The Half-Duplex Gaussian Two-Way Relay Channel with Direct Links

Lawrence Ong
School of Electrical Engineering and Computer Science, The University of Newcastle, Australia

***Abstract*—We study the half-duplex Gaussian two-way relay channel with direct user-to-user links. In this setup, two users exchange data via a relay and via direct user-to-user links. Due to the half-duplex constraint, the channel can be in one of eight different states at any time (two of which are useless: no node transmitting and no node listening). Restricting to only four states, we propose a scheme that utilizes lattice codes to improve upon existing four-state schemes. Using all six states, we propose another scheme that utilizes lattice codes and coherent combining, and show that it can outperform existing schemes.**

***Index Terms*—Two-way relay channel, bi-directional relaying, lattice codes, half duplex, Gaussian**

I. Introduction

We study a class of relay-aided networks where two users exchange data through a relay. This network configuration—commonly found in the cellular mobile network, the satellite network, and the WiFi network—is referred to as the two-way relay channel or the bi-directional relaying channel. Common assumptions for the two-way relay channel are that the nodes operate in the full duplex mode and/or that there is no user-to-user link (i.e., data transfer is done only via the relay) [1], [2], [3], [4, Sec IV.A]. In this paper, we consider the case where (i) the nodes operate in the half-duplex mode,[1] and (ii) there are user-to-user links.

Under the half-duplex constraint, the channel operates in one of eight *states* at any time, depending on which nodes transmit and which listen.[2] Using only four states, Kim et al. [6] designed a coding scheme where the relay completely decodes all the messages, bins the messages, and forwards the addition of the bin indices. Also using only four states, Ghasemi-Goojani and Behroozi [7] built on the idea of nested lattice codes (which was designed for the full-duplex two-way relay channel [1]) by proposing an *intermediate* lattice to achieve a better rate region. In this paper, we use the combination of functional-decode-forward (which neither uses binning nor requires the relay to decode all messages) using lattice codes (but without needing the intermediate lattice) and simultaneous decoding to design a new four-state scheme. We show that this scheme can achieve a strictly larger rate region than that by Kim et al. and that by Ghasemi-Goojani and Behroozi.

Using all six states (ignoring the two states where no node listens, and no node transmits), Gong et al. [8] proposed a scheme that utilizes partial decode-forward. In this paper, we combine the techniques used in our proposed four-state scheme and the idea of coherent combining [9] to design a new six-state scheme. We show that this scheme can achieve rate regions strictly larger than all aforementioned schemes.

In this paper, we also point out some issues/errors in the scheme proposed by Ghasemi-Goojani and Behroozi [7] and that by Gong et al. [8].

Lawrence Ong is the recipient of an Australian Research Council Future Fellowship (FT140100219).

[1]There are fundamental differences between a full-duplex channel and its half-duplex counterpart. See the single-relay channel [5] for example.

[2]Without the user-to-user links, there are only two useful states: (1) the relay transmits and both users listen, and (2) the relay listens and both users transmit.

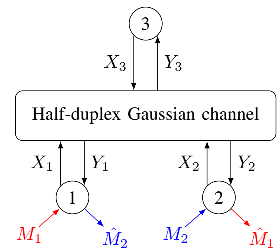

Fig. 1. The half-duplex Gaussian two-way relay channel, where two users (nodes 1 and 2) exchange data with the help of a relay (node 3). Under the half-duplex constraint, at any time, for each node i, either $X_i = 0$ (node i is listening) or $Y_i = 0$ (node i is transmitting).

II. Channel Model

In the two-way relay channel (see Figure 1), node 1 wishes to send a message, denoted by M_1, to node 2; node 2 wishes to send a message, M_2 to node 1. Node 3, who has no message to transmit, facilitates the message exchange. In this setup, nodes 1 and 2 are the *users*, and node 3 the relay. Let $X_i \in \mathbb{R}$ and $Y_i \in \mathbb{R}$ be the channel input and channel output, respectively, of node i, for $i \in \{1, 2, 3\}$.

