Oscillation, Orientation, and Locomotion of Mechanical Rectifier Systems

Lijun Zhu

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Doctor of Philosophy

School of Electrical Engineering and Computer Science

The University of Newcastle
Callaghan, N.S.W, 2299
Australia

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Declaration

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I hereby certify that the work embodied in this thesis is the result of original research, completed subsequent to admission to candidature for the degree. During the course of the candidature several papers have been coauthored with my academic supervisors based on a normal candidate-supervisor practice.

Lijun Zhu
August, 2013
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List of Symbols

\( a \) Coefficient in locomotion equation

\((\cdot)^*\) Conjugate transpose

\( \bar{\cdot} \) \( T \)-average value of periodic signal

\( \beta \) Constant matrix for orientation control

\( \chi \) Natural orientation

\( \Im \) Imaginary part of a complex number

\( \Re \) Real part of a complex number

\( \Gamma \) A complex coefficient matrix for natural orientation

\( \gamma \) Phase vector of the oscillation

\( h \) Function that determines the natural velocity

\( \iota_i \) Perturbation terms related to control performance

\( \mu \) Scalar damping factor

\( \kappa \) Nonlinear strictly decreasing function for entrainment control

\( k \) Dimension of mass center’s position \( p \)

\( n \) Dimension of torque input \( \tau \) and body shape \( \phi \)

\( l_o \) Length of each link for fliptail and snake-like locomotors

\( \mathbb{M} \) The set of generalized eigenvalue, eigenvector and left eigenvector of \( M \)
\( l \)  Total system length for fliptail and snake-like locomotors
\( m \)  Total system mass
\( m_b \)  Mass of the flapping body
\( m_o \)  Mass of each link for fliptail and snake-like locomotors
\( m_p \)  Mass of each point mass for flapping-wing locomotor
\( \nu \)  Auxiliary bias signal for orientation control
\( \mathbb{C} \)  Complex set
\( \mathbb{R} \)  Real set
\( \mathbb{R}^+ \)  Set of real positive number
\( \mathbb{O} \)  Orbit of the natural oscillation
\( \omega \)  Oscillation frequency
\( D_o \)  Damping coefficient matrix in orientation equation of \( \varphi \)
\( J_o \)  Inertia coefficient matrix in orientation equation of \( \varphi \)
\( D_s \)  Damping coefficient matrix \( \phi \) in orientation equation of \( \varphi \)
\( J_s \)  Inertia coefficient matrix of \( \phi \) in orientation equation of \( \varphi \)
\( K_s \)  Stiffness coefficient matrix \( \phi \) in orientation equation of \( \varphi \)
\( \phi \)  Vector of local shape variables
\( \phi_n \)  Natural oscillation pattern
\( \phi_o \)  Optimal oscillation pattern
\omega \quad \text{Global velocity variable for mass center}

v \quad \text{Global velocity variable for mass center along the main locomotion direction}

\varphi \quad \text{Vector of orientation variables}

q \quad \text{Generalized coordinate of robotic systems}

\Psi \quad \text{Linear quadratic function in locomotion equation}

\Xi \quad \text{Linear quadratic function in locomotion equation}

\alpha_r \quad \text{Parameter for amplitude of serpenoid curve}

\beta_r \quad \text{Parameter for phase lags of serpenoid curve}

\gamma_r \quad \text{Parameter for direction of serpenoid curve}

\rho \quad \text{Damping factor related to the natural oscillation}

D \quad \text{Damping matrix of shape equation } \phi

J \quad \text{Inertia matrix of shape equation } \phi

K \quad \text{Stiffness matrix of shape equation } \phi

\theta \quad \text{Vector representing local shape and orientation variables}

\tau \quad \text{Torque input of the robotic system}

\theta \quad \text{Vector representing local shape and orientation variables after system approximation}

p \quad \text{Dimension of } \theta

B \quad \text{Coefficient matrix of } \tau
Damping matrix in equation of $\theta$

Inertia matrix in equation of $\theta$

Stiffness matrix in equation of $\theta$

Asymmetric matrix in equation of $\theta$

Coefficient matrix in locomotion equation

$\mu_n$ Coefficient of the friction or fluid force in normal direction

$\mu_t$ Coefficient of the friction or fluid force in tangential direction

$u$ System input for shape equation related to torque input $\tau$

p Position of mass center

$\zeta$ Nonlinear functions for exact entrainment to natural oscillation

$Z$ Amplitude matrix of the oscillation

$z$ Mode shape of the oscillation
Abstract

Animal locomotion possesses advantageous characteristics in animal morphology, motion efficiency and environment adaptivity, hence furnishing a natural learning template for design and optimization of robotic vehicles. Inspired by their natural analogues, the increasing number of animal-like robots have emerged in recent years. This thesis considers a class of mechanical rectifiers arising from dynamics of animal locomotion. The thesis starts with introduction of the general system modeling method of the mechanical rectifiers, followed by the description of three typical robotic implementations, namely fliptail, snake-like and flapping-wings locomotors. The dynamics of mechanical rectifiers then is revealed to be under-actuated and essentially have three components: oscillation, orientation and locomotion dynamics.

The oscillation dynamics describes how the oscillatory motion of the body shape results from the interaction with environment and body actuation. The orientation dynamics shows the under-actuation property of the locomotion system where the system orientation does not receive direct control. But it is coupled with oscillation dynamics. The locomotion dynamics demonstrates the effect of the mechanical rectification where the periodic body motion is converted to the movement on the mass center.

Regarding three components, the main achievement of this work has three folds. For oscillation dynamics, we introduce the natural oscillation patterns for body shape and proposed a biologically inspired controller to make the closed-loop system exactly entrain the oscillation. For orientation dynamics, we construct a orientation controller on top of the entrainment controller manipulating the average orientation to be aligned with locomotion direction, though at price of the accuracy of body oscillation. The oscillation accuracy
however can be tuned by controller parameters. For locomotion dynamics, we prove that the induced body oscillation and orientation can globally achieve a specific locomotion velocity at mass center which is called natural velocity.

This three efforts compose a control design framework for mechanical rectifiers. Throughout the thesis, these three aspects of the framework would be detailed by applications to the three aforementioned robotic locomotors. The effectiveness of control design framework is supported theoretically by rigorous proof and practically validated by numerical simulation.
Chapter 1

Introduction

This thesis is focused on the control of a class of locomotion systems called mechanical rectifiers, that mimick the morphology and motion of animals in nature. Due to the complexity of mechanical design, nonlinearity in dynamics and hyper-redundant degree of freedoms, the control design of biologically inspired robotic system is quite challenging. The intention of the research is to propose a general framework for designing a feedback controller that can achieve an efficient locomotion behavior in closed loop. The framework is expected to be as general as possible so that it is applicable to many life-like robotic systems that imitate the animal locomotion such as snake crawling, fish swimming and bird flying.

This introductory chapter will present an overview of locomotion systems covering a range of biologically inspired robotic locomotors. It then emphasizes the advantages that can benefit from the mechanical rectifiers. It also introduces the challenging tasks that one can come across in the control design, namely gait optimization and gait control. Three gait optimization method then are summarized. A review on the control design method is provided and shows that the feedback control based on dynamics model is rare, revealing the motivation and contribution of the thesis. Besides, a short review of the biologically inspired control design is included. Finally, the organization of the thesis will be introduced by a brief outline.
1.1 Locomotion Systems

Locomotion, as an act of self-propulsive animal motion in wild environment, is one of the defining characteristics of animals, which demonstrating elegance, diversity and sheer athleticism. Animals manifest diverse locomotion abilities in terrestrial, aerial and aquatic environment for a variety of purposes, such as foraging, migration, escaping, etc. Energy is consumed to overcome friction, drag, viscosity, inertia and gravity and it is believed by biologists that animals have evolved to use the minimum energy during some locomotion [3]. The capability of autonomous movement in complex environment and low energy consumption enthrall engineers’ attention to pursue in building up efficient human made machines. Some forms of animal locomotion exhibit capabilities useful for engineering applications, including agility of flapping flight, propulsive efficiency of oscillating fish fins [4], and robustness of snake crawling in rugged environments. These properties have inspired engineers to design and optimize robotic vehicles under water [5, 6, 7], on land [8, 9, 10, 11, 12], as well as in air [13, 14]. However, the underlying mechanism embedded in animal locomotion is often difficult to completely unveil due to the complexity of biological systems, but in terms of physical biology and biomechanics some foundation has been laid for understanding the principles [15, 16].

The early development of life-like robotic system can be exemplified by the invention of the airplane, which was an attempt to build human made machine mimicking flying bird. In the same way, building life-like robots can offer engineers a novel means of locomotion that could be capable in complex environment and could be used to accomplish challenging tasks. For instance, by taking inspiration of the morphology snake-like robots could be able to travel in tight and cluttered environments that are not accessible for human. For the development of the life-like robots, researchers are motivated to gain better understanding of physiological design and underlying control strategies employed by animals in order to
improve the capability and efficiency of robots. Therefore, the life-like robots could serve as a platform: (1) to take inspiration from versatile forms of animal locomotion for the design of unmanned robotic vehicles; (2) to investigate biological hypotheses such as how central nervous systems implement these abilities within animals.

“Walking” machines with legs have taken a significant part of research on life-like robots performing artificial locomotion. The underlying mechanism of legged robots is characterized by pushing against the environmental media such as ground through the alternating rhythmic movement of multiple legs. According to the number of legs, it can be roughly categorized into biped locomotion in humanoid robots (e.g. [17, 18, 19, 20]), quadruped robots (e.g. [21, 22, 23]), and hexapod locomotion inspired by insects (e.g. [24, 25]). In contrast to walking and running creatures, some animals, due to lack of appendages, use their bodies to generate propulsive force. This type of locomotion is called limbless locomotion and could be found in animals such as snake [26, 27], lamprey [28], fish [29] and bird [30]. A number of life-like robots have been built to take advantage of the limbless locomotion including snake-like robots [8, 9, 31, 32], robotic fish [33, 7, 6], lamprey [34, 35] and flying bird [36]. Fig 1.1 demonstrates some of them. There are a range of potential applications for life-like robots, such as exploring and inspection of complex environment inaccessible to human for instance down in a narrow tunnel or deep in the ocean, urban search and rescue (see more in [9]), underwater exploration, monitoring, surveillance and military use.

1.1.1 Mechanical Rectifiers

Animal locomotion in general can be viewed as a process of mechanical rectification [37, 38], in which oscillatory motion of limbs or bodies is converted (rectified) to sustained thrust force through dynamical interactions with the environment. Rather than
CHAPTER 1. INTRODUCTION

legged locomotion, the focus of the thesis are on the limbless locomotion that relies on a large amount of *continual* contact between the body and the surrounding media (e.g. fluid or surface). The mechanical systems that fulfill this kind of locomotion are called mechanical rectifiers. This class of mechanical rectifier includes various engineering examples such as disk-mass system [39, 40] (e.g. human arm operating on the disk), snake robots, flapping-wing system (e.g., bird flying or ray swimming) and carangiform swimming of non-slender fishes (e.g., jacks, mackerel, and snapper), but excludes legged robots.

A rich body of the literature of mechanical rectifier locomotion is occupied by snake locomotion and snake-like locomotors. In [41], Gray gave one of the earliest analytical studies of snake locomotion. The pioneering research on snake-like robot was conducted by Hirose and his colleagues by observing actual snakes and designing snake-like robots. The snake robot built by Hirose, named active cord mechanism (ACM), was the first success.
1.1. LOCOMOTION SYSTEMS

in demonstrating serpentine locomotion by employing the same locomotion principle of a real snake in 1972 [8]. The design of snake-like robots continued with a series of improvements in terms of mechanical design, increasing mobility from two-dimensional to three-dimensional movement [42], etc. It progressed recently towards the realization of the amphibious snake-like robot that can crawl on the ground and swim in the water [9]. The research line of snake-like robot was also paralleled by other robotics researchers. Chirikjian et al. constructed a 30 degree-of-freedom (DOF) planar hyper-redundant snake-like robot [43]. The Jet Propulsion Laboratory of the NASA presented a 12 DOF robot used for the remote inspection [44]. Crespi et al. built a snake-like robot that achieved the locomotion using a biologically inspired controller [10]. [45, 12] gives a lengthy review on the snake-like robots that are successfully implemented in the laboratory. Among those, [46] appeared to be the first research on multiple snake-like robots operating as a swarm, which could find applications such as team rescue or inspection and “bait-ball” rotation formations. Recently, snake robots capable of three-dimensional motion have appeared to fulfill 3-D serpentine motion patterns such as sinus lifting and sidewinding [47, 48, 49]. [50, 51, 52] proposed control approaches to drive snake robots in environments with obstacles enabling snake robots to move in cluttered environments.

Relevant to research on snake-like robots are robotic fish (mainly anguilliform locomotion) and eel-like mechanism. An underwater eel-like robot was built and controlled with vision system providing position feedback signal in [53, 6]. [54] presented three-dimensional serial eel-like robots with 12 spherical joints. In [55], a sea lamprey robot actuated by shape memory alloy artificial muscles was developed as a underwater autonomous vehicle. A three-link robotic fish was constructed [56] to study carangiform-like swimming. A radiocontrolled, 4-link and free-swimming biomimetic robot fish was developed in [57, 58], with the speed adjusted by oscillation frequency. Refer to [7] for a complete review of robotic underwater locomotion. Being different type of fish locomotion and related to
snake locomotion, batoid fishes have been of recent interest to the robotics researcher. The preliminary research has been conducted in [59] to develop a powerful analytical model describing the swimming motions of batoid fishes and attempt to emulate their abilities in the next generation of bio-inspired underwater vehicles. [60] derived a nonlinear model for batoid fish locomotion and concluded that the optimal gaits were very similar to real batoid fish.

1.1.2 Advantages and Challenges

Mechanical rectifiers as a class of legless locomotion offers several advantages over legged locomotion. Firstly, a large amount of constant contact with the environment ensures that limbless robots are essentially stable especially on rough terrain. With limited contact points on the environment, the legged machines with less supporting legs tend to be more unstable. Thus, the mechanical design and motion planning of legged machines have to explicitly take into consideration the stability issue. Taking the snake-like robots crawling on the ground as an example, as there are no legs to support the body, the center of the mass is relatively low and thus the potential energy remains low in most situations. In addition to the stability, the large contact surface area gives the robots good traction characteristics in variable environments. Moreover, taking advantage of the long and slender body and lack of legs, limbless locomotors (such as snake and fish) sometimes possess compact cross-sections compared with animals of equivalent sizes and capabilities, and hence are competent to operate in a confined environment. In terms of energy consumption, the limbless locomotors usually consume a comparable amount of energy to other biological forms with similar sizes, weights, and speeds [61, 62]. This could be explained by the fact that limbless locomotors take less energy to maintain stability and do not perform a significant amount of lifting of their body in the motion.
1.2 Locomotion Gaits

The term 'hyper-redundant' [63] can be used to characterize mechanical rectifiers, as their robotic implementations generally employ a large number of simple segments in sequence. It is also common in nature that snakes for instance can contain as many as 400 vertebrae, each of which constitutes two degrees of freedom. The property could enable the robots to preserve mobility and maneuverability even in case of failure of a few segments or actuators although at price of motion efficiency [64]. However hyper-redundancy brings up difficulties of fulfilling two fundamental tasks toward locomotion control, namely gait optimization and gait control. Gait optimization considers how to plan trajectory for every body segment in order to achieve efficient motion. The dimension of configuration space for the gait increases with the number of the body segments making gait optimization a complex problem. Gait control considers how to use the outputs of actuators to achieve a specific gait. For a hyper-redundant system with actuators decentralized and equipped at every body segment, how to design a control law to coordinate the movement of multiple body segments is also a difficult problem.

