (A_i + B_i) + \frac{t_i + i}{2} \log (1 + \lambda_{i,i}), \quad (20) \\
R_i &< (C_i + D_i) + \frac{t_i + i}{2} \log \left(1 + \frac{\lambda_{i,3}}{1 + \lambda_{i,i}}\right) + F_i, \quad (21) \\
R_1 + R_2 &< (A_1 + A_2 + E) + F_1 + F_2, \quad (22)
\end{align*}

for all $i \in \{1, 2\}$.

Setting $t_5 = t_6 = 0 \Rightarrow F_1 = G_4 = 0$, we recover the region $\mathcal{R}_{HBC}$. This means $\mathcal{R}_{HBC} \subseteq \mathcal{R}_{G-Y-W}$. Here, $F_i$ represents the rate of $M_{ci}$, i.e., $R_{ci}$, transmitted on the direct user-to-user link in state $4+i$, and $G_i$ is the additional information (of the messages $M_{ai}$ and $M_{bi}$) that the relay forwards in states 5 and 6.

Issue: The encoding scheme described by Gong et al. [8] is incorrect, though the region $\mathcal{R}_{G-Y-W}$ is actually achievable. To see this, let $t_5 = t_6 = 0$, meaning that $F_1 = G_4 = 0$ and $M_{a1} = M_{b2} = \emptyset$. We get the region $\mathcal{R}_{HBC}$. However, using the message-splitting technique described by Gong et al., we see that the relay must decode $M_{a1}$ over state 1, which necessarily imposes $R_{a1} < A_1$. In addition, $M_{b3}$ is decoded by user 2 over only state 4 (since we have set $t_5 = 0$), which necessarily imposes $R_{b1} < D_1$. So, there should be another constraint $R_1 < A_1 + D_1$ for $\mathcal{R}_{G-Y-W}$ when $t_5 = t_6 = 0$. This issue can be easily rectified using the coding technique in the HBC scheme, where (i) user $i$ sends the same message in states 1 and 3, i.e., the message is not split, and (ii) the relay sends the sum of binned message indices in states 4–6. The splitting of message $M_{ci}$ for direct user-to-user transmissions in states 5 and 6 is fine here, because it does not impose additional rate constraint (as $\{M_{ci}\}$ are not decoded by the relay).

B. Our Proposed Scheme 2

We will now improve upon the G-Y-W scheme by using lattice codes as in our proposed scheme 1, which removes the sum rate constraint (22). In addition, we will also utilize coherent combining [9, Sec. IV] to get power gain in states 5 and 6. Our proposed scheme 2 is as follows:

- The message $M_i$, $i \in \{1, 2\}$, is split into $M_{a1}$, $M_{b1}$, and $M_{c1}$ with rates $R_{a1}$, $R_{b1}$, and $R_{c1}$ respectively.
- In states 1–4, we used our proposed scheme 1.
- As in scheme 1, the relay decodes $V_{b1}$-sum (instead of both $M_{b1}$ and $M_{b2}$ as in the G-Y-W scheme) and $\{M_{ai}\}$ over states 1–4.
- In state $(4+i)$ for $i \in \{1, 2\}$:
  - The relay sends $M_{a1}$ using a random Gaussian codeword.
  - User $i$ sends two codewords. Instead of sending only the new sub-message $M_{ci}$ (as in the G-Y-W scheme),
user $i$ splits its power to send (i) same signal as the relay using $\alpha_i$ fraction of its power, and (ii) $M_i$ using $(1 - \alpha_i)$ fraction of its power, for some $0 \leq \alpha_i \leq 1$.

Decoding at the relay (over states 1–4) is the same as that in scheme 1. So, the relay can decode $V_{b_{\text{sum}}}$ if (12) holds, and $M_{ai}$ if (13) holds.

Using simultaneous decoding over states $i, 4$, and $(4 + i)$, user $i$ (knowing $M_{ai}$ a priori) can decode $V_{b_{\text{sum}}}$ and $M_{ai}$ if

$$R_{ai} < C_i + D_i + \frac{t_{k+i}}{2} \log \left(1 + \frac{\sqrt{\lambda_{ai}} + \sqrt{\alpha_i \lambda_{ai}}}{1 + (1 - \alpha_i) \lambda_{ai}} \right),$$

$$R_{bi} < D_i,$$

$$R_{ai} + R_{bi} < C_i + D_i + H_i,$$

where $H_i$ is due to the coherently-combined signals (carrying $M_{ai}$) transmitted by both the relay and user $i$ in state $(4 + i)$, by treating the signals carrying $M_{ai}$ as noise. The analysis is the same as that for scheme 1. Given (25), (23) is redundant.

In state $(4 + i)$, after decoding $M_{ai}$, user $i$ subtracts the signals carrying $M_{ai}$ off its received signals. It can decode $M_{ci}$ if [11, Thm. 9.1.1]

$$R_{ci} < \frac{t_{k+i}}{2} \log \left(1 + (1 - \alpha_i) \lambda_{ai} \right).$$

Combining (12), (13), (24), (25), and (26), we have the following:

**Theorem 2 (Proposed scheme 2):** For the half-duplex Gaussian two-way relay channel, the rate region $R_2 \in \mathbb{R}^2$, consisting all non-negative rate pairs $(R_1, R_2)$, each satisfying the following, is achievable:

$$R_i < A_i + B_i'' + I_i,$$

$$R_i < A_i + D_i + I_i,$$

$$R_i < C_i + D_i + H_i + I_i,$$

for all $i \in \{1, 2\}$.

Note that setting $t_5 = t_6 = 0$, we recover $R_1$. Hence, $R_1 \subseteq R_2$.

### C. Comparison

Setting $\alpha_i = 0$ for scheme 2, we have $H_i|_{\alpha_i=0} = G_i$ and $I_i|_{\alpha_i=0} = F_i$, and thus (27) equals (20), and (29) equals (21). While scheme 2 does not impose the sum-rate constraint (22) of the G-Y-W scheme, it includes an additional constraint (28). So, one scheme could perform better under some channel parameters, and worse under some.

Figure 4 compares the rate region achievable using two four-state schemes (the HBC scheme and our proposed scheme 1) and two six-state schemes (the G-Y-W scheme and our proposed scheme 2). We use the same channel parameters as in Figure 3. We see that for this channel configuration, our proposed scheme 2 outperforms all other schemes. As expected, $R_{\text{HBC}} \subseteq R_{\text{G-Y-W}}$ and $R_1 \subseteq R_2$.

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**References**