We consider the half-duplex channel, where the *channel state* (a description of which node transmits and which node listens) is pre-determined and made known to all nodes a priori. Denote the channel state by a triplet (s_1, s_2, s_3), where $s_i = 0$ if node i listens, and $s_i = 1$ otherwise (i.e., if node i transmits). It is convenient to represent the channel state by an integer $s = \sum_{i=1}^{3} s_i 2^{i-1}$. With this notation, the channel states for all $s \in \{0, 1, \dots, 7\}$ are depicted in Figure 2.

Consider n uses of the memoryless Gaussian two-way relay channel, where the t-th channel use is defined as follows:

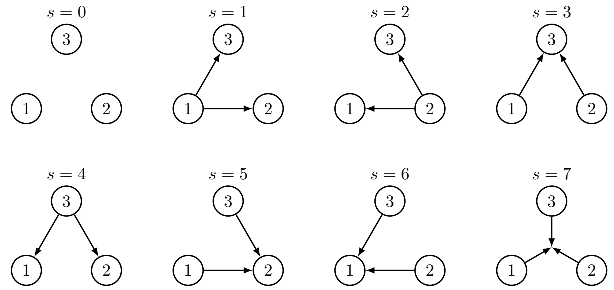

Fig. 2. All possible channel states, where an arrow from a node indicates that it is transmitting, and no arrow, listening.

$$Y_j[t] = \begin{cases} \sum_{\substack{i \in \{1,2,3\} \setminus \{j\} \\ \text{s.t. } s_i[t]=1}} h_{i,j} X_i[t] + Z_j[t], & \text{if } s_j[t] = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

for $j \in \{1, 2, 3\}$ and $t \in \{1, 2, \dots, n\}$. Here, $h_{i,j}$ is the channel gain from node i to node j, which is constant for all channel uses. We do not impose the restriction that $h_{i,j} = h_{j,i}$. The noise term $Z_j[t]$ is a zero-mean Gaussian random variable with variance σ_j^2, and is independent for each j and t. $(s_1[t], s_2[t], s_3[t])$ is the pre-determined channel state of the t-th channel use.

For simplicity, for any user $i \in \{1, 2\}$, we denote the other user by $\bar{i} \triangleq (i \mod 2) + 1$. Let n_k be the number of channel uses (out of n) when the channel is in state k, i.e., $n_k \triangleq |\{t \in \{1, 2, \dots, n\} : s[t] = k\}|$.

An (R_1, R_2, n) block code is defined as follows:

- A set of channel states: $\boldsymbol{s} = (s[1], s[2], \dots, s[n])$.
- Two message sets: $M_i \in \{1, 2, \dots, 2^{nR_i}\}$, for $i \in \{1, 2\}$.
- Three sets of encoding functions: $X_i[t] = f_{i,t}(M_i, Y_i[1], Y_i[2], \dots, Y_i[t-1], s_i[t])$, for $i \in \{1, 2, 3\}$ and $t \in \{1, 2, \dots, n\}$, subject to the constraint that $X_i[t] = 0$ if $s_i[t] = 0$, and the following *per-state*[3] transmitted power constraint:
$$\frac{1}{n_k} \sum_{\substack{t \in \{1,2,\dots,n\} \\ \text{s.t. } s[t]=k \text{ and } s_i[t]=1}} E[X_i^2[t]] \le P_i, \tag{2}$$
for each node $i \in \{1, 2, 3\}$ and each state k. We set $M_3 = 0$.
- Two decoding functions: $\widehat{M_{\bar{i}}} = g_i(M_i, Y_i[1], Y_i[2], \dots, Y_i[n], \boldsymbol{s})$, for $i \in \{1, 2\}$.

Assuming that each message is uniformly distributed, the error probability is defined as $P_\mathrm{e} = \Pr\{\widehat{M_1} \neq M_1 \text{ or } \widehat{M_2} \neq M_2\}$. A rate pair (R_1, R_2) is said to be achievable if given any $\epsilon > 0$, there exists an (R_1, R_2, n) code such that $P_\mathrm{e} \le \epsilon$.