1.2 Locomotion Gaits

Gait is generally defined as the rhythmic movement pattern of body segments or limbs of animals. Depending on several factors such as speed, terrain and energy, most animals tend to use a variety of gaits and switch between them. Fascinated by the fact, the engineers are stimulated to design life-like robots with the basic shape that however offer the capability and flexibility of changing gaits to fulfill certain aims during the locomotion. For instance, energy consumption is a big issue for the autonomous mobile robots carrying their own battery source, and thus designing an energy-efficient gait to save the energy allows robots to work continually without power supply in a long period. In addition, it is desirable capability for robots to adjust velocity during locomotion via gait transition such as walking,
jogging and running to human beings. Thus, how to design and transition between gaits are important and fundamental problems for the development of the life-like robots. Developing gaits for life-like robots is essentially trajectory planning for the body segments. The movement of the body segments shapes the final gaits. Thus, a mechanical rectifier system of many degrees of freedom (or with a number of body segments) could have millions of options for the gaits selection. Selecting an efficient gait for the robotic system is indeed an optimization problem of finding a set of gaits that meet certain requirements and performance characteristics.

The gaits optimization of life-like robots can be generally categorized mainly into three approaches. The first approach is based on the standard formulations of optimal control problems, and it has been used in the area of optimal trajectory generation in manipulator robotics. The formulation of finding optimal gaits is to determine system input, that brings the system from a known initial state to a desired final state while minimizing a integral cost function specifying the objective to be optimized. Along with expanding control signals over a finite set of basis function, this method has been used for robotic manipulators [65] and biped walking machines [66, 67]. For limbless locomotion system, the analysis of nonholonomic dynamic systems can be simplified using tools from differential geometry [68], and calculus of variations is also used to reduce the problem to a two-point boundary-value problem. With these techniques, the optimal gaits can be found for robotic eel [69], snakeboard [70] and snake-like locomotor [71].

The second approach is based on biological inspirations, wherein a particular gait, observed in animal locomotion, is parametrized and examined for optimality with respect to a cost function. For the snake locomotion, Hirose [8] discovered that the shape of a biological snake can be approximated by a planar curve during lateral undulation. He named it the
serpenoid curve that was mathematically described as follows

\[
x(s) = \int_0^s \cos(a_s \cos(b_s \sigma) + c_s \sigma) d\sigma,
\]
\[
y(s) = \int_0^s \sin(a_s \cos(b_s \sigma) + c_s \sigma) d\sigma,
\]

where \((x(s), y(s))\) are the coordinates of the point along the curve at arc length \(s\) from the origin, and where \(a_s, b_s\) and \(c_s\) are positive scalars that determine the shape of the curve. [11] practically discretized the continuous serpenoid curve in terms of successively connected \(n\) segments. The undulation of the snake was described by the relative angles of segments in the following manner:

\[
\phi_i(t) = \alpha_r \sin(\omega t + (i - 1)\beta_r) + \gamma_r, \quad (i = 1, \ldots, n - 1)
\]  

(1.1)

where \(\omega\) specifies how fast the serpentine wave propagates along the body, and \(\alpha_r\) and \(\beta_r\) specify the amplitude and phase lag between segments, respectively. The speed and direction of the serpentine locomotion are mainly determined by \(\omega\) and \(\gamma_r\), respectively. Then, the task of finding the optimal gait is to find the best parameters \(\alpha_r, \beta_r\), and \(\omega\) that maximize or minimize the given cost function. While tuning these parameters, the evaluation of the cost function concerning the locomotion performance can be conducted online or off-line.

The off-line optimization explicitly formulates the cost function in a mathematical form, and it requires system modeling served as the dynamic constraint to the optimization problem. [11] formulated and evaluated the average power loss for given average locomotion velocity, while gridding the parameter space for \((\alpha_r, \beta_r, \gamma_r)\), and found optimal undulation motion for robotic snake corresponding to the minimum of the function. The online optimization measures the performance index based on the data collected from the real-time
experiments. System modeling is not necessary, what is needed is to change trajectory parameters at every experiment setup, set up controller that can follow the trajectory, implement and use sensors wisely to obtain measurement. Prior to the experiments, a feasible controller has to be proposed. In [10, 72], experiments were carried out to identify types of traveling waves that maximized speed, as was relatively easy to measure. The target trajectory that was achieved by controller in [10] took the form of (1.1), while [72] used a different formulation for the serpenoid curve.

The optimal gaits derived from the second approach can mimic the movement of the animal well, since parametrized trajectory of gaits are given based on biological observation. However, the underlying mechanism that why animals adopt these gaits seems not clear. The third approach believes that the locomotion system, when undulating at mechanical resonance, could require less energy input but obtain large output efficiently. In [73], through the experiments by tuning the gain of actuators, optimal gaits close to the natural vibration mode of robotic snake were generated, which was believed to produce high speed and high efficiency. This self-excitation principle were also adapted to a biped walking robot [74, 75], a fish robot and so on. But the linear controller thus proposed can not guarantee the stability of the periodic movement.

In this thesis, we will adopt the third approach to design a gait called natural oscillation pattern. It will be compared with the optimal gait that is derived using the second approach with off-line computation.
1.3 Control Method

1.3.1 Controller Design Approaches

It is the central task to design a controller for life-like robots achieving the selected gaits. Considerable research effort has been dedicated to the control design for the mechanical rectifiers. These methods are to be classified into two categories: model-based and non-model-based control. The methods differ on whether a mathematical model depicting the locomotion system is involved. We believe that mathematical models, at various levels and complexities, can play a critical role in exploring the underpinning locomotion mechanism, gait optimization, stability analysis, robustness and perturbations analysis. Thus, the review has bias toward the model-based control.

The reasons that the development of the model is important to design control laws have several aspects. Firstly, the explicitly modeling is useful for understanding the locomotion mechanism. For instance, Gray [41] concluded external forces acting in the normal direction to the snake body was vital to the forward motion of a planar snake, and the conclusion was drawn based on the mathematical model of snake locomotion. [76, 60] showed that the bilinear terms in the dynamics model of mechanical rectifier system was essential to the thrust generation and they resulted from anisotropic forces from environment. It is also stated in the second gaits optimization approach in Section 1.2 that off-line optimization requires a dynamic or kinematic model. In addition, the existence of the system models can allow for the stability analysis of the control design. Although the model complexity limits the formal stability analysis on the locomotion system, but with model simplification techniques it is still possible. For instance, [77] arrived at a simplified model with partial feedback linearization, and gave controllability and stability analysis for planar snake-like robots.
Most of the model-based approaches are based on kinematics of the robot’s locomotion, as it is usually simpler than dynamics models. For snake-like robots, kinematic models are developed under explicit assumption that the body cannot move sideways, and this assumption introduces nonholonomic constraints [78] in the equations. Exploiting tools from differential geometry, [68] presented the kinematics of snake robots where passive wheels were installed to impose the slipslide constraints in implementation. It also established the relation between shape changes and the overall displacement of robot, which allows the control input to be specified directly in terms of the desired propulsion of the robot (see Remark 2 in [12]), thus simplifying the controller design. [79, 80, 81] employed this approach to propose path and/or position following controller, among which [79] used Lyapunov function candidate to prove the stability of the controller.

The development of the dynamics model is primarily employed to analyze the optimality of the gaits. As they are normally very complicated, however it is difficult to incorporate dynamics model in model-based control design. A model of the two-dimensional dynamics of a wheeled snake with slipslide constraints was developed in [82] from Lagrange’s equation of motion and in [73] form first principle. In addition, there are also models that assume that the friction exhibit anisotropic properties similar to biological snake, such as [83] and [11]. The main difference between various dynamics model lies on the utility of different friction model, namely, Coulomb and viscous friction. Few work has been dedicated to the control design based on pure dynamics model. In [82], the velocity constraints between each link, as the kinematic constraints, were used to reduce the dynamics model. It then proposed a PD control based on the Lyapunov function method. To minimize the lateral constraint force resulting from kinematic constraints in [82], [84] proposed a new control strategy to yield smooth locomotion.

As a non-model-based method, it is not necessary to obtain a system model for the control design. [85, 11] use (1.1) as reference for joint angles and a PID controller is then
designed to minimize the error between the reference joint angles and the calculated joint angles. Another non-model-based method is inspired by biological control structure. In animal locomotion, rhythmic movements are controlled by mechanisms consisting of the neuronal circuit called the central pattern generator (CPG) [86, 87, 88]. In [10, 35], the locomotion of the robots were controlled by a CPG network (mimicked by coupled nonlinear oscillators) that produced traveling waves of oscillations as limit cycle behavior. The generated trajectory of relative angles followed (1.1). These open-loop control do not need implicit modeling and it can be considered as a particular feed-forward controller. A similar approach has been presented in [89]. The main difference is that [89] use coupled phase oscillators instead of nonlinear oscillators, which rather have explicit amplitude state variable. However, these controllers are typically open-loop; the set points of the joints are calculated and sent to the motor controllers. The dynamics including the interaction between the body segments and environment is out of the control loop, therefore there is no feedback loop to sense the environment changes. Thus, such controller is not robust and adaptive to the perturbation.

Due to highly nonlinearities, hyper-redundant degrees of freedom and complicated body/environmental interaction, the dynamics of the locomotion system are complex, while kinematics model with slipslide constraints is relatively easy to deal with. An appropriate model simplification sometimes is necessary to reduce the complexity of the problem. The common approach is linearization. [73] applied the linear approximation of link angle around the nominal position to analyze the self-excited locomotion of snake. One should be aware of the notation “appropriate”, as inappropriate or over-simplified model simplification process could possibly lead to missing the essential part of locomotion system. For instance, [76, 60] showed Taylor’s series expansion of the nonlinear terms down to order of one would fail to capture the mechanical rectification effect. The partial feedback linearization is another controller design tool. It was used in [77] to simplify the nonlinear
model from [11]. Moreover, the mild assumption can be used for model simplification. [90] assumed a linear displacement model for the body shape changes of the snake robot and thus proposed a simplified model of snake locomotion, which then was employed in [91] to derive properties of the locomotion velocity. At last, robot’s mechanical and structural design can simplify the mathematical model from very beginning. For instance, installing the passive wheel to impose the kinematic constraints could result in a simple kinematics model leading to an easy control design.

In this thesis, we will take a different approach to propose a dynamics-mode-based controller to achieve designed oscillation pattern. The approach is based on dynamical modeling of the class of mechanical rectifiers followed by model simplification. We would like to propose a feedback controller that is inspired by the structure of biological central pattern generators (CPGs), which makes the closed-loop system generate the periodic oscillation autonomously and does not require explicitly time-dependent periodic input.

### 1.3.2 CPG Inspired Control

Made of multiple coupled oscillatory centers, CPGs are distributed networks of neural circuits that can produce coordinated oscillatory signals without oscillatory inputs. Clear evidence showed that rhythmic movement can be generated centrally without requiring sensory information [86, 87]. The useful properties, including distributed control, the ability to deal with redundancies and fast control loops, make CPGs interesting building blocks for locomotion controllers in robots. The CPG inspired controllers have been widely applied into control of life-like robots, such as humanoid robots [18, 20], quadruped robots [23, 92] and hexapod robots [25, 93]. Tightly related to the limbless locomotion are anguilliform swimming of lamprey/eel robots [5, 94, 95] and snake-like robots [89, 96, 49, 97].
Controller design for the life-like robots that is inspired by the structure of CPGs offers several advantages: (1) CPG models can exhibit limit cycle behavior, i.e. to produce stable rhythmic patterns for periodic robots movement; (2) the control scheme can be implemented in a distributed fashion; (3) sensory feedback can be incorporated in order to take perturbations into account.

In implementation, CPGs are mathematically modeled as a network of coupled nonlinear oscillators. How to analyze and design the limit cycle behavior with desirable property are difficult problems, due to limited number of suitable nonlinear analysis tools. In addition, the method of incorporating the CPGs control structure in the feedback controller can rarely be found in the literature. Since the complexity of the CPG and dynamics models, CPGs are mostly used as an open loop controller. Furthermore, it is even more difficult to derive the stability of the CPG-robot coupling system.

As the building blocks for locomotion controllers, CPG can be modeled as a network of nonlinear oscillators that could exploit relevant theories. Nonlinear oscillator theories have a long history and various analysis methods have been developed in the literature, including the Poincaré-Bendixson theorem [98, 99], Hopf bifurcation theorem [100, 101], perturbation theory and averaging [102, 103], harmonic balance [104, 105], and the contraction analysis for global convergence [106, 107, 108].

These methods have been applied into the work related to robotics for analysis and generation of limit cycle. In [109], the contraction analysis approach was used to design a CPG-based controller that were composed of Hopf oscillators, achieving control of turtle-like underwater vehicle. The multivariable harmonic balance method was employed to characterize the condition for a class of collocated mechanical systems to have limit cycle as its trajectory [110] in closed-loop. The method was also used to analyze a group of weakly coupled interconnected oscillators [111] and to construct a network of oscillators...
with prescribed oscillation profiles [112], [113] exploited dissipativity theory for the global analysis of limit cycles resulting from hopf and pitchfork bifurcations. Other methods such as Poincaré map were exploited to characterize the orbital stability of the snake-like robots [77] where the straight line path following controller was proposed.

Most of the results so far have focused on the analysis of limit cycles to prove existence, assess stability, and estimate frequency and amplitude, and there are very few general theories for the feedback control design of limit cycles. The work by Shiriaev et al. [114, 115] has developed one of such design theories. The virtual constraint approach was proposed to reduce the problem complexity, but the problem of setting a desired oscillation profile has not been addressed.

### 1.4 Thesis Outline

Chapter 1 provided an introduction to the background knowledge of the animal locomotion and life-like robots. A literature review about control and gaits optimization of locomotion system were given. A few comments to the current research status were exposed so that the reader may appreciate the contributions in the work.

Chapter 2 presents the general system modeling of the mechanical rectifiers detailed with the description of three typical robotic locomotors. The dynamical structure of such kind of locomotion systems is clearly revealed to have three components: oscillation, orientation and locomotion dynamics. The breakdown of dynamical structure furnishes a vital foundation for controller design and locomotion analysis later.

As to the oscillation dynamics, Chapter 3 introduces the definition of optimal and natural oscillation patterns. The existing control design was introduced but it could only achieve the natural oscillation approximately. Inspired by biological CPG network, a controller that
achieves the natural oscillation exactly will be proposed with a sound mathematical analysis on the stability analysis. The comparison of the optimal and natural oscillation patterns for three robotic locomotors is illustrated to show that the natural oscillation pattern is a sub-optimal pattern.

Due to the under-actuation of the locomotion system, the system orientation cannot be manipulated by the oscillation controller proposed in Chapter 3. Chapter 4 therefore proposes an orientation controller on top of the previous controller making the average orientation to be aligned with locomotion direction, though at the price of the accuracy of the body oscillation. Two variation of orientation control design are devoted to snake-like and flapping-wing locomotors, while fully actuated fliptail locomotors is not considered.

Chapter 5 analyzes locomotion behavior based on the locomotion dynamics of the mechanical rectifiers. A technique of balancing thrust and drag force exerted on the mass center over a period will reveal the relation between the locomotion velocity, body oscillation and orientation. The analysis goes further and it is proved that a specific velocity called natural velocity can be achieved by the induced natural oscillation pattern and orientation.

Combining the results from the previous three chapters, a unified framework of control design for mechanical rectifier systems has been formed. To verify the effectiveness of the framework and proposed controllers, Chapter 6 is populated by a group of numerical simulations for three typical robotic locomotors. Some of the additional properties that are not revealed previously is introduced.

Chapter 7 concludes this thesis with a brief summary and a discussion on future research. Following that are appendix, including derivation of the models for robotic locomotors.
1.5 Publication

The research on which this thesis is based has led to a number of articles in specialized journals and conference proceedings. Details of publications having a direct connection to the work in this thesis are given below:

**Journal Papers:**


**Conference Papers:**


Chapter 2

Mechanical Rectifier Locomotion

The chapter presents a general class of animal locomotion, whose structure is composed of multiple body segments in continuous contact with the surrounding environment. The mass center of the locomotor is driven by persistent and periodic undulation of body segments, thus the locomotor is called mechanical rectifier and the mechanism of animal locomotion is called mechanical rectification. The mechanical rectifier can be modeled mathematically as a group of complex nonlinear differential equations, that fully captures the essence of the mechanical rectification effect. A model simplification that retains the embedded rectification mechanism, is somewhat necessary to make it a tractable problem to propose an effective model-based controller.

The dynamical structure of the mechanical rectifier will be revealed to include three important parts: oscillation, orientation and locomotion dynamics. The dynamical breakdown provides an insightful viewpoint to the interpretation of the mechanical rectification, furnishing a vital foundation for controller design and locomotion analysis. The mechanical rectifier can be herein exemplified by three typical robotic locomotors, namely fliptail, snake-like and flapping-wing locomotors, that are inspired by the morphology and motions of animals. In particular, the chapter introduces general modeling of a class of mechanical rectifiers and extensively provides dynamical models of three robotic locomotors in detail. The system modeling is devoted to offering the basis towards a unified framework for controller design and hence this chapter is of importance to chapters that follow.
2.1 Introduction

Animal locomotion can be viewed as a process of mechanical rectification [37, 38] in which oscillatory motion of limbs or bodies is converted (rectified) to sustained thrust force on the mass center through dynamical interactions with the environment. For instance, a snake crawls by undulating its body constantly on the ground, while a bird flies by flapping its wings repeatedly in the air. Taking inspiration from their natural analogues, the increasing number of human-made machines have emerged in recent years. As a class of the robotic systems, mechanical rectifiers are generally considered to consist of multiple body segments (or bodies simply) and placed in an environment of up to three spatial dimensions. The multiple bodies are rigid, and connected to each other through rigid or flexible mechanisms (e.g., rotational joints as in manipulator arms [124], and flexible wires as in tensegrity structures [125, 126, 127, 128].) The salient feature of mechanical rectifiers is that no global propulsion devices exist such as a jet engine, although each joint is equipped with an actuator, which provides the power input acting like muscular contraction and relaxation. The bodies are driven by the actuators, while in the contact with environmental media, generating the interactive (frictional or viscous) forces and torques. The forces and torques so generated and distributed across multiple bodies lead to a global net thrust on the mass center propelling the system forward along. Therefore, the movement of mechanical rectifiers is the global effect caused by the motion and particularly periodic motion of every local contributing body segments. The motion coordination among the body segments is of vital importance to the mobility of mechanical rectifiers, thus requiring delicate trajectory planning and sophisticated control of the actuators.

The aim of the thesis is to propose a unified framework tackling control design for a class of mechanical rectifiers based on approximately accurate dynamical model. The system
modeling is a tool that provides mathematical description of the dynamical system. Modeling of mechanical rectifiers helps a good understanding of the systems and unravels the principles hidden in the biological manifestation. In a broad sense, system modeling offers a tool that bridges the biological and robotic research, not only giving biology researcher an engineering viewpoint of biological systems, but also helping engineers to understand better and take advantage of the biological system for robot design and optimization. In the control level, a decent modeling of the system is helpful to design a feedback controller that is robust and adaptive to the environment change.

Euler-Lagrange formulation and Newton’s law are two of the common modeling methods for obtaining equations of motion. Essentially all methods are equivalent in terms of resulting equations. Euler-Lagrange formulation treats the system as a whole, and disregards all interactive and constraint forces that do not perform work for the whole system. Its analysis is based on the derivative of its kinetic and potential energy. The Newton’s law takes into consideration each part of the system in turn and thus yields good physical understanding, but requires to obtain every interactive and constraint force. The favor of different methods sometimes depends on the system complexity and topology of the mechanical structures, etc. [1, 2] applied Euler-Lagrange formulation, while Newton’s law was used in [11].

It is known that the interactive force between the bodies and environment plays an important role in the thrust generation on the mass center. Many modeling methods assumed that the bodies can not move sideways (sideslip constraints) [82, 84, 129, 63, 130], and introduced nonholonomic constraints [78]. In practice, the assumption does not hold all the time, but sideslip can be minimized and the constraints can be relaxed by optimal controller in [84]. In addition, many models does not enforce such constraints, but instead only assume that the force models demonstrate anisotropic property, namely the friction coefficient in one direction significantly differs from that in other directions. Two contact models are generally adopted to describe the interactive force, especially for snake-like robots. They
are Coulomb and viscous contact model. The Coulomb friction contact model was applied in [131, 132], while the viscous contact model was used in [133]. Sometimes combination of viscous/Coulomb friction models was considered [11, 73]. In this thesis, the modeling of the mechanical rectifier adopts viscous contact model.

The chapter will take readers through the modeling of mechanical rectifiers. The rest of the chapter is organized as follow. Section 2.2 will introduce the modeling procedure for a general class of mechanical rectifiers. Section 2.3 will present models of three specific robotic systems, namely fliptail, snake-like and flapping-wing locomotors. Given the detailed dynamical models, Section 2.4 will summarize optimization and control problems needed to be solved in the thesis.

2.2 General Mechanical Rectifier Locomotion

2.2.1 General Modeling

Consider a class of mechanical rectifiers that is characterized by a multiple-body mechanical system placed in an environment of up to three spatial dimensions. Fig 2.1 illustrates the multi-body robot in $x$-$y$ plane inspired by the swimming frogs (adapted from [60]). The bodies are normally made up of the mechanical parts such as links and connected to each other by joints. Note that for the multi-link system as in Fig 2.1, it is only two neighboring links that one joint can bind. The number of joints is just enough to connect bodies in order to avoid redundant degrees of freedom. As a result, $n_l - 1$ joints exist at the junction where $n_l$ links join together. In Fig 2.1, with six joints seen though, there are actually eight joints. The structure of the joints might be flexible (probably by adding spring) so that the energy can be stored and released at different phase of the movement aiming at improving
the efficiency. Torque can be applied by the actuator equipped at each joint to change the relative shape formed by neighboring links.

![Diagram](image)

Figure 2.1: Multi-body swimming system in two-dimensional space with different orientation (adapted from [1].)

It is notable that the frictional or viscous force resulting from the relative motion of bodies and environment plays an important role in thrust generation. Without friction and viscosity, the mechanical system would just slip on the ground incapable of moving forward. In particular, the anisotropy of environmental force is a crucial property for mechanical rectifiers. According to analytical studies of snake locomotion [41], it emphasized the importance of environmental forces acting in the normal direction during body undulation. More recently, it has been shown experimentally [8, 134] and analytically [11, 1] that the dominance of the environmental force in the normal direction over that in the tangential direction is crucial to thrust generation. The anisotropic condition enables smooth movement along the tangential direction and prevents sliding sideways. In the thesis, the environmental forces are assumed to result from continual interactions between bodies and the surrounding media (e.g. fluid or surface), and are modeled by static functions of the relative velocity without slipping [11, 135] for reason of simplification.
The variable $\theta(t) \in \mathbb{R}^p$ is defined to contain the information specifying the local body shape $\phi$ and global orientation of the bodies $\varphi$. The body shape determines the shape or appearance of the system regardless of the orientation. To illustrate this, two figures in Fig 2.1 have exactly the same shape but their global orientations are different. This can be noted from the fact that $\theta_i$ changes while $\phi_i$ keeps the same with variation of the orientation. The equations of motion for the general class of mechanical rectifiers were developed in [60, 1] using the Euler-Lagrange method. The generalized coordinate $q$ is defined as

$$q = \begin{bmatrix} \theta \\ p \end{bmatrix},$$

where $p(t) \in \mathbb{R}^k$ is the global position vector for the mass center relative to the environment and $\varpi := \dot{p}$ is the global velocity vector. The equations of motion are given by the following form:

$$I_n(\theta) \ddot{\theta} + C_n(\theta, \dot{\theta}) \dot{\theta} + k_n(\theta) + d_n(\theta, \dot{\theta}) + R_n(\theta)^T \gamma_n(R_n(\theta) \dot{\theta} + N_n(\theta) \varpi) = T(\theta) \tau, \tag{2.1}$$
$$m \varpi + mg + N_n(\theta)^T \gamma_n(R_n(\theta) \dot{\theta} + N_n(\theta) \varpi) = 0,$$

where the terms $I_n(\theta) \ddot{\theta} + C_n(\theta, \dot{\theta}) \dot{\theta}$ and $m \varpi$ are the inertial torques and forces, $k_n(\theta) + d_n(\theta, \dot{\theta})$ are the torques due to body stiffness and damping, $\tau(t) \in \mathbb{R}^n$ is the applied input, the terms involving the function $\gamma_n$ capture the effect of environmental forces. $\gamma_n : \mathbb{R}^\sigma \rightarrow \mathbb{R}^\sigma$ is a nonlinear function that maps the relative motion to the forces and torques. The torque input $\tau$ can be applied to generate variation of the shape and orientation, that then are rectified through the second equation to result in a “locomotion” with velocity $\varpi$. The $\varpi$-dynamics takes no direct input from $\tau$, verifying the lack of global propulsion devices.
The nonlinear model can be simplified by assuming symmetric body oscillations with small amplitudes around a nominal posture [60]. Nominal posture can be regarded as the particular posture that many biological systems use for relaxed cruising between active locomotion phases. Normally, the whole body structure is symmetric about a certain axis (or a plane) when at the nominal posture and the direction of locomotion is often chosen to be aligned with the axis of symmetry $V$. Let the nominal posture be specified by $\theta = \theta_{nom}$ at velocity $\dot{\varpi}(t) = \varpi_o$. Let the first axis of the global coordinate frame be aligned with $V$ and there is not significant locomotion on the directions except $V$ due to the symmetry. Then we can reduce independent variable from $\varpi$ to $v$ where $v \in \mathbb{R}$ and $e_1 \in \mathbb{R}^k$ with the first entry being one and the others being zero due to symmetric structure. Consider a periodic body motion $\theta(t)$ about the nominal posture $\theta_{nom}$, and assume that small oscillation of $\theta := \theta - \theta_{nom}$ maintains the locomotion velocity $\varpi(t)$ near $\varpi_o \simeq v_o e_1$. Given the assumptions, a model approximation of the nonlinear equation (2.1) can be conducted for further analysis. It was revealed that a direct linear approximation failed to capture the rectifying dynamics [136]; the simplest model should contain a bilinear term of $\theta$ and their derivatives, allowing for insightful analysis of efficient gait. The small amplitude assumption makes it reasonable to approximate nonlinearities by the Taylor series with truncation of higher order terms. The method partially retains the bilinear term and simplifies (2.1) to

\[
\begin{align*}
J\ddot{\theta} + D\dot{\theta} + K\theta &= B\tau, \quad K = K_o + \Lambda v_o, \\
m\ddot{v} + (a + \theta^TQ\theta)v + \theta^T\Lambda^T\dot{\theta} &= 0,
\end{align*}
\]

(2.2)

where $J$, $D$, $K$, $Q$, $\Lambda \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times n}$ and $a$, $m \in \mathbb{R}$. Although the approximation could introduce a potentially large error in general, qualitative characteristics of the environmental forces can be well captured [60]. The essence of mechanical rectification is embedded in the bilinear term $\theta^T\Lambda^T\dot{\theta}$ in second equation with a asymmetric matrix $\Lambda$ (refer to [1, 76] and Chapter 5 for further explanation). The bilinear mechanism in (2.2) is a generalization of the canonical dynamics for rectification studied by Brockett [37, 38].
Two features of the mechanical rectifier are exposed in the modeling process: redundancy and decentralization. As to redundancy, multi-body structure requires a large number of actuators that is more than the degree of freedom for the motion in the space. The redundancy of multi-body swimming system in Fig 2.1 can be shown by the fact that it has three degrees of freedom for motion with two for translation and one for rotation, while equipped with eight actuators. On the other hand, decentralization depicts that the actuators are not centralized but distributed across the multi-body system at each joint. Robotic implementation of mechanical rectifiers needs to take into consideration the coordination, communication and synchronization issues between body segments. These properties make the control of the mechanical rectifier very challenging and bring up several problems that need to be tackled. What principle of choosing optimal motion pattern for multiple bodies and how to design a distributed control strategy are two major concerns to be addressed in following chapters.

### 2.2.2 Dynamical Structure of Mechanical Rectifier

With regard to (2.2), the number of actuators equipped on the system $n$ and the number of the variable $\theta$ to be controlled $p$ vary with the specific setup of the mechanical rectifiers. For $p = n$, the bodies and orientation can be fully controlled by the torque input, i.e., $\theta$ representing desired body shape and orientation can be achieved by carefully designed periodic input $\tau$. It is called fully actuated mechanical rectifier. For $n < p$, the number of the torque input is not sufficient to generate an arbitrary oscillation for $\theta$, thus it is called under-actuated. Sometimes, the robotic design ensures the dimension of the body shape $\phi$ equals to that of the torque input, which enables the full controllability of the body shape at least. Multi-body swimming system in Fig2.1 is one of the cases, where actuators are just enough to manipulate the relative angle of the links keeping the body shape fully controlled.
For mechanical rectifiers of concern, let us assume there exists a nonsingular real matrix 
\( T \in \mathbb{R}^{p \times p} \) corresponding to the following coordinate transformation:

\[
\begin{bmatrix}
\phi \\
\varphi
\end{bmatrix} = T \theta, \quad \theta = T^{-1} \begin{bmatrix}
\phi \\
\varphi
\end{bmatrix}, \quad T = \begin{bmatrix}
W \\
U
\end{bmatrix}, \quad (2.3)
\]

where \( W \in \mathbb{R}^{n \times p} \) and \( U \in \mathbb{R}^{(p-n) \times p} \) such that

\[
TJT^{-1} = \begin{bmatrix}
J & 0 \\
-J_s & J_o
\end{bmatrix}, \quad TDT^{-1} = \begin{bmatrix}
D & 0 \\
-D_s & D_o
\end{bmatrix},
\]

\[
T(K + v_o \Lambda)T^{-1} = \begin{bmatrix}
K_o + v_o \Lambda & 0 \\
-K_s & 0
\end{bmatrix}, \quad TB = \begin{bmatrix}
B
\end{bmatrix},
\]

where non-singular matrices \( J, D, K \) and \( B \in \mathbb{R}^{n \times n} \), \( \phi \in \mathbb{R}^n \) and \( \varphi \in \mathbb{R}^{p-n} \). The coordinate transformation turns (2.2) into

\[
J \ddot{\phi} + D \dot{\phi} + K(v_o)\phi = u, \quad K(v_o) = K_o + v_o \Lambda, \quad u = B \tau, \quad (2.4)
\]

\[
J_o \ddot{\varphi} + D_o \dot{\varphi} - \begin{bmatrix}
J_s \dot{\phi} + D_s \dot{\phi} + K_s(v_o)\phi
\end{bmatrix} = 0, \quad (2.5)
\]

\[
m \ddot{v} + \Psi(\phi, \varphi)v + \Xi(\phi, \dot{\phi}, \varphi, \dot{\varphi}) = 0, \quad (2.6)
\]

where \( \Psi \) and \( \Xi \) are the linear quadratic functions of the variables inside brackets. The breakdown of equation (2.2) reveals that the dynamical structure of mechanical rectifiers is made up of three differential equations, i.e., shape or oscillation (2.4), orientation (2.5) and locomotion equations (2.6). The shape equation shows how the oscillatory motion of the body shape \( \phi \) results from the interaction with environment (represented by the velocity \( v_o \) with respect to the inertial frame) and the body actuation (represented by the torque vector \( u \in \mathbb{R}^n \) applied on the links). The body shape \( \phi \) is directly controlled by the full actuation
The orientation equation shows how the orientation $\varphi$ is passively driven by $\phi$ through coupling terms $\left[J_s \ddot{\phi} + D_s \dot{\phi} + K_s \phi\right]$ and hence indirectly from $u$ (as $\dot{\phi}$ depends on $u$ indirectly). Finally, the mechanism of mechanical rectification is embedded in the locomotion equation. The locomotion equation shows that the locomotion velocity $v$ is controlled not directly by actuation $u$ but through the interaction of the shape change $\phi$ with the environment. In particular, periodic body movements $\phi$ are rectified to generate the locomotion velocity $v$. It is notable that the model (2.4)-(2.5) is obtained at the expense of potential inaccuracy due to the slender body (i.e., small $\theta$) assumption. When the body undulation involves large curvature, the model may not accurately predict the orientation and locomotion velocity. However, the model qualitatively captures the essential dynamics of locomotion and is reasonably accurate for the range of undulations observed in animals. It is worthwhile to state and justify explicitly the characteristics of the shape equation (2.4).