To simplify expressions in our analyses, we define pair-wise signal-to-noise ratio (SNR) $\lambda_{i,j} \triangleq h_{i,j}^2 P_i / \sigma_j^2$ for all node pairs (i, j), and $t_k \triangleq n_k / n$ for all states k.

[3]Under this type of power constraint, a node i is not allowed to transmit at power higher than P_i in one state and at power lower than P_i in another state to achieve the overall average of P_i. This simplifies the computation for the achievable rate region as the we need not optimize the transmit power of the nodes in each state.

Note that $Y_i[t] = 0$ for all $i \in \{1, 2, 3\}$ in states 0 and 7. So, it suffices to consider only states 1 to 6, and set $t_0 = t_7 = 0$.

III. Achievable Regions Using only Four States

In this section, we study two existing schemes that use only four channel states (i.e., states 1–4). We identify their shortcomings, and propose a new scheme that utilizes index coding and simultaneous decoding.

A. The Hybrid Broadcast Scheme using Decode-Forward and Binning

Kim et al. [6] proposed the hybrid broadcast (HBC) scheme, which uses functional-decode-forward and binning, described briefly as follows:

- In state $i \in \{1, 2\}$, user i broadcast its message M_i to the relay and user $\bar{i}$.
- In state 3, both users simultaneously send $\{M_i\}$ again to the relay, using another independently generated codebook.
- The relay decodes M_1 and M_2 over states 1–3.
- In state 4, the relay randomly assigns the message M_i into $2^{nR_i'}$ bins. Let the bin indices be $b_i(M_i)$. The relay broadcasts $s_1(M_1) + s_2(M_2) \mod 2^{n \max\{R_1', R_2'\}}$.
- User $\bar{i} \in \{1, 2\}$ decodes $b_i(M_i)$ over state 4, and then M_i over state i with the help of $b_i(M_i)$.

Lemma 1 (The HBC Scheme [6]): The HBC scheme achieves a rate region $\mathcal{R}_\mathrm{HBC} \in \mathbb{R}^2$, consisting of all non-negative rate pairs (R_1, R_2), each satisfying

$$R_i < \underbrace{\frac{t_i}{2}\log(1 + \lambda_{i,3})}_{\triangleq A_i} + \underbrace{\frac{t_3}{2}\log(1 + \lambda_{i,3})}_{\triangleq B_i}, \tag{3}$$

$$R_i < \underbrace{\frac{t_i}{2}\log(1 + \lambda_{i,\bar{i}})}_{\triangleq C_i} + \underbrace{\frac{t_4}{2}\log(1 + \lambda_{3,\bar{i}})}_{\triangleq D_i}, \tag{4}$$

$$R_1 + R_2 < A_1 + A_2 + \underbrace{\frac{t_3}{2}\log(1 + \lambda_{1,3} + \lambda_{2,3})}_{\triangleq E}, \tag{5}$$

for all $i \in \{1, 2\}$.

B. The Ghasemi-Behroozi Scheme using Lattice Codes

Recently, Ghasemi-Goojani and Behroozi [7] proposed a coding scheme, which extends the lattice-based coding scheme for the full-duplex two-way relay channel [1] using their proposed *intermediate* lattice. They claimed that their scheme achieves a larger region than the HBC scheme.

The scheme, referred to as the G-B scheme, is as follows:

- Form four nested lattices $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_b \subseteq \Lambda_c \subset \mathbb{R}^{n_3}$, where Λ_b is the intermediate lattice.
- User $i \in \{1, 2\}$ maps a message M_i to a lattice point $V_i \in \{\Lambda_c \cap \mathcal{V}_i\}$, where $\mathcal{V}_i$ is the Voronoi region of Λ_i [10]. This lattice point is decomposed into two sub-messages: (i) $V_{ai} \in \{\Lambda_c \cap \mathcal{V}_b\}$ of rate R_{ai}, and (ii) $V_{bi} \in \{\Lambda_b \cap \mathcal{V}_i\}$ of rate R_{bi}, where $R_{ai} + R_{bi} = R_i$.
- In state $i \in \{1, 2\}$, user i sends V_{ai} and V_{bi} using the superposition of two *random Gaussian codewords*.