**Assumption 2.1** Let $v_o$ be a fixed constant in a desired range of locomotion velocity.

a. $J = J^T > 0$;

b. $D = \mu J$ for some $\mu \in \mathbb{R}$;

c. $K(v_o) = K_o + v_o \Lambda$ where $K_o$ is a symmetric positive definite matrix and $\Lambda$ is an asymmetric matrix;

d. All the eigenvalues of $M(v_o) := J^{-1} K(v_o)$ are simple and have positive real parts;

e. The eigenvalue $\lambda$ of $M(v_o)$ that minimizes $\Im(\lambda)/\sqrt{\Re(\lambda)}$ is unique;

The assumptions are not overly restrictive. The moment of inertia matrix $J$ for a general mechanical rectifier is always symmetric positive definite as stated in Assumption 2.1.a.
The Rayleigh damping \( D = \mu J \) in Assumption 2.1.b typically arises from modeling of fish fins as a set of flat plates subject to resistive hydrodynamic forces [137, 1]. In particular, the fluid force on a small plate segment is approximately modeled by a linear function of the plate velocity relative to the fluid, with its magnitude proportional to the plate area. Such model results in the fluid damping matrix \( D \) proportional to the moment of inertia matrix \( J \) with proportionality constant \( \mu := c/\rho \) where \( c \) is the fluid drag coefficient and \( \rho \) is the body density. On the other hand, the stiffness matrix \( K(v_o) \) is not necessarily symmetric due to the force from the environment. In particular, it is given by \( K(v_o) = K_o + v_o \Lambda \) where \( K_o \) is a symmetric positive definite matrix representing the body stiffness, and \( \Lambda \) is an asymmetric matrix representing the skewed stiffness arising from the locomotion at velocity \( v_o \) relative to the environment. We see that all the eigenvalues of \( M(v_o) \) are real positive when \( v_o = 0 \) since \( J \) and \( K_o \) are positive definite. Hence, by continuity, there exists \( \epsilon > 0 \) such that the real parts of the eigenvalues of \( M(v_o) \) are positive for all \( |v_o| < \epsilon \).

When the model represents animal locomotion, velocities typically observed in biology are well within this range [138, 137]. Thus, it seems reasonable to assume Assumption 2.1.d for robotic locomotors. For simplicity, we introduced the additional assumption that no eigenvalues are repeated, which is generically satisfied. Finally, Assumption 2.1.e is a technical assumption, which will turn out to give uniqueness of the natural oscillation. This assumption is always satisfied except for a very special case where there is a parabolic region \( y^2 \leq cx \) with \( c > 0 \) on the complex plane \( x + jy \in \mathbb{C} \), containing all the eigenvalues of \( M(v_o) \) with more than one pair exactly on the boundary.
2.3 Robotic Locomotors

The section will focus on three typical robotic implementations of mechanical rectifiers, namely fliptail, snake-like and flapping-wing locomotors. They adopt the idea of the mechanical rectification but are different in terms of actuator number, appearance and complexity in control. The equations of motion can be obtained by following the modeling process introduced in previous section (which was proposed in [60, 1]) and also using Newton’s laws as in [11]. The equations take the final form of the nonlinear differential equations (2.1) regardless of the modeling methods, which can be further simplified to the equations similar to (2.2) with the essential bilinear terms kept. The fliptail locomotor is considered to be fully actuated, while the snake-like and flapping-wing locomotors are under-actuated. The fliptail locomotor and snake-like locomotor are multi-link mechanical systems placed in two-dimensional space, while fliptail locomotor is simpler in dynamics, since the movement of the mass center is constrained along a line due to the symmetry of the structure. The flapping-wing locomotor is placed in three-dimensional environment and free for translation and rotation. It is subject to three-dimensional surface contact with the environment leading to a complex external force model. Depending on the dimension $p$ of the motion variable $\theta$ and $n$ of control input $u$, the differences of the three locomotor systems are listed in Table (2.1).

<table>
<thead>
<tr>
<th>Locomotor</th>
<th>$p$ and $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fliptail Locomotor</td>
<td>$p = n$</td>
</tr>
<tr>
<td>Snake-like Locomotor</td>
<td>$p = n + 1$</td>
</tr>
<tr>
<td>Flapping-wing Locomotor</td>
<td>$p = n + 2$</td>
</tr>
</tbody>
</table>

Table 2.1: Comparison of fliptail, snake-like and flapping-wing locomotors.
2.3. **ROBOTIC LOCOMOTORS**

2.3.1 **Fliptail Locomotor**

The fliptail locomotor consists of multiple links connected with each other by the joint which is equipped with actuator. The fliptail locomotor maintains its forward propulsion velocity $v$ by flipping (or undulating) the link chain (tail links), while a particular part of the locomotor acting as the head is constrained not to undulate. One of the example is illustrated in Fig 2.2 [121], where two chains of tail links are attached to the head. The system is considered to be a typical fliptail locomotor, and the analysis and controller design method are also applicable to other fliptail locomotors such as the one in [136] due to the similarity of mechanical structures. Consider the scenario that two chains undulate symmetrically about x-axis, which is also common in animal locomotion (such as jellyfish). In the case, the force acting on the y-axis of the head link is balanced out so that the head only moves along the x-axis without incurring displacements in the y-direction. Due to the symmetry, the head loses the degree of freedom for rotation and needs not to be manipulated by actuators, and the global orientation becomes fixed to align with x-axis. Then, the body shape is fully actuated and the variable $\theta$ coincides with body shape $\phi$ as in Fig 2.2. Fliptail Locomotor is an engineering analogue of various locomotion behaviors of multi-body animals including jellyfish, octopus, etc.

Let $(x_o, y_o)$ be the coordinate of the head link instead of mass of the center, as their velocities along x-axis are the same and velocities along y-axis are zero due to symmetry about x-axis. $v := \dot{x}_o \in \mathbb{R}$ denotes the head velocity resulting from the undulation. Assume that there are $n$ identical links in each chain, and each link has mass $m_o$, length $2l_o$ and moment of inertia $m_o l_o^2 / 3$, and each joint has torsional stiffness $k_o$. The angle between the $i$th link and the negative x-axis is denoted by $\phi_i$. Let $\tau_i$ be the torque applied at the $i$th joint. Each link is subject to directional environmental forces. In particular, the environmental force on each link is modeled approximately as linear functions $f_n = -\mu_n v_n$ and $f_t = -\mu_t v_t$, where $f_n$ and $f_t$ are the force components in the direction tangent and normal to the link,
Figure 2.2: A fliptail locomotor with symmetric mechanical structure moving in $x$-axis only.

$u_n$ and $v_t$ are the components, in the respective directions, of the velocity of the link mass center, and $\mu_n$ and $\mu_t$ are proportionality constants [11] (see Fig (2.3)).

The equations of motion can be derived using Newton’s third laws (see Appendix A.1 and [11] for the detailed derivation). The links are considered to oscillate symmetrically about the nominal position ($\phi = 0$), in which all the two link chains are aligned with $x$-axis. The equations of motion can be simplified around the nominal position and are given...
by

\[ J\ddot{\phi} + D\dot{\phi} + K(v)\phi = u, \quad (2.7) \]
\[ \dot{v} + \Psi(\phi)v + \Xi(\phi, \dot{\phi}, \ddot{\phi}) = 0, \quad (2.8) \]

with

\[ D = \mu J, \quad K(v) = v\Lambda + k_oBB^T, \quad u = B\tau, \quad (2.9) \]
\[ \Psi(\phi) = c + \phi^T C\phi, \quad \Xi(\phi, \dot{\phi}, \ddot{\phi}) = \dot{\phi}^T S\dot{\phi} + \theta^T S\dot{\theta} + \dot{\theta}^T Q\theta, \]

for some scalar \( c \) and square matrices \( S, C \) and \( Q \) such that \( c \geq 0 \) and \( C = C^T > 0 \). Refer to Appendix A.1 for the definitions of the coefficients and nominal parameters.

In this model, \( u \in \mathbb{R}^n \) is the input force uniquely determined by the muscle (or actuator) \( \tau \), \( \phi \in \mathbb{R}^n \) is a vector variable describing the shape of the body, and \( v \in \mathbb{R} \) is the locomotion velocity of the body (e.g., the mass center or the head) along \( x \)-axis. The mechanism of mechanical rectification is captured by term \( \Xi(\phi, \dot{\phi}, \ddot{\phi}) \) with asymmetric matrices \( S \) and \( Q \) in (2.8). In comparison to general mechanical rectifier (2.4)-(2.6), the orientation equation (2.5) is dropped, as the orientation of fliptail locomotor does not have to be manipulated. In addition, body shape \( \phi \) has the same number as torque input \( \tau \), thus is fully actuated. These simplicity properties of the model (2.7)-(2.8) make the fliptail locomotor a good starting platform for investigating other mechanical rectifiers. It is worthwhile to note that the stiffness matrix \( K(v) \) is dependent on velocity \( v \) rather than the nominal velocity \( v_o \) as in general model (2.4). By doing so, the bilinear term \( v\Lambda \) is kept during the procedure of model simplification, with which the resulting model resembles the original nonlinear model more approximately. Chapter 6 will show that this bilinear can be handled in the control design due to the model simplicity of fliptail locomotor.
2.3.2 Snake-like Locomotor

Snake-like locomotor is similar to flitptail locomotor, but with the head removed. It can represent a class of robotic model of undulatory locomotion systems inspired by crawling and swimming slender-body animals (e.g., snakes, eels, and leeches) better than flitptail locomotor. We adopt the fully nonlinear equations of motion for a robotic snake derived in [11, 122, 116], and its second order (rather than linear) approximation developed in [60]. Consider the mechanical system comprises a chain of $p = n + 1$ identical rigid links connected by $n$ rotational joints as depicted in Fig (2.4). In comparison to flitptail locomotor, a constrained and inert head link and an active input torque are removed together, leaving actuators one less than links to be controlled.

Figure 2.4: The schematic diagram of a snake-like locomotor.

For this model, undulatory body movements are supposed to occur within the $(x, y)$ plane, where the $x$-axis is taken to be the direction of locomotion. Let $\theta_i$ be the absolute angle between the $i$th link and the negative $x$-axis, and $(x_i, y_i)$ be the coordinate of the mass center of the $i$th link. Each joint is flexible with torsional stiffness $k_o$, and the $i$th joint is driven by an actuator that generates torque $\tau_i$. We use the same environmental force model from the flitptail locomotor. Each link is subject to the environmental force (e.g. ground friction for crawling, and fluid drag for swimming), which is approximately modeled as linear functions of link velocities. In particular, the force components tangential and normal to the $i$th link are given by $f_{t_i} = -\mu_t v_{t_i}$ and $f_{n_i} = -\mu_n v_{n_i}$, respectively, where $\mu_t$ and $\mu_n$
are constant coefficients, and $v_{t_i}$ and $v_{n_i}$ are the components of the link velocity in the respective directions. Directional preference ($\mu_n \gg \mu_t$, i.e., tendency of the link to slide much more easily in the tangential direction than in the normal direction) is known to be important for thrust generation, hereby we assume that $\mu_t = 0$. The generalized coordinates can be chosen as follows:

$$x_c = \sum_{i=1}^{n+1} \frac{x_i}{n+1}, \quad y_c = \sum_{i=1}^{n+1} \frac{y_i}{n+1}, \quad \varphi = \sum_{i=1}^{n+1} \frac{\theta_i}{n+1}, \quad \phi_i := \theta_{i+1} - \theta_i, \quad i = 1, \ldots, n,$$

where $(x_c, y_c) \in \mathbb{R} \times \mathbb{R}$ are the coordinate of the mass center of the whole body, $\varphi \in \mathbb{R}$ is the average link angle representing the body orientation with respect to the inertial frame, and $\phi \in \mathbb{R}^n$ is the vector of relative angles between two adjacent links, representing the body shape. We define the forward velocity $v := \dot{x}_c$, side velocity $w := \dot{y}_c$, and their vector $\varpi := [v \ w]^T \in \mathbb{R}^2$.

The equations of motion for the multi-link system have been derived using the Newton’s law [11] and the Euler-Lagrange equation [60, 1]. The fully nonlinear model has been simplified for analytical studies to gain insights into the locomotion mechanisms [60]. In particular, a nominal locomotion condition is considered in which a nearly constant velocity $\varpi \simeq \varpi_o := [v_o \ 0]^T$ with an almost fixed body orientation $\varphi \simeq 0$ is achieved through body undulation with a small amplitude $\phi \simeq 0$, where $v_o \in \mathbb{R}$ is a given constant nominal velocity. Note that we choose the inertial frame so that the $x$-axis is aligned with the direction of locomotion $v$, and further restrict our attention to the case where the body orientation is also nominally (on average) aligned with this direction. The restriction seems reasonable since there is no asymmetry in the mechanical system or the environment. In asymmetric cases, the situation can be different. For instance, if the undulation occurs in a vertical plane (with the gravity along the $y$-axis) as is the case for leech swimming, negative buoyancy of the leech body makes it appropriate to slant the body to have a positive angle of attack on average to counteract the gravity [137]. For our symmetric locomotion
CHAPTER 2. MECHANICAL RECTIFIER LOCOMOTION

system, however, it would be natural to consider body undulation that is symmetric about the direction of locomotion so that \( \phi = 0 \) on average.