- The relay recovers only V_{ai} over state i, $i \in \{1,2\}$.
- In state 3, the two users simultaneously send $\{V_{bi}\}$.
- Knowing V_{a1} and V_{a2}, the relay decodes the lattice-modulo sum [10] of V_{a2} and V_{b2}, denoted as $V_{b\text{-sum}}$, over states 1–3.
- In state 4, the relay broadcasts the lattice-modulo sum of $V_{b\text{-sum}}$, V_{a1}, and V_{a2} using a random Gaussian codeword.
- User $\bar{i} \in \{1,2\}$ decodes V_i over states i and 4. From V_i, user $\bar{i}$ obtains M_i.

Lemma 2 (The G-B Scheme [7]): The G-B scheme achieves a rate region $\mathcal{R}_{\text{G-B}} \in \mathbb{R}^2$, consisting of all non-negative rate pairs (R_1, R_2), each satisfying

$$R_i < A_i + \underbrace{\left[\frac{t_3}{2}\log(\lambda_{i,3})\right]^+}_{\triangleq B_i'}, \tag{6}$$

$$R_i < C_i + D_i, \tag{7}$$

for all $i \in \{1,2\}$, where $[x]^+ \triangleq \max\{0, x\}$.

Note that while the G-B scheme got rid of the sum-rate constraint (5) in the HBC scheme (this is because using lattice codes, the relay is not required to decode both messages M_1 and M_2), its constraint (6) is tighter than (3) in the HBC, as $B_i' < B_i$.

Issue: It is not clear if the rate bound (6) can be obtained with the aforementioned coding scheme. The authors have shown that considering state $i \in \{1,2\}$, the relay can decode V_{ai} and V_{bi} (using successive decoding) if

$$R_{ai} \leq \frac{1}{2}\log\left(1 + \frac{\alpha\lambda_{i,3}}{1 + (1-\alpha)\lambda_{i,3}}\right), \tag{8}$$

$$R_{bi} \leq \frac{1}{2}\log(1 + (1-\alpha)\lambda_{i,3}), \tag{9}$$

for some $0 \leq \alpha \leq 1$. Considering state 3, having decoded V_{a1} and V_{a2}, the relay can decode $V_{b\text{-sum}}$ if

$$R_{bi} \leq \left[\frac{1}{2}\log(\lambda_{i,3})\right]^+, \tag{10}$$

for $i \in \{1,2\}$. The authors then argued that by using *time sharing* between states i and 3, (9) and (10) give

$$R_{bi} \leq \frac{t_i}{2}\log(1 + (1-\alpha)\lambda_{i,3}) + B_i', \tag{11}$$

which is then combined with (8) to give (6).

An issue with this argument is that using time sharing, different messages (say V_{bi}' and V_{bi}'') are transmitted in different states, and this necessarily imposes additional rate constraints on individual components. These components also impose more rate constraints for them to be decoded at user $\bar{i}$.

On the other hand, if the same message V_{bi} is transmitted in states i and 3 (which is intended by the authors), one cannot invoke time sharing, but one can use simultaneous decoding. However, there is still an issue of how the relay can simultaneously decode the required lattice-modulo sum $V_{b\text{-sum}}$ from two different codewords, namely a Gaussian codeword in state i (where only V_{bi} has been transmitted) and a lattice codeword in state 3 (where both V_{b1} and V_{b2} have been transmitted), if (11) is satisfied.

C. Our Proposed Scheme 1

In addition to the aforementioned issues related to the G-B scheme, we also note that using the intermediate lattice Λ_b incurs some rate loss in B_i' (c.f. Nam et al. [1]).