Assuming that \( \phi, \varphi, v - v_o, \) and \( w \) are small, the Taylor series expansion of the model equations, followed by truncation of higher order terms, yields

\[
\ddot{\phi} + \mu \dot{\phi} + M(v_o)\phi = u, \tag{2.10}
\]
\[
\dot{\varphi} + \mu \dot{\varphi} + p(v_o) \dot{\phi} = q^T u, \tag{2.11}
\]
\[
\dot{\omega} + \Psi(\phi, \varphi) = \Xi(\phi, \dot{\phi}, \dot{\varphi}), \tag{2.12}
\]

where \( \mu \in \mathbb{R} \) is a constant and \( u(t) \in \mathbb{R}^n \) is the control input uniquely determined by the joint torque vector \( \tau(t) \in \mathbb{R}^n \) through a nonsingular linear transformation, and

\[
\Psi(\phi, \varphi) := \mu \begin{bmatrix} \phi^T P \phi + \varphi^2 & -\varphi \\ -\varphi & 1 \end{bmatrix}, \quad \Xi(\phi, \dot{\phi}, \dot{\varphi}) := \mu \begin{bmatrix} \phi^T Q \dot{\phi} + \dot{\phi}^T c \dot{\varphi} \\ 0 \end{bmatrix}. \tag{2.13}
\]

The definitions of the coefficients and detailed development of the model are given in Appendix A.2. Corresponding to the notation \( J \) and \( D \) in (2.4), \( J = I \) and \( D = \mu I \).

The simplified model (2.10)-(2.12) has linear dynamics for the body oscillation and orientation, and nonlinear dynamics for the locomotion velocity. The function \( \Xi \) is kept to retain the rectification effect for thrust generation. The simple model is obtained at the expense of potential inaccuracy due to the slender body (i.e., small \( \phi \)) assumption. However, the model qualitatively captures the essential dynamics of locomotion and is reasonably accurate for the range of undulations observed in animals [1]. The simplified system defined by (2.10) and (2.11) is controllable whenever \( p \neq M^T q \) (see [77] for a complete discussion of this issue in a general nonlinear setting), and it is not difficult to see that a constant torque input
2.3. ROBOTIC LOCOMOTORS

$u$ gives a constant shape $\phi$ at equilibrium, which in turn leads to a constant rate of change of the orientation $\dot{\phi}$.

In comparison to the general model (2.6), (2.12) retains two degrees of freedom for the mass center of the system instead of one, that is the movement on $y$-axis $w$. The inclusion of the dynamics on $y$-axis can be used to show that the controller to be designed is capable of changing the orientation of the system in the numerical examples. On the other hand, the orientation equation (2.11) is slightly different from general form (2.5), but substitution of $u$ from (2.10) would result in the equation in the form of (2.5) exactly. It is worthwhile to note that $M(v_o)$ and $p^T(v_o)$ are dependent on the nominal velocity $v_o$ instead of $v$ as a result of model simplification.

2.3.3 Flapping-wings Locomotor

Flapping-wings locomotor focuses on robotic systems arising from animal locomotion such as bird flying or ray swimming. A typical flapping-wings locomotor configuration consists of two independent wings and a rigid main body of six degrees of freedom (rotation and translation) to which the wings are attached. The system is expected to locomote primarily in the translation coordinate through the periodic flapping of the wings (or fins), while in constant contact with the surrounding environment. These systems are of particular interest for underwater vehicles due to their expected maneuverability, silence, and efficiency. The nonlinear model and bilinear approximation of a simple flapping-wings rectifier system was originally developed in [2], where a simple point-mass model is assumed for the wing geometry allowing independent wing motion. The schematic diagram is shown in Fig 2.5.

The locomotor consists of a main body (marked red in Fig 2.5) which is capable of free rotation and translation in a three dimension space, and a pair of symmetric wings attaching to the body. The coordinate system is defined so that, at the nominal position, the body lies
on the \((x, y)\) plane with the \(y\)-axis being the direction of motion (swimming or flying) and the \(z\)-axis being vertical. The body can be viewed as rigid and the body-attached Cartesian coordinate frame is denoted by \((x^b, y^b, z^b)\). The wings (marked as light blue area with many separate rectangles) is described by the displacement of \(n = 2n_p\) discrete points, located at the intersections of a body-fixed grid and representing the joints between neighboring segments. Each segment is composed of neighboring joints and approximated as a point mass placed half way between the joints (shown as a dot in Fig 2.5). The \(i\)th point mass in the body frame is given by \((x_i^b, y_i^b, z_i^b)\), and \(x^b, y^b, z^b\) are the vectors stacked by \(x_i^b, y_i^b, z_i^b\), respectively. By small angle approximation, the joints and point masses are constrained to move in the \(z^b\) direction only (i.e. \(x_i^b\) and \(y_i^b\) are constant). To represent the yaw, roll and pitch angles of the body orientation, the \((z-y-x)\) convention is used to define Euler angle vector as \([\alpha_b, \beta_b, \gamma_b]^T \in \mathbb{R}^3\). The vector \(p_b := [p_x, p_y, p_z]^T \in \mathbb{R}^3\) is the Cartesian coordinate of the center of mass of the system in the global frame, and \(\varpi = \dot{p}_b := [v_x, v_y, v_z]^T \in \mathbb{R}^3\) the corresponding velocity. It is assumed that the rigid body is of mass \(m_b\), each wing point mass is \(m_p\), and hence the total system mass is \(m = m_b + nm_p\).

The forces exerted on the wings consist of actuation torques and fluid forces from the environment. Torque is applied at joints driving the point mass upward and downward, and
causes the curvature changes in the local wing surface. At each mass, the plane normal and
tangential to the local wing surface is determined from the coordinates of the surrounding
masses (see Fig 2.6). With the relative movement between wings and fluid, friction or
viscosity is generated along the normal and tangential directions of the local wing surface
when fluid or air passes by, and approximated as linearly proportional to the normal and
tangential velocities, with coefficients $\mu_n$ and $\mu_t$, respectively [2]. The anisotropy property
$\mu_n \gg \mu_t$ (easier to slide in the tangential direction than in the normal direction) is also
assumed for flapping-wings locomotor. The drag friction coefficient for the body is $\mu_b$.

![Figure 2.6: Surface normal and velocity projections of flapping-wings locomotor [2].](image)

In the case that the pair of wings moves symmetrically about the plane formed by $x^b$ and
$z^b$ axis, it is reasonable to assume $\alpha_b = 0$ and $v_x = 0$. Also, it is assumed that the system
gravity and buoyancy are balanced and there is no outstanding motion in the $z$-axis, i.e.,
$v_z = 0$. As a result, the system state is defined as $[\delta^T, v]^T$ with $\delta := [(z^b)^T, \beta_b, \gamma_b]^T \in \mathbb{R}^{n+2}$
and $v := v_y$. The equation of motion can be derived by the same techniques used for
fiptail and snake-like locomotors. Using the Euler-Lagrange method and the Taylor series
expansion of the model equations around $\delta = 0$ and $v = v_o$ for a constant $v_o$, followed by
truncation of higher order terms, gives

\begin{align*}
J\ddot{\phi} + D\dot{\phi} + K(v_o)\phi &= u, \quad (2.14) \\
J_o\ddot{\phi} + D_o\dot{\phi} - [J_s\ddot{\phi} + D_s\dot{\phi} + K_s(v_o)\phi] &= 0, \quad (2.15) \\
m\dot{v} + (a + \Theta^TQ\Theta)v + \Theta^T\Lambda^T\dot{\Theta} &= 0, \quad (2.16)
\end{align*}

where

\[ \Theta = \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} K_c & 0 \\ 0 & I \end{bmatrix} \theta, \]

with \( \phi = K_c\varepsilon^b \), \( \varphi = [\beta_b, \gamma_b]^T \) and \( a = \mu_b \). In particular, \( \phi_i \) is called the curvature of wing shape at the \( i \)th point mass and \( \varphi \) represents the orientation of the system. \( K_c \) is the matrix that maps the position of the mass points into curvature of wing shape. \( u \in \mathbb{R}^n \) is input force, from which the real torque \( \tau \) is uniquely determined. For a rectangular wing shape (i.e., any four neighboring point masses form a rectangular grid), it can be verified, using the derivation in [2], that the matrices in (2.14) have the following properties:

\[ K(v_o) = k_oI + v_o\Lambda_s, \quad D = \mu J, \quad (2.17) \]

for some asymmetric matrix \( \Lambda_s \) and \( \mu = \mu_n/m_p \). The definitions of the coefficients and detailed development of the model are given in Appendix A.3.
2.4 Objectives of Locomotion Control

The equations of motion for flippers, snake-like and flapping-wings locomotor are more or less in the form of the general mechanical rectifier systems (2.4)-(2.6). Through observation, it is obvious that (2.7), (2.10) and (2.14) for shape variable $\phi$ are body shape equation; The dynamics (2.11) and (2.15) for orientation variable $\varphi$ are orientation equation; The dynamics (2.8), (2.12) and (2.16) for locomotion variable $v$ or $\varpi$ are locomotion equation.

Our aim of the locomotion control is to develop a unified framework that gives a solution for an effective feedback controller $u$ that achieves a desired behavior in terms of three actions: body shape oscillation $\phi$, orientation $\varphi$ and locomotion $v$. To this end, the framework of locomotion control should incorporate three basic tasks corresponding to equations (2.4)-(2.6), respectively, namely body shape control, orientation control and locomotion analysis (there is no orientation control for the flippers locomotor.) Regarding the body shape control, the following problem is proposed.

**Entrainment to Oscillation Pattern Problem:** Consider the system (2.4), a reasonable periodic motion pattern of the body shape should be proposed based on certain standard or principle, so that the locomotion could be optimized or resemble the one observed in nature. Then, a state feedback control $u$ needed to be designed in order to achieve the proposed periodic motion. The desired closed-loop behavior should be realized as a stable autonomous motion, rather than a forced response to a fixed and periodic trajectory command. In this way, the controller design would be more robust against disturbances, noises, and unmodeled dynamics.

With the above problem solved though, the dynamical model (2.5) reveals that orientation $\varphi$ is out of direct control of $u$. Specifically, the undulatory body motion in certain oscillation pattern does not automatically guarantee that the body orientation is always as expected.
In a locomotion behavior, we normally expect the body orientation be consistent with its forward velocity. Taking the snake-like locomotor moving in the $x$-$y$ frame for example, the forward direction is along the positive $x$-axis, it is required that the essential component (the average) of $\varphi$ should be regulated at zero. The orientation regulation problem can be stated as follows:

**Orientation Regulation Problem:** Consider the system (2.4) and (2.5), apart from the oscillation entrainment controller proposed in solution to *Entrainment to Oscillation Pattern Problem*, an extra controller is required so that body orientation $\varphi$ is consistent with its locomotion direction.

Note that the solution to the above problems are given based on the system (2.4) and (2.5), where the velocity-related term $K(v_o)$ is given at $v_o$ as result of model simplification (except for fliptail locomotor). Hence, the control design would be valid only if the actual velocity $v$, achieved through the dynamics (2.6) for the mass center, turns out to be equal to or approximated at the nominal velocity $v_o$. To enforce the consistency, it is desired that the oscillation pattern of the body shape and orientation through the control of $u$ can generate a stable forward locomotion at the desired velocity $v_o$ (at least in average). The locomotion analysis problem can be stated as follows:

**Locomotion Analysis Problem:** Consider the system (2.4)-(2.6) and define a velocity $v_o$. The locomotion is stated to be achieved at velocity $v_o$ if the forward velocity $v$ in dynamics (2.6) in the steady state oscillates around $v_o$ with sufficiently small ripple, provided that the oscillation pattern achieved for (2.4) with $v_o$ is achieved and the body orientation in (2.5) is regulated properly.

In summary, the dynamic controller needs to be designed such that the closed-loop system (2.4)-(2.6) achieves the following three properties:
2.4. OBJECTIVES OF LOCOMOTION CONTROL

a. The body shape $\phi$ displays the reasonable oscillation pattern for the locomotion behavior.

b. The body orientation is regulated to match its forward direction, i.e. $\varphi$ is periodic with zero average (the average orientation is regulable).

c. A desired forward velocity $v = v_o$ is generated from (2.6) (for snake-like locomotor while the side velocity $w$ is zero on average).
Chapter 3

Oscillation Pattern and Entrainment Control

The chapter is intended to solve the Entrainment to Oscillation Pattern Problem proposed in the previous chapter. Given the mechanical rectifier system (2.4)-(2.6), two oscillation patterns for the body shape will be introduced: optimal and natural oscillation patterns. Both oscillation patterns can fit into a definition for a general class of oscillations that have certain properties. The optimal oscillation patterns define a set of oscillations that can explicitly optimize the locomotion performance, while the definition of natural oscillation exploits the idea of mechanical resonance. In this way, both oscillation patterns, when applied, are believed to improve the efficiency of the locomotion.

Inspired by biological Central Pattern Generator (CPG) network, a systematic method for designing a feedback controller for (2.4) was introduced, but it lacked the guarantee of the stability of oscillation. Therefore, a new controller that achieves the entrainment to natural oscillation pattern exactly will be proposed with a sound mathematical analysis on the stability analysis. The controller will be extended by a dynamic compensator in next chapter to solve Orientation Regulation Problem. At last, the natural and optimal oscillation patterns for three different robotic locomotors will be illustrated. The comparison will reach the conclusion that natural oscillation would be a sub-optimal oscillation pattern, as it is close to one of optimal oscillation patterns.
CHAPTER 3. OSCILLATION PATTERN AND ENTRAINMENT CONTROL

3.1 Introduction

Oscillatory movement is prevalent in nature and is of importance for animals to achieve complex locomotion behavior. The rhythmic motion pattern of limbs or body segments is called gait (or motion pattern). Most Animals are capable of using different gaits and switch depending on the motion velocity, terrain, and energy efficiency. Understanding the principles underlying the choice of gaits could be helpful for designing efficient robotic locomotors. As to mechanical rectifiers, coordinated oscillatory movements of the multiple bodies are rectified to generate thrust and drag force on the mass center. A well designed oscillation pattern could generate thrust that balances out drag force as well as requires low energy input. It is a fundamental problem to determine an efficient gait for robotic locomotors, and the implementation of the gait in the robots could be helpful for understanding locomotion mechanisms in return.

In some cases, the gaits observed in the animal locomotion can be mathematically formulated and parametrized (as serpenoid curve for snakes in [8]). Among the set of parametrized gaits, one particular gait that optimizes a quantity representing the locomotion cost and/or performance such as input energy can be called optimal gait. By performing gridding of the parameter space, the parametric optimization approach has been taken to search for optimal gaits for robots that mimic human walking [139], snake crawling [11], and anguilliform swimming [6]. [1] formulates the optimal gait problem as a minimization of a quadratic cost function over the set of periodic functions representing different choices of locomotion performance. The optimization problem is formulated for the general mechanical rectifier system (2.2), thus it is also applicable for fliptail, snake-like and flapping-wings locomotors. However, the control design to realize the optimal gait has not been discovered yet.
The resonance phenomena is very common in mechanical system. It is the tendency of a mechanical system to absorb more energy when the frequency of its oscillations matches the system’s natural frequency of vibration than it does at other frequencies. At resonance frequency, a small periodic driving forces can produce oscillations of large amplitude. It can cause violent swaying motions and even catastrophic failure in reality but also can be utilized to build efficient engineering system. It is conjectured that the energy consumption during locomotion is minimized by exploiting the mechanical resonance between the body and the surrounding environment [140, 141]. The idea of exploiting the natural rhythmic patterns of mechanical resonance has led to many efficient robotic locomotors that are robust against and adaptive to environmental changes (see, e.g., [142, 143, 144, 145, 146].) Multi-body system can be generally regarded as a mechanical system with multiple mass-spring-damping subsystems possessing a number of resonance frequencies, but the standard definition of natural oscillation can not apply to mechanical rectifiers which normally have an asymmetric stiffness matrix. [136] firstly adapted the original definition and applied it to mechanical rectifier locomotion. It will be shown that so defined natural oscillation exhibits coordinated body movements similar to those observed in biology (e.g. traveling waves of eel swimming).

A CPG can be modeled as a nonlinear oscillator, and when placed in a feedback loop, provides a basic control architecture to achieve coordinated oscillations of engineered systems [18, 23, 21]. Within the CPG framework, feedback control laws to achieve entrainment to a resonance have been studied for standard mechanical systems [143, 144, 147, 145, 146]. In particular, references [110, 142] analytically showed how simple structure of CPGs can be used to provide nonlinear damping and achieve a stable oscillation near a resonance. Inspired by these results, nonlinear feedback controllers shall be proposed to achieve oscillation pattern by simplifying the CPG structure. In this chapter, we aim at introducing a feedback controller that is inspired by CPG structure.
CHAPTER 3. OSCILLATION PATTERN AND ENTRAINMENT CONTROL

The rest of the chapter is organized as follow. Section 3.2 will present the definition for a general class of sinusoidal oscillation pattern. Definitions for optimal and natural oscillation pattern will be revisited in Section 3.2.2 and Section 3.2.3. Section 3.3 will introduce and propose two controllers inspired by CPGs network that achieve the natural oscillation approximately and exactly, respectively. Then, natural and optimal oscillation patterns for fliptail, snake-like and flapping-wings locomotors will be illustrated and compared in Section 3.4.

3.2 Oscillation Patterns

3.2.1 Definition of General Oscillation Patterns

Normally, the natural environments such as fluid, ground surface are locally isotropic, i.e., the physical properties along any direction on the local contact surface are the same. Therefore the dynamics of the robotic locomotor which is placed in an isotropic environment is mainly dependent on the motion pattern of the body shape despite that the orientation plays a trivial role. It is important to distinguish the difference between the isotropic environment and anisotropic environmental force. Anisotropic environmental force in isotropic environment is caused by the material property at the robot side of the contact surface. For instance, the nanostructure of the snake’s skin was revealed to be responsible for the generation of anisotropic friction force in [148]. In this chapter, the gait of mechanical rectifier (2.4)-(2.6) is defined as the motion pattern of body shape $\phi$ by temporarily ignoring the effect of orientation $\varphi$, and then any gaits can be achieved by the input $u$ due to full actuation for body shape.
3.2. OSCILLATION PATTERNS

Mathematically, the gaits can be described by a collection of periodic vector valued functions of time called oscillation pattern. The focus of the thesis is on the oscillation pattern whose vector functions are sinusoidal signals (oscillation pattern will stand for sinusoidal oscillation pattern for simplicity hereafter). The information of rhythmic movement can been encoded in the amplitudes, frequencies and relative phases of oscillation pattern. The following lemma introduces the mathematical description of oscillation pattern and associates it with a complex vector.

Lemma 3.1 Let \( z \in \mathbb{C}^n \) be \( z := Ze^{j\gamma} \), where \( Z \) is the diagonal matrix with \( |z_i| \) on the \( i \)th diagonal entry, \( \gamma \) the vector with \( \angle z_i \) in the \( i \)th entry and \( e^{j\gamma} \) the vector with \( e^{j\gamma_i} \) stacked in a column. Let \( \vartheta(t) \) be a vector-valued function of \( t \), representing \( T \)-periodic oscillation pattern, where the \( i \)th entry is \( \vartheta_i(t) = Z_i \cos(\omega t + \gamma_i) \). Then the oscillation pattern can be described as

\[
\vartheta(t) = \Re(ze^{j\omega t}),
\]

where \( z \) and \( \gamma \) are the amplitude matrix and phase vector respectively and \( \omega = 2\pi/T \) the frequency. Define

\[
R := [\Re(z) - \Im(z)], \quad \varphi(t) := [\cos \omega t \; \sin \omega t]^T,
\]

then

\[
\vartheta(t) = R\varphi(t).
\]

The oscillation pattern is denoted \((\omega, z)\), where \( \omega \) and \( z \) are referred to as frequency and mode shape of the oscillation.
Proof: It is easy to show that

\[
\Re(ze^{j\omega t}) = \Re\{\Re(z) + i\Im(z)\}(\cos\omega t + i\sin\omega t) \]
\[
= \Re(z)\cos\omega t - \Im(z)\sin\omega t
\]

which indicates (3.3). We have the association by noting the \(i^{th}\) entry of \(\Re(ze^{j\omega t})\) is \(\Re(z_i)\cos\omega t - \Im(z_i)\sin\omega t = Z_i\cos(\omega t + \gamma_i)\), which coincides with \(\vartheta_i(t)\).

The Lemma 3.1 shows the oscillation pattern \(\vartheta(t)\) is associated with the complex vector \(z\) in (3.1). In (3.2), \(\varphi(t)\) is clockwise circular motion in two-dimensional state space and the linear matrix \(R\) maps the planar oscillation into \(n\)-dimensional space. The complex vector \(z\) encodes the phase and amplitude of oscillation in its module and argument. The following example simply illustrates the utility of Lemma 3.1.

Example 3.1 Consider the mass-spring-damper system with two degrees of freedom in Fig3.1. The two masses \(m_1\) and \(m_2\) are connected with each other and fixed to wall by springs and viscous damper. \(k_i\) and \(c_i\) are the stiffness and damping coefficients for each subsystem respectively. Find the complex vector \(z\) that is associated with the undamped natural oscillation of the system.
3.2. OSCILLATION PATTERNS

The equations of motion of the system can be described by

\[ M\ddot{x} + C\dot{x} + Kx = F, \quad (3.4) \]

where \( M, C \) and \( K \) are symmetric matrices related to system mass, damping and stiffness respectively,

\[
M = \begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}, \quad D = \begin{bmatrix}
c_1 + c_2 & -c_2 \\
-c_2 & c_2 + c_3
\end{bmatrix}, \quad K = \begin{bmatrix}
k_1 + k_2 & -k_2 \\
-k_2 & k_2 + k_3
\end{bmatrix},
\]

and \( F = [f_1, f_2]^T \) is the external force applied onto the masses. The natural oscillation defined for undamped and unforced mass-spring-damper system can be represented by the oscillation pattern \((\omega, z)\), where \( \omega \) is natural frequency and \( z \) is the generalized eigenvalue of \( M \) and \( K \). To see this, let us assume the solution is a sinusoidal signal like

\[ x = \Re(z e^{j\omega t}), \quad (3.5) \]

and substitute (3.5) into (3.4) with \( C = 0 \) and \( F = 0 \). Then, the natural oscillation can be obtained by calculating from generalized eigenvalue/eigenvector of \( M \) and \( K \)

\[ [-\omega^2 M + K]z = 0, \]

where \( \omega^2 \) and \( z \) are eigenvalue and eigenvector respectively. As \( M \) and \( K \) are symmetric and positive definite matrices, two generalized eigenvalues are positive real number. Thus, there are two valid \((\omega > 0)\) natural oscillation \((\omega_1, z_1)\) and \((\omega_2, z_2)\) corresponding to the two eigenvalues, with natural frequencies \( \omega_1 \) and \( \omega_2 \), where eigenvalue \( z_1 \) and \( z_2 \) are the corresponding eigenvectors.
Definition 3.1 restricts the oscillation pattern to be sinusoidal but it embodies a class of locomotion behavior, such as snake in serpentine locomotion [11] and harmonic oscillator (as in Example 3.1). Besides, there are two oscillation patterns which are found to be useful for animal locomotion. In [136], the natural oscillation pattern is defined for the mechanical rectifier system with asymmetric stiffness. It is the free response under partial damping compensation and is sinusoidal that can fit in the Definition 3.1. Interestingly, the optimal oscillation pattern defined in [1] are periodic sinusoidal signal rather than harmonics up to a high order and therefore can be represented as the oscillation pattern $(\omega, z)$. In particular $z$ is the generalized eigenvector of two specific matrices that are determined by the system dynamics.

### 3.2.2 Optimal Oscillation Pattern

[1] presented a tractable algorithm of finding optimal oscillation pattern for mechanical rectifier systems. Before the definition and algorithm are revisited, some notations are necessary to be introduced. $\mathbb{P}_T$ denotes the set of $T$-periodic signals. Let $\Pi$ be the set of transfer functions $\Pi(s)$ of the form $\Pi(s) = F(-s)^T \Phi F(s)$, where $\Phi$ is a constant Hermitian matrix and $F(s)$ is a linear combination of stable (proper) transfer functions and differentiators. If an input $\mu_T \in \mathbb{P}_T$ is applied to $F(s)$, the output is given by $y + \tilde{y}$ where $y$ is the steady state response that is $T$-periodic and $\tilde{y}$ is the transient response that eventually dies out. With a slight abuse of notation, we denote the $T$-periodic signal $y^T \Phi y$ by $\mu_T^T \hat{\Pi} \mu_T$.

The optimal oscillation pattern is the locomotion gait that minimizes a quadratic cost function, given that the mechanical rectifier (2.2) locomotes at velocity $v_o$. The problem of
finding the optimal gait for mechanical rectifiers can be formulated as the following optimization over the set of $T$-periodic signals $\mathbb{P}_T$:

$$
\min_{T \in \mathbb{R}^+} \quad f(\theta, \tau) = \frac{1}{T} \int_0^T \begin{bmatrix} \theta \\ \tau \end{bmatrix}^T \Pi \begin{bmatrix} \theta \\ \tau \end{bmatrix} \, dt, \quad \text{subject to}
$$

$$
\begin{align*}
\int_0^T \left( (a + \theta^T Q \theta) v_o + \theta^T \Lambda \dot{\tau} \right) dt &= 0, \\
J \ddot{\theta} + D \dot{\theta} + (K + v_o \Lambda) \theta &= B \tau,
\end{align*}
$$

where $\tau$ and $\theta$ are signals satisfying (2.2) and $\Pi(s) \in \Pi$ is given transfer function. $f(\theta, \tau)$ is the quadratic objective function in $\theta$ and $\tau$, through the choice of $\Pi(s)$ representing many physical quantities such as energy consumption, input torque or shape magnitude. Table 3.1 provides a list of such quantities and their associated weighting functions $\Pi(j\omega)$ (see (2.3) that $W$ is the matrix that relates $\theta$ to the body shape $\phi$, i.e., $\phi := W \theta$). The second equation follows from averaging the second equation of (2.2) under the optimization condition for the locomotion velocity $v(t) \simeq v_o$. The third equation gives constraint on the relation between $\theta$ and $\tau$ in steady state. To see this, Laplace transform of the linear equation in the frequency domain yields $(s^2 J + s D + K + v_o \Lambda) \hat{\theta} = B \hat{\tau}$ where $\hat{\theta}$ and $\hat{\tau}$ are phasors of $\theta$ and $\tau$, respectively. Due to $B \in \mathbb{R}^{p \times n}$ with rank $n \ (n < p)$, $\hat{\theta}$ is uniquely determined by $\hat{\tau}$ from $\hat{\theta} = (s^2 J + s D + K + v_o \Lambda)^{-1} B \hat{\tau}$ and thus is in the vector space of dimension $n$ spanned by columns of matrix $(s^2 J + s D + K + v_o \Lambda)^{-1} B$. 

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Table 3.1: Objective functions specified by $\Pi(j\omega)$.

The optimization problem (3.6) can be converted into a problem formulated in the frequency domain. To this end, define

$$X(\omega) := \frac{1}{2} \left[ \begin{array}{c} P(\omega) \\ I \end{array} \right]^* \Pi(j\omega) \left[ \begin{array}{c} P(\omega) \\ I \end{array} \right],$$

$$Y(\omega) := P(\omega)^* (S(\omega) - \nu_0 Q) P(\omega)/(2\omega),$$

$$S(\omega) := j\omega(\Lambda - \Lambda^T)/2,$$

$$P(\omega) := (K + \nu_0 \Lambda + j\omega D - \omega^2 J)^{-1} B,$$

where $\omega \in \mathbb{R}$. The $T$-periodic signals $\tau$ and $\theta$ can be represented by Fourier series with harmonics of order up to $h$:

$$\tau(t) = \sum_{k=1}^{h} \Re \left[ \hat{\tau}_k e^{j\omega_k t} \right], \quad \hat{\tau} = \text{col}(\hat{\tau}_1, \ldots, \hat{\tau}_h),$$

$$\theta(t) = \sum_{k=1}^{h} \Re \left[ \hat{\theta}_k e^{j\omega_k t} \right], \quad \hat{\theta} = \text{col}(\hat{\theta}_1, \ldots, \hat{\theta}_h),$$
3.2. OSCILLATION PATTERNS

where \( \hat{\theta}_k \) and \( \hat{\tau}_k \) are the \( k \)th phasors of \( \hat{\theta} \) and \( \hat{\tau} \), due to (2.4)

\[
\hat{\theta}_k = P(\omega)\hat{\tau}_k.
\] (3.8)

Then the optimal problem (3.6) is equivalent to the tractable optimization problem:

\[
\min_{\omega \in \mathbb{R}, \hat{\tau} \in \mathbb{C}^{nh}} \{ \hat{\tau}^* X^h(\omega)\hat{\tau} : \hat{\tau}^* Y^h(\omega)\hat{\tau} = 1 \}. \tag{3.9}
\]

where for an integer \( h \) and matrix function \( N(x) \) is defined as

\[
N^h(x) := \text{diag}(N(x), N(2x), \ldots, N(hx)).
\]

The problem is then to find the optimal frequency \( \omega \) and torque \( \tau \) that minimize the quadratic function in (3.9) while subject to a quadratic constraint. The optimal \( \theta \) in the steady state can be later determined from (3.8). [1] presents that (3.9) can be solved by the following theorem through frequency sweeping and finding generalized eigenvalue and eigenvector.

**Theorem 3.1** [1] Define \( X(\omega), Y(\omega), S(\omega) \) and \( P(\omega) \) in (3.7) for frequency \( \omega \). Consider the system given in (2.2) and the optimal locomotion problem in (3.6). Let \( \Upsilon_o \) be the optimal value of the objective function \( f(\theta, \tau) \) in (3.6). Then the optimal problem in (3.9) is equivalent to

\[
\Upsilon_o = \min_{\omega \in \mathbb{R}} \max_{\Upsilon \in \mathbb{R}} \{ \Upsilon : X(\omega) \geq \Upsilon Y(\omega) \}. \tag{3.10}
\]

Let \( \omega_o \) and \( \Upsilon_o \) be the optimizers. Then, the optimal period is \( T_o = 2\pi/\omega_o \), and the optimal torque input \( \tau_o \) and \( \theta_o \) are given by

\[
\tau_o(t) = \Re[\hat{\tau}_o e^{j\omega_o t}], \quad \theta_o = \Re[P(\omega_o)\hat{\tau}_o e^{j\omega_o t}], \tag{3.11}
\]
where \( \hat{\tau}_o \in \mathbb{C}^n \) is the eigenvector of the pair \((X(\omega_o), Y(\omega_o))\) associated with the generalized eigenvalue \( \Upsilon_o \), normalized to satisfy \( \hat{\tau}^* Y(\omega_o) \hat{\tau} = 1 \). The optimal oscillation pattern for the body shape in terms of \( \hat{\tau}_o \) is

\[
\phi(t) := \phi_o(t) = \Re[W P(\omega_o) \hat{\tau}_o e^{j\omega_o t}],
\]

(3.12)

where \( W \) is defined in (2.3) and \( W P(\omega_o) \in \mathbb{C}^{n \times n} \).