We propose the following scheme, which uses nested lattice codes without needing the intermediate lattice, and simultaneous decoding:

- Create nested lattices as per the full-duplex case [1], $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_c \in \mathbb{R}^{n_3}$.
- Each user i splits its message into two parts: M_{ai} of rate R_{ai}, and M_{bi} of rate R_{bi}. It then maps its second sub-message to a lattice point $M_{bi} \to V_{bi} \in \{\Lambda_c \cap \mathcal{V}_i\}$.
- In state $i \in \{1,2\}$, user i sends M_{ai} using a random Gaussian codeword, and the relay decodes M_{ai}.
- In state 3, the users send lattice codewords simultaneously, and the relay decodes the lattice-modulo sum $V_{b\text{-sum}}$ (the lattice-modulo sum of V_{b1} and V_{b2}).[4]
- In state 4, the relay encodes $V_{b\text{-sum}}$ and $(M_{a1} + M_{a2} \mod 2^{n\max\{R_{a1},R_{a2}\}})$ together using a random Gaussian codeword.
- User $\bar{i}$ decodes $V_{b\text{-sum}}$ and M_{ai} *simultaneously* over states i and 4.

Over state 3, the relay can decode $V_{b\text{-sum}}$ if [1, Sec. IV.A]

$$R_{bi} < \underbrace{\left[\frac{t_3}{2}\log\left(\frac{\lambda_{i,3}}{\lambda_{i,3} + \lambda_{\bar{i},3}} + \lambda_{i,3}\right)\right]^+}_{\triangleq B_i''}, \tag{12}$$

for both $i \in \{1,2\}$; over state $i \in \{1,2\}$, the relay can decode M_{ai} if [11, Thm. 9.1.1]

$$R_{ai} < A_i, \tag{13}$$

where A_i has been defined in (3).

Note that without using an intermediate lattice, we regain the missing term in red (compare (12) with (6)), as $B_i'' > B_i'$.

Using simultaneous decoding (see, e.g., Asadi et al. [12]) over states i and 4, we can show that user $\bar{i} \in \{1,2\}$ (knowing its own message $M_{a\bar{i}}$) can decode $V_{b\text{-sum}}$ and M_{ai} if

$$R_{ai} < C_i + D_i, \tag{14}$$

$$R_{bi} < D_i, \tag{15}$$

$$R_{ai} + R_{bi} < C_i + D_i. \tag{16}$$

Given (16), (14) is redundant.

From M_{ai} and $V_{b\text{-sum}}$, user $\bar{i} \in \{1,2\}$ can obtain its required M_i. Combining (12), (13), (15), (16) using Fourier-Motzkin elimination, we have the following:

Theorem 1 (Proposed scheme 1): For the half-duplex Gaussian two-way relay channel, utilizing only states 1–4, a rate region $\mathcal{R}_1 \in \mathbb{R}^2$, consisting all non-negative rate pairs (R_1, R_2), each satisfying the following, is achievable:

$$R_i < A_i + B_i'', \tag{17}$$

$$R_i < A_i + D_i, \tag{18}$$

$$R_i < C_i + D_i, \tag{19}$$

[4] Note that the definition of $V_{b\text{-sum}}$ here is different from that in the G-B scheme, which uses an intermediate lattice.