**Remark 3.1** (3.12) indicates that the optimal oscillation pattern is sinusoidal, although the optimization problem (3.9) is looking for the \( T \)-periodic oscillation pattern with harmonics of order up to \( h \). Therefore, the optimal oscillation pattern can fit into the Definition 3.1.

**Remark 3.2** Denote \( \Upsilon \) the optimizer in (3.10) for a given frequency \( \omega \). \( \Upsilon \) is the optimal value of the objective function \( f(\theta, \tau) \) in (3.6). In this case, [1] reveals that the optimizer \( \Upsilon \) is one of the generalized eigenvalue of \( X(\omega) \) and \( Y(\omega) \). To find the optimal oscillation pattern, it is necessary to sweep \( \omega \) in the domain of \( \mathbb{R}^+ \) to find the minimum value from a group of \( \Upsilon \)'s that points out the optimal value \( \Upsilon_o \) and the corresponding optimal frequency \( \omega_o \). Then, calculation of the eigenvector of the pair \((X(\omega_o), Y(\omega_o))\) associated with the generalized eigenvalue \( \Upsilon_o \) obtains optimal torque phasor \( \hat{\tau}_o \).

**Remark 3.3** Whether the equations of the motion are like (2.2) or not, once the optimal oscillation problem can be formulated as (3.9), Theorem 3.1 can provide the solution to optimal problem.
According to the above remark, it is able to solve the optimal oscillation pattern for flittail locomotor, whose equations are slightly different from the one in (2.2).

**Remark 3.4** Theorem 3.1 is applicable to solving the optimal problem (3.6) only if \( a \neq 0 \), since \( Y(\omega) \) in (3.7) is not well defined at \( a = 0 \).

**Remark 3.5** Let the optimal oscillation pattern for body shape be \((\omega, z)\) satisfying dynamics (2.2), where \( \omega \) is frequency and \( z \) is the mode shape of body shape movement \( \phi \). The constraint on the optimal torque \( \hat{\tau} Y(\omega) \hat{\tau} = 1 \) then is equivalent to

\[
\left( (WP(\omega))^{-1} z \right)^* Y(\omega) (WP(\omega))^{-1} z = 1, \tag{3.13}
\]

by noting (3.12) and \( \hat{\tau} = (WP(\omega))^{-1} z \).

### 3.2.3 Natural Oscillation Pattern

The natural oscillation of standard lightly damped mechanical systems is a well established concept. In general, a natural oscillation is defined to be a free response of the modified system obtained by removing all the damping effects to achieve marginal stability for sustained oscillation. This definition is valid for the standard case where the stiffness matrix is symmetric positive (semi)definite (see Example 3.1). However, this idea does not directly apply to the body shape dynamics (2.4) because simple removal of damping effect \( D \dot{\phi} \) does not result in marginal stability of \( J \ddot{\phi} + K \phi = 0 \) due to the asymmetry of \( K \) (see Assumption 2.1.a). In this section, we use \( K \) instead of \( K(v_o) \) by dropping \( v_o \) for the reason of simplicity hereafter in this chapter.
Clearly, the systems (2.4) is stable when the velocity is zero, provided $K$ for $v_o = 0$ is symmetric positive definite according to Assumption 2.1.c. By continuity, the system would be stable for any value of $v$ smaller than a threshold, and oscillations, if any, would die out in the steady state. However, if the damping effect $D\dot{\phi}$ is gradually reduced, the oscillations may eventually become unstable. When the eigenvalues with the largest real part go across the imaginary axis, the system becomes marginally stable and the oscillations can be sustained. The natural oscillation can thus be defined as a free response of the modified system obtained by reducing the damping effect by an appropriate amount to achieve marginal stability for sustained oscillations. Let (2.4) be adjusted by the amount of damping with a parameter $\epsilon \in \mathbb{R}$ so that the resulting uncontrolled system

$$J\ddot{\phi} + (\mu - \epsilon)J\dot{\phi} + K\phi = 0,$$

is marginally stable. Then, a precise definition of the natural oscillation for the locomotor system is given as follows.

**Definition 3.1** Consider body dynamics of mechanical rectifiers described by (2.4) with assumptions listed in Assumption 2.1. Let the damping effect be adjusted by a parameter $\epsilon \in \mathbb{R}$ so that the characteristic equation of the resulting system is

$$\left(\lambda^2 J + (\mu - \epsilon)\lambda J + K\right) z = 0, \quad \lambda \in \mathbb{C}, \quad z \in \mathbb{C}^n. \quad (3.14)$$

Suppose, for a specific value $\epsilon = \rho$, the characteristic roots $\lambda$ are all in the open left half plane except for a complex conjugate pair on the imaginary axis. Let the one on the positive imaginary axis be denoted by $\lambda = j\omega$, $\omega > 0$, with associated eigenvector $z \neq 0$. Then the free response $\phi(t) = \phi_n(t)$ where

$$\phi(t) = \phi_n(t) := \Re[ze^{j\omega t}], \text{ i.e., } \phi_i^j(t) = |z_i| \cos(\omega t + \gamma_i), \quad i = 1, \ldots, n, \quad (3.15)$$
3.2. OSCILLATION PATTERNS

is called the natural oscillation pattern \((\omega, z)\) of the original system (2.4), where \(\omega\) and \(z\) are referred to as the natural frequency and mode shape of the natural oscillation and \(\gamma_i = \angle z_i\). The set of points \((\phi_n(t), \dot{\phi}_n(t))\) in the state space for \(t \in \mathbb{R}\) is called the natural oscillation orbit.

**Remark 3.6** When \(J\) and \(K\) are symmetric positive definite, this definition reduces to the standard notion of natural oscillations, where the damping factor is given by \(\rho := \mu\) and the choice \(\epsilon := \rho\) cancels the damping effect exactly (see Example 3.1). In this case, the system is marginally stable with all the eigenvalues on the imaginary axis, defining multiple modes of natural oscillations.

For the class of systems (2.4) we consider, it is not obvious how to characterize the critical value \(\rho\) nor how the system behaves under a choice of \(\epsilon := \rho\). The following result will provide characterization of \(\rho\) and explicitly state the stability properties in the neighborhood \(\epsilon \simeq \rho\). First let \(\mathbb{M}\) be the set of generalized eigenvalue, eigenvector and left eigenvector of \(M = J^{-1}K\):

\[
\mathbb{M} := \{ (\varsigma, z, \ell) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n : \quad (\varsigma I - M)z = 0, \quad \ell^* (\varsigma I - M)z = 0, \quad \ell^* z = 1 \}, \tag{3.16}
\]

where the superscript \(*\) is the conjugate transpose operator.

**Lemma 3.2** Consider body shape dynamics of mechanical rectifiers described by (2.4) with assumptions listed in Assumption 2.1. Denote the minimizer and optimal value of

\[
q := \min_{(\varsigma, z, \ell) \in \mathbb{M}} \mu + \frac{\Im(\varsigma)}{\sqrt{\Re(\varsigma)}}, \tag{3.17}
\]
by $\varsigma$ and $\varrho$, respectively. Let $z$ be the right eigenvector of $M$ associated with eigenvalue $\varsigma$, and define $\omega := \sqrt{\Re(\varsigma)}$. Then, $(\omega, z)$ is the natural oscillation of (2.4), and the adjusted system (3.14) is exponentially stable if $\epsilon < \varrho$, marginally stable if $\epsilon = \varrho$, and exponentially unstable if $\epsilon > \varrho$.

Assumption 2.1.e guarantees that the minimizer of (3.17) is unique, which in turn implies that, when $\epsilon = \varrho$, the system is marginally stable with a single pair of unrepeated eigenvalues on the imaginary axis. For a given mechanical rectifier system, the natural oscillation can be found from the eigenvalue/eigenvector pair $(\varsigma, z)$ of $M$ that solves (3.17). The natural frequency is given by $\omega = \sqrt{\Re(\varsigma)}$, and the mode shape $z$ is uniquely determined up to the magnitude scaling. Since the system (2.4) is linear, the amplitude of the natural oscillation $\|z\|$ can take an arbitrary value, depending on the initial condition.

### 3.3 Entrainment Control to Natural Oscillation

We would like to develop a systematic method for designing a feedback controller for (2.4) to achieve the natural oscillation with a prescribed amplitude. The control objective is precisely defined as follows.

**Definition 3.2 (Entrainment to Natural Oscillation)** Consider the body shape dynamics (2.4). Denote the state vector by $x := (\phi, \dot{\phi}) \in \mathbb{R}^{2n}$. Let the orbit of the natural oscillation $(\omega, z)$ be defined by

$$
\mathcal{O} := \{ \Omega(t) \in \mathbb{R}^{2n} \mid t \in \mathbb{R} \}, \quad \Omega := (\phi_n, \dot{\phi}_n),
$$

(3.18)

where $\phi_n$ is defined in (3.15). A feedback controller is said to achieve entrainment to the natural oscillation $(\omega, z)$ if the following property holds: When the initial condition $x(0)$
is sufficiently close to the orbit $\mathcal{O}$, the trajectory of the closed-loop system $x(t)$ converges to the orbit $\mathcal{O}$, i.e., there exists $t_o$, dependent upon the initial condition, such that

$$\lim_{t \to \infty} ||x(t) - \Omega(t + t_o)|| = 0.$$ 

The simplest approach is to exactly cancel the damping factor by implementing a controller $u = \varrho J \dot{\theta}$ with a properly selected initial condition. Obviously, this approach is not practical because it is critically sensitive to the parameter $\varrho$ and the initial condition. For practical purposes, the controller should be designed to achieve the natural oscillation as a stable limit cycle of the closed-loop system as described in Definition 3.2. Since the plant is linear, the controller is necessarily nonlinear to create a structurally stable limit cycle.

### 3.3.1 CPG Controller

The main objective of the controller design is to apply a sinusoidal input $u$ at the natural frequency $\omega$ with appropriate amplitude and phases to drive the stable system so that the response converges to the natural oscillation with the prescribed amplitude in the steady state. The input $u$ is to be generated by a nonlinear feedback controller of the following form:

$$u = G\psi(q), \quad q = f(s)H\theta,$$

where $G$ and $H$ are $n \times n$ real matrices, $f(s)$ is a scalar transfer function, and $\psi : \mathbb{R} \to \mathbb{R}$ is a static nonlinearity satisfying acting on a vector argument element-wise. The structure
in (3.19) is motivated by biological control mechanisms. In particular, the simplest input-output model of neuronal dynamics is given by $v_{\text{post}} = \psi(f(s)v_{\text{pre}})$ from the presynaptic potential $v_{\text{pre}}$ to the postsynaptic potential $v_{\text{post}}$ where $\psi$ and $f(s)$ represent the threshold nonlinearity and dynamics (time lag, adaptation, etc.) associated with synaptic and cell membrane processes. The controller in (3.19) is a network of multiple neurons with the interconnections specified by $G$ and $H$.

Let us assume the nonlinear function $\psi$ is a static nonlinearity satisfying the following properties:

- $\psi$ is odd, bounded, and strictly increasing.
- $\psi(x)$ is strictly concave on $x > 0$, and $\psi'(0) = 1$.

Then idea based on CPG network inspires the following theorem concerning the controller for the entrainment problem.

**Theorem 3.2** [118] Consider body dynamics of the mechanical rectifier described by (2.4) with assumptions listed in Assumption 2.1. Let $(\omega, z)$ be the natural oscillation with damping factor $\varrho$ define in Lemma 3.2 and $\eta \in \mathbb{R}$ be a positive number. Define

$$r := \frac{\varrho}{\kappa(\eta\omega)\eta},$$

where $\kappa$ is the describing function of $\psi$. Then the controller

$$u = rJZ\psi(q), \quad q = \eta Z^{-1} \dot{\phi},$$

(3.20)

can approximately achieve the entrainment to natural oscillation described in Definition 3.2.
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3.3.2 Exact Controller

The insufficiency of the controller given in Theorem 3.2 is two fold. First, the proof of Theorem 3.2 is based on an approximate analysis method multivariable harmonic balance (MHB), thus the trajectory $\phi(t)$ of the closed-loop system converges to the desired $\phi_n(t)$ only in the approximate sense where the higher order harmonic terms of $\phi(t)$ are ignored. In particular, in the harmonic balance analysis, the nonlinear term is approximated as $\psi(x) \approx \kappa(a) x$ for $x(t) = a \sin(\omega t)$, where $\kappa$ is the describing function of $\psi$. Secondly, numerical simulation can show that the asymptotic trajectory of the closed-loop is a stable limit cycle, but it lacks a rigorous stability analysis. It has been proven that once $\psi(q)$ is quasi-linearized at the desired trajectory $\phi_n(t)$ using MHB, the resulting linear system is marginally stable, which hence induces a natural oscillation $\phi_n(t)$. However, it is important to know whether $\phi_n(t)$ is a stable limit cycle of the original nonlinear system. Once $\phi(t)$ is deviated from $\phi_n(t)$, it should be asymptotically attracted to $\phi_n(t)$, but this property has yet to be well addressed.

We will propose a controller in the form of (3.19) by appropriately choosing the nonlinear function such that the exact entrainment of $\phi(t)$ to $\phi_n(t)$ can be achieved with a rigorous proof for stability.

Theorem 3.3 Consider body dynamics of the mechanical rectifier described by (2.4) with assumptions listed in Assumption 2.1. Let $(\omega, z)$ be the natural oscillation with damping factor $\varrho$ defined in Lemma 3.2. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable, strictly decreasing function such that $\kappa(\omega) = \varrho$. Then, the controller

$$ u = \zeta(\dot{\phi}) := \kappa(||R^t\dot{\phi}||) J \dot{\phi}, $$

$$ R = \begin{bmatrix} \Im(z) & \Re(z) \end{bmatrix} \in \mathbb{R}^{n \times 2}, \quad R^t = 2\begin{bmatrix} \Im(\ell) & \Re(\ell) \end{bmatrix} \in \mathbb{R}^{2 \times n}, $$

(3.21)
achieves the entrainment to the natural oscillation exactly, where $\ell$ is the left eigenvector of $M$ defined in (3.16).