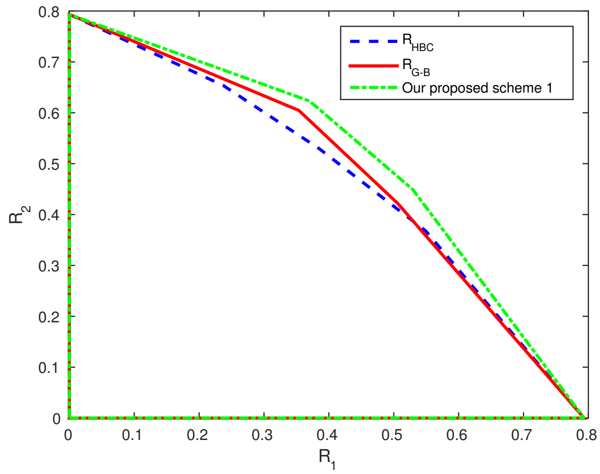

Fig. 3. A comparison of the HBC scheme, the G-B scheme, and our proposed scheme 1, where $\text{SNR}_1 = \lambda_{1,3} = \lambda_{3,1} = 6\text{dB}$, $\text{SNR}_2 = \lambda_{2,3} = \lambda_{3,2} = 5\text{dB}$, and $\text{SNR}_3 = \lambda_{1,2} = \lambda_{2,1} = 2\text{dB}$. The regions are obtained via linear programming.

for all $i \in \{1, 2\}$.

Here, (19) is common among all three schemes. (17) is looser than (6) in the G-B scheme (due to not using the intermediate lattice). There is an additional constraint (18) (due to rate splitting), but there is no sum rate constraint (as the relay does not decode both messages).

D. Comparison

Figure 3 compares the rate regions achievable by these three schemes for a specific channel configuration. It can be seen that our proposed scheme achieves a strictly larger rate region than the other two for this set of channel parameters. Also, it shows that $\mathcal{R}_{\text{G-B}}$ is not always strictly larger than $\mathcal{R}_{\text{HBC}}$.

IV. Achievable Regions Using All Six States

Instead of using only four states as in the previous section, we may be able to enlarge the achievable rate region by using all six states (states 1–6).

A. The Gong-Yue-Wang Scheme

A coding scheme using all six states has been proposed by Gong, Yue, and Wang [8].[5] The coding schemes are as follows:

- Each user i splits its message into three parts, $M_i = (M_{ai}, M_{bi}, M_{ci})$ with rates (R_{ai}, R_{bi}, R_{ci}) respectively, where $R_{ai} + R_{bi} + R_{ci} = R_i$.
- In state $i \in \{1, 2\}$, user i sends M_{ai}, and the relay decodes M_{ai}.
- In state 3, both users send $\{M_{bi}\}$ simultaneously, and the relay decodes both M_{b1} and M_{b2}.
- In state 4, the relay send $(M_{a1}, M_{b1}) \oplus (M_{a2}, M_{b2})$ (the messages are converted into binary bits and then XORed).
- In state $(4 + i)$, for $i \in \{1, 2\}$, user i sends M_{ci}, while the relay sends (M_{ai}, M_{bi}).

The scheme, referred to as the G-Y-W scheme achieves the following rate region:

Lemma 3 (The G-Y-W Scheme [8]): The G-Y-W scheme achieves a rate region $\mathcal{R}_{\text{G-Y-W}} \in \mathbb{R}^2$, consisting of all non-negative rate tuples (R_1, R_2), each satisfying the following:

$$R_i < (A_i + B_i) + \underbrace{\frac{t_{4+i}}{2} \log(1 + \lambda_{i,\bar{i}})}_{\triangleq F_i}, \tag{20}$$

$$R_i < (C_i + D_i) + \underbrace{\frac{t_{4+i}}{2} \log\left(1 + \frac{\lambda_{3,\bar{i}}}{1 + \lambda_{i,\bar{i}}}\right)}_{\triangleq G_i} + F_i, \tag{21}$$

$$R_1 + R_2 < (A_1 + A_2 + E) + F_1 + F_2, \tag{22}$$

for all $i \in \{1, 2\}$.

Setting $t_5 = t_6 = 0 \Rightarrow F_i = G_i = 0$, we recover the region $\mathcal{R}_{\text{HBC}}$. This means $\mathcal{R}_{\text{HBC}} \subseteq \mathcal{R}_{\text{G-Y-W}}$. Here, F_i represents the rate of M_{ci}, i.e., R_{ci}, transmitted on the direct user-to-user link in state $4 + i$, and G_i is the additional information (of the messages M_{ai} and M_{bi}) that the relay forwards in states 5 and 6.

Issue: The encoding scheme described by Gong et al. [8] is incorrect, though the region $\mathcal{R}_{\text{G-Y-W}}$ is actually achievable. To see this, let $t_5 = t_6 = 0$, meaning that $F_i = G_i = 0$ and $M_{c1} = M_{c2} = \varnothing$. We get the region $\mathcal{R}_{\text{HBC}}$. However, using the message-splitting technique described by Gong et al., we see that the relay must decode M_{a1} over state 1, which necessarily imposes $R_{a1} < A_1$. In addition, M_{b1} is decoded by user 2 over *only* state 4 (since we have set $t_5 = 0$), which necessarily imposes $R_{b1} < D_1$. So, there should be another constraint $R_1 < A_1 + D_1$ for $\mathcal{R}_{\text{G-Y-W}}$ when $t_5 = t_6 = 0$.[6] This issue can be easily rectified using the coding technique in the HBC scheme, where (i) user i sends the same message in states i and 3, i.e., the message is not split, and (ii) the relay sends the sum of binned message indices in states 4–6. The splitting of message M_{ci} for direct user-to-user transmissions in states 5 and 6 is fine here, because it does not impose additional rate constraint (as $\{M_{ci}\}$ are not decoded by the relay).

B. Our Proposed Scheme 2

We will now improve upon the G-Y-W scheme by using lattice codes as in our proposed scheme 1, which removes the sum rate constraint (22). In addition, we will also utilize *coherent combining* [9, Sec. IV] to get power gain in states 5 and 6. Our proposed scheme 2 is as follows:

- The message M_i, $i \in \{1, 2\}$, is split into M_{ai}, M_{bi}, and M_{ci} with rates R_{ai}, R_{bi}, and R_{ci} respectively.
- In states 1–4, we used our proposed scheme 1.
- As in scheme 1, the relay decodes $V_{b\text{-sum}}$ (instead of both M_{b1} and M_{b2} as in the G-Y-W scheme) and $\{M_{ai}\}$ over states 1–4.
- In state $(4 + i)$ for $i \in \{1, 2\}$:
  - The relay sends M_{ai} using a random Gaussian codeword.
  - User i sends two codewords. Instead of sending only the new sub-message M_{ci} (as in the G-Y-W scheme),

[5] We follow the state definition by Kim et al. [6] and Ghasemi-Goojani and Behroozi [7], which is different from that by Gong et al. [8].

[6] This is similar to constraint (18) in scheme 1 where message splitting is used, and both parts are decoded by both the relay and the user.

user i splits its power to send (i) same signal as the relay using α_i fraction of its power, and (ii) M_{ci} using $(1-\alpha_i)$ fraction of its power, for some $0 \leq \alpha_i \leq 1$.

Decoding at the relay (over states 1–4) is the same as that in scheme 1. So, the relay can decode $V_{b\text{-sum}}$ if (12) holds, and M_{ai} if (13) holds.

Using simultaneous decoding over states i, 4, and $(4+i)$, user $\bar{i}$ (knowing $M_{a\bar{i}}$ a priori) can decode $V_{b\text{-sum}}$ and M_{ai} if

$$R_{ai} < C_i + D_i + \underbrace{\frac{t_{4+i}}{2}\log\left(1+\frac{(\sqrt{\lambda_{3,\bar{i}}}+\sqrt{\alpha_i\lambda_{i,\bar{i}}})^2}{1+(1-\alpha_i)\lambda_{i,\bar{i}}}\right)}_{\triangleq H_i}, \tag{23}$$

$$R_{bi} < D_i, \tag{24}$$

$$R_{ai} + R_{bi} < C_i + D_i + H_i, \tag{25}$$

where H_i is due to the coherently-combined signals (carrying M_{ai}) transmitted by both the relay and user i in state $(4+i)$, by treating the signals carrying M_{ci} as noise. The analysis is the same as that for scheme 1. Given (25), (23) is redundant.

In state $(4+i)$, after decoding M_{ai}, user $\bar{i}$ subtracts the signals carrying M_{ai} off its received signals. It can decode M_{ci} if [11, Thm. 9.1.1]

$$R_{ci} < \underbrace{\frac{t_{4+i}}{2}\log\left(1+(1-\alpha_i)\lambda_{i,\bar{i}}\right)}_{I_i}. \tag{26}$$

Combining (12), (13), (24), (25), and (26), we have the following:

Theorem 2 (Proposed scheme 2): For the half-duplex Gaussian two-way relay channel, the rate region $\mathcal{R}_2 \in \mathbb{R}^2$, consisting all non-negative rate pairs (R_1, R_2), each satisfying the following, is achievable:

$$R_i < A_i + B_i'' + I_i, \tag{27}$$

$$R_i < A_i + D_i + I_i, \tag{28}$$

$$R_i < C_i + D_i + H_i + I_i, \tag{29}$$

for all $i \in \{1, 2\}$.

Note that setting $t_5 = t_6 = 0$, we recover $\mathcal{R}_1$. Hence, $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

C. Comparison

Setting $\alpha_i = 0$ for scheme 2, we have $H_i|_{\alpha_i=0} = G_i$ and $I_i|_{\alpha_i=0} = F_i$, and thus (27) equals (20), and (29) equals (21). While scheme 2 does not impose the sum-rate constraint (22) of the G-Y-W scheme, it includes an additional constraint (28). So, one scheme could perform better under some channel parameters, and worse under some.

Figure 4 compares the rate region achievable using two four-state schemes (the HBC scheme and our proposed scheme 1) and two six-state schemes (the G-Y-W scheme and our proposed scheme 2). We use the same channel parameters as in Figure 3. We see that for this channel configuration, our proposed scheme 2 outperforms all other schemes. As expected, $\mathcal{R}_{\text{HBC}} \subseteq \mathcal{R}_{\text{G-Y-W}}$ and $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

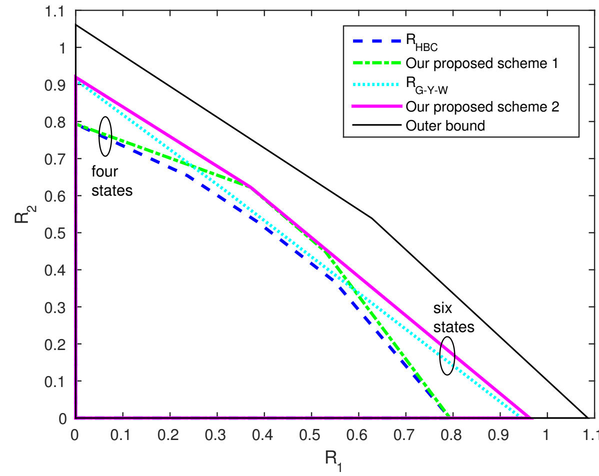

Fig. 4. A comparison of the HBC scheme, the G-Y-W scheme, and our proposed schemes 1 and 2, where $\text{SNR}_1 = \lambda_{1,3} = \lambda_{3,1} = 6$dB, $\text{SNR}_2 = \lambda_{2,3} = \lambda_{3,2} = 5$dB, and $\text{SNR}_3 = \lambda_{1,2} = \lambda_{2,1} = 2$dB. The region $\mathcal{R}_2$ is obtained via linear programming and exhaustive search over $0 \leq \alpha_1, \alpha_2 \leq 1$.

An outer bound to the capacity region is calculated based on the cut-set argument [11, Thm. 15.10.1]. It consists of all non-negative rate pairs (R_1, R_2), each satisfying the following:

$$R_i \leq \frac{t_i}{2}\log(1+\lambda_{i,3}+\lambda_{i,\bar{i}}) + B_i + I_i|_{\alpha_i=0}, \tag{30}$$

$$R_i \leq C_i + D_i + H_i|_{\alpha_i=1}, \tag{31}$$

for all $i \in \{1, 2\}$.

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