Proof: For the convenience of the proof, we first define some notations. Denote a matrix

$$\Gamma := \begin{bmatrix} j & -j \\ 1 & 1 \end{bmatrix}.$$  

It is easy to show that

$$\begin{bmatrix} z & z^* \end{bmatrix} \Gamma^{-1} = R, \quad \Gamma \begin{bmatrix} l^* \\ l \end{bmatrix} = R^\dagger,$$

and hence

$$R^\dagger R = \Gamma \begin{bmatrix} l^* \\ l \end{bmatrix} \begin{bmatrix} z & z^* \end{bmatrix} \Gamma^{-1} = I.$$

That is, $R^\dagger$ is the Moore-Penrose inverse of $R$. It is easy to find matrices $N \in \mathbb{R}^{n \times (n-2)}$ and $N^\dagger \in \mathbb{R}^{(n-2) \times n}$ such that

$$N^\dagger N = I, \quad R^\dagger N = 0, \quad N^\dagger R = 0.$$

In particular, the columns of $N$ (row of $N^\dagger$) can be the real and imaginary parts of the $(n - 2)$ right (left) eigenvectors of $M$ corresponding to the eigenvalues other than $\varsigma$ or $\varsigma^*$. Some simple calculation gives

$$R^\dagger MR = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \varsigma := a + bj,$$

$$R^\dagger MN = 0, \quad N^\dagger MR = 0.$$
Finally, we define a nonsingular matrix $T$ as follows:

$$
T = \begin{bmatrix}
R^\dagger \\
N^\dagger
\end{bmatrix}, \quad T^{-1} = \begin{bmatrix}
R & N
\end{bmatrix}.
$$

The closed-loop system under consideration is

$$
\ddot{\phi} + (\mu - \epsilon - \kappa(||R^\dagger \dot{\phi}||)) \dot{\phi} + M \phi = 0. \quad (3.22)
$$

Under the new coordinate

$$
\varphi := \begin{bmatrix}
\varphi_1 \\
\varphi_2
\end{bmatrix} := T \phi, \quad \varphi_1 \in \mathbb{R}^2, \quad \varphi_2 \in \mathbb{R}^{n-2}.
$$

The closed-loop system (3.22) becomes

$$
\ddot{\varphi}_1 + (\mu - \epsilon - \kappa(||\dot{\varphi}_1||)) \dot{\varphi}_1 + R^\dagger MR \varphi_1 = 0, \quad (3.23)
$$

$$
\ddot{\varphi}_2 + (\mu - \epsilon - \kappa(||\dot{\varphi}_1||)) \dot{\varphi}_2 + N^\dagger MN \varphi_2 = 0. \quad (3.24)
$$

In what follows, we will investigate these two subsystems (3.23) and (3.24), respectively. For the upper subsystem (3.23), we write $\varphi_1$ in the polar coordinate:

$$
\varphi_1 = \xi \begin{bmatrix}
\cos \alpha \\
\sin \alpha
\end{bmatrix},
$$

with the radius $\xi \in \mathbb{R}$ and the angle $\alpha \in \mathbb{R}$. For the convenience, we define two vectors

$$
v_1 = \begin{bmatrix}
\cos \alpha \\
\sin \alpha
\end{bmatrix}, \quad v_2 = \begin{bmatrix}
-\sin \alpha \\
\cos \alpha
\end{bmatrix}.$$

Now, we have

\[ \dot{\varphi}_1 = \dot{\xi} v_1 + \xi v_2 \dot{\alpha}, \]

and hence, together with (3.23),

\[ \ddot{\varphi}_1 = \ddot{\xi} v_1 + \dot{\xi} v_2 \dot{\alpha} - \xi v_1 \dot{\alpha}^2 + \xi v_2 \dot{\alpha} \]
\[ = -(\mu - \epsilon - \kappa(\|\dot{\varphi}_1\|))(\dot{\xi} v_1 + \xi v_2 \dot{\alpha}) - R^\dagger M R \xi v_1. \]

Multiplying \( v_1^T \) and \( v_2^T \) from left on the above equation gives the following two equations, (noting that \( v_1^T v_2 = v_2^T v_1 = 0 \) and \( v_1^T v_1 = v_2^T v_2 = 1 \)

\[ \ddot{\xi} - \left( \frac{b}{\omega} \right) \dot{\xi} - \delta \dot{\xi} + \xi \left( \omega^2 - \dot{\alpha}^2 \right) = 0, \]
\[ \ddot{\alpha} + 2\left( \frac{\dot{\xi}}{\xi} \right) \dot{\alpha} - \left( \frac{b}{\omega} \right) (\dot{\alpha} - \omega) - \delta \dot{\alpha} = 0. \] 
(3.25)

where \( \delta := \kappa(\|\dot{\varphi}_1\|) - \kappa(\omega) \) and the following facts are used

\[ v_1^T R^\dagger M R v_1 = a = \omega^2, \quad v_2^T R^\dagger M R v_1 = b, \]
\[ \mu - \epsilon - \kappa(\omega) = -\frac{b}{\omega}. \]

We consider the system (3.25) as a three dimensional system with states \((\xi, \dot{\xi}, \dot{\alpha})\). Obviously, the system has an equilibrium point \((\xi, \dot{\xi}, \dot{\alpha}) = (1, 0, \omega)\). Next, we will show this equilibrium point is asymptotically stable. To this end, we define \((\bar{\xi}, \dot{\bar{\xi}}, \dot{\bar{\alpha}}) = (\xi - 1, \dot{\xi}, \dot{\alpha} - \omega)\)
and the linearized system at \((\bar{\xi}, \dot{\bar{\xi}}, \dot{\omega}) = (0, 0, 0)\) is

\[
\begin{align*}
\ddot{\bar{\xi}} - g_1 \dot{\bar{\xi}} - 2\dot{\bar{\alpha}}\dot{\omega} &= 0, \\
\dot{\bar{\alpha}} + 2\omega \dot{\bar{\xi}} - g_1 \dot{\bar{\alpha}} - g_2 (\omega^2 \bar{\xi} + \omega \dot{\alpha}) &= 0,
\end{align*}
\]

(3.26)

where

\[
g_1 = b/\omega < 0, \quad g_2 := \kappa'(\omega) < 0,
\]

and

\[
\begin{align*}
\delta &= \kappa(||\dot{\phi}_1||) - \kappa(\omega) \\ &= (1/\omega)\kappa'(\omega)\dot{\phi}_1(\dot{\phi}_1 - v_2\dot{\omega}) \\ &\approx \kappa'(\omega)(\dot{\xi}\omega + \dot{\alpha}).
\end{align*}
\]

The characteristic equation for (3.26) is

\[
a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0,
\]

with \(a_3 = 1, \ a_2 = -(2g_1 + g_2\omega) > 0, \ a_1 = g_1^2 + g_1 g_2 \omega + 4\omega^2\) and \(a_0 = -2g_2\omega^3 > 0\). We can easily check that \(b_1 = (a_2a_1 - a_3a_0)/a_2 > 0, \ b_2 = 0, \) and \(c_1 = (b_1a_0 - a_2b_2)/b_1 > 0\). It is ready to use Routh table to conclude the stability of (3.26). In other words, we have \(\lim_{t \to \infty} \xi(t) = 1\) and \(\lim_{t \to \infty} \dot{\alpha}(t) = 0\) exponentially. The latter implies

\[
\lim_{t \to \infty} \int_0^t \dot{\alpha}(\tau)d\tau = \lim_{t \to \infty} \alpha(t) - \omega t - \alpha(0),
\]
exists, or, there exists a constant

$$\alpha_o = \alpha(0) + \lim_{t \to \infty} \int_0^t \dot{\alpha}(\tau) d\tau,$$

such that

$$\lim_{t \to \infty} \alpha(t) - \omega t - \alpha(0) = 0.$$

As a result, we have

$$\lim_{t \to \infty} \wp_1(t) - \left[ \begin{array}{c} \cos(\omega t + \alpha_o) \\ \sin(\omega t + \alpha_o) \end{array} \right] = 0.$$

With $\xi = 1$ and $\alpha(t) = \omega t + \alpha_o$, the lower subsystem (3.24) becomes

$$\ddot{\varphi}_2 - \left( \frac{b}{\omega} \right) \dot{\varphi}_2 + N^\dagger M N \varphi_2 = 0,$$

which has a characteristic equation of

$$(\lambda^2 - \left( \frac{b}{\omega} \right) \lambda + N^\dagger M N) \omega = 0.$$ \hspace{1cm} (3.27)

Because $N$ is spanned by the $n-2$ eigenvectors of $M$, for any eigenvalue $\beta \neq \varsigma$ or $\varsigma^*$, there exists a vector $w$ such that $M(Nw) = \beta(Nw)$, and hence, $N^\dagger M N w = \beta N^\dagger N w = \beta w$. Accordingly, $\lambda$ to the equation (3.27) is determined by

$$\lambda^2 - \left( \Im(\varsigma) / \sqrt{\Re(\varsigma)} \right) \lambda + \beta = 0.$$

For Lemma (3.2), the polynomial is Hurwitz since $\varsigma$ is given by (3.17). As a result, we have

$$\lim_{t \to \infty} \varphi_2(t) = 0.$$
Let
\[ \phi(t) = R \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} = Z \sin(\omega t + \gamma), \quad t_o = \alpha_o/\omega. \]

It is ready to show that
\[ \lim_{t \to \infty} \phi(t) - \phi_n(t + t_o) = \lim_{t \to \infty} T^{-1} \varphi(t) - \vartheta(t + t_o) = \]
\[ \lim_{t \to \infty} R \varphi_1(t) + N \varphi_2(t) - R \begin{bmatrix} \cos(\omega t + \alpha_o) \\ \sin(\omega t + \alpha_o) \end{bmatrix} = 0. \]

The proof is thus complete.

Remark 3.7 In comparison to (3.19), the controller (3.21) follows the basic structure of the CPG network, although the nonlinear function \( \psi \) is replaced by \( \kappa(||R^\dagger \dot{\phi}||) J \dot{\phi} \) and \( G \) by identity matrix.

The key idea behind the control law \( u = \zeta(\dot{\phi}) \) is to provide a nonlinear damping with the following properties: (a) when the system trajectory is on the orbit \( O \), the control provides the amount of damping exactly required for the natural oscillation, and (b) when the oscillation frequency increases/decreases, the control provides more/less damping so that the trajectory tends to return to the orbit \( O \). To see these mechanisms in (3.21), note that the natural oscillation \((\omega, z)\) satisfies
\[ \phi_n(t) = R \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}, \quad \omega = ||R^\dagger \dot{\phi}_n||. \quad (3.28) \]

Hence, on the orbit \( O \), we have \( u = \kappa(\omega) \dot{\phi} = \varrho \dot{\phi}, \) leading to the closed-loop system described by (3.14) with \( \epsilon = \varrho \), achieving the natural oscillation as a solution. If the frequency
If $\phi$ is increased/decreased, $||R^\dagger \dot{\phi}||$ is greater/smaller than $\omega$ and therefore $u = \epsilon \dot{\phi}$ with $\epsilon$ smaller/greater than $\varrho$, providing more/less damping to reduce/increase the frequency.

The rigorous proof can be summarized as follow. The trajectory of the natural oscillation can be expressed as (3.28), therefore $\phi_n(t)$ is in the space spanned by the rows of $R$, whose dimension is two. Using the property of vector space of the reduced dimension, a proper coordinate transform is introduced so that the closed-loop system with proposed exact entrainment controller (3.21) becomes two subsystems with variables $\varphi_1 \in \mathbb{R}^2$ and $\varphi_2 \in \mathbb{R}^{n-2}$, respectively. The natural oscillation $\phi_n(t)$ within $\varphi_1$ subsystem presents as a planar oscillator acting as anti-clockwise circular motion along unit circle. As a consequence, the property of a planar oscillator facilitates the analysis of the attractiveness of the natural oscillation. Then it is relatively easy to prove that the trajectory of $\varphi_1$ subsystem is closely attracted by circular trajectory representing the natural oscillation, while the trajectories of $\varphi_2$ subsystem converge to zero. Equivalently in the original coordinate, the trajectory of the closed-loop system with proposed controller (3.21) is proved to be attracted by natural oscillation orbit $\mathbb{O}$ and thus it is a locally stable limit cycle.

### 3.4 Oscillation Patterns of Three Locomotors

In this section, the natural oscillation and optimal oscillation patterns of three robotic locomotors will be illustrated by a sequence of snapshots at different phase of a period or by graphical representation of amplitude and phase variation along the system. The optimal oscillation patterns we consider here are the oscillation patterns with minimum input energy, torque and shape derivative whose objective functions are listed in Table 3.1. The definition of optimal oscillation patterns inherently indicates their optimality, while it is still obscure if the natural oscillation pattern is optimal. To see this, we would like to calculate the performance index $f(\theta, \tau)$ in (3.6) for the natural oscillation and then compared
3.4. OSCILLATION PATTERNS OF THREE LOCOMOTORS

with that of optimal oscillation patterns. If two corresponding quantities are very close, the natural oscillation pattern is approximate to the optimal oscillation patterns in the sense of system performance and could be said at least sub-optimal. Note that the optimal oscillation pattern is the one that the amplitude must satisfy the constraints in (3.9). When calculating the performance index $f(\theta, \tau)$ for the natural oscillation pattern, the same constraint should apply and the constraint equation for the natural oscillation pattern $\phi = \phi_n$ can follow from (3.13) in Remark 3.5. Thus, for natural oscillation pattern $(\omega, z)$, the amplitude $\|z\|$ which although could take an arbitrary value by the definition, shall be certain value after considering the constraint. The finding in [1] speculated that the optimal oscillation pattern with minimum shape derivative may be closely related to movement that is observed in nature. Surprisingly, it will be shown that the natural oscillation pattern is tightly similar to such optimal oscillation pattern rather than oscillation with minimum input energy and torque.

The calculation of natural oscillation can follows from Definition 3.1 and Lemma 3.2, while optimal oscillation pattern can be obtained following the instruction in Remark 3.2 for three different objective functions listed in Table (3.1). Thus, for a given system, we will show a single natural oscillation and three different optimal oscillation patterns. The calculation of oscillation patterns are based on a set of parameters. The default parameters used in the examples can refer to the Appendix A, unless they are specified. In order to make $a \neq 0$ in (2.2), the coefficients of friction or fluid force along tangential direction $\mu_t$ should not be zero. Otherwise, it violates the requirement in Remark 3.4 for calculation of optimal oscillation patterns. For flapping-wings locomotor, $\mu_t$ or the drag friction coefficient for the body $\mu_b$ should not be zero at the same time.
3.4.1 Fliptail Locomotor

The fliptail locomotor in the example consists of $p = 5$ links. For the natural oscillation pattern, the profiles are calculated for several cases of locomotion velocity $v_o$ and joint stiffness $k_o$ (linearly parametrized by $\kappa$, see Appendix A.1), and the result is summarized in Table 3.2. For each case, the phase of the $i$th joint angle $\gamma_i$ lags behind its anterior neighbor $\gamma_{i-1}$, indicating traveling waves that propagate from head to tail. We see that the amount of phase lag (or the number of waves expressed by the fliptail) and the period of oscillation depend on $v_o$ and $k_o$. It is observed that, for a faster $v_o$ or a softer $k_o$, a body snapshot during locomotion exhibits more traveling waves. When $v_o = 0.15 \text{m/s}$, the snapshots with $\kappa = 0.5$ and $\kappa = 5$ are shown in Fig 3.2, where the green dot represents the mass center of each link. The snapshots only display one of the chains and the head is fixed at $x = 0$. We see that more traveling waves are exhibited if the tail is softer in Fig 3.2.

Next, we will show the comparison between natural oscillation and optimal oscillation patterns. As optimal oscillations are defined at certain locomotion velocity $v_o$, we take $v_o = 0.3 \text{ m/s}$ as the nominal velocity. Let $\kappa = 5$, and the friction coefficient be $\mu_t = 0.05$ and $\mu_n = 0.5 \text{ Ns/m}$. To obtain the natural oscillation comparable to the optimal oscillations, the amplitude of the natural oscillation should satisfy equation (3.13). Fig 3.3 shows performance indices of the oscillation patterns. The three curves give the minimum value of three different objective functions by sweeping frequency $\omega$, i.e., it is one of the generalized eigenvalue of the pair in $(X(\omega), Y(\omega))$ in (3.7), which are marked by circle, star and pentagram right on the curves. These markers on the curves show the numerical solution to optimization problem (3.7), indicating the best performance indices for different objective functions. For the natural oscillation pattern, the corresponding performance indices are shown by three corresponding markers located at the natural frequency $\omega = 4.9 \text{ rad/s}$. The red line representing the objective function of input energy cuts off at frequency
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| $v_o$ (m/s) | $\kappa$ | Period (s) | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ | $|z_1|$ | $|z_2|$ | $|z_3|$ | $|z_4|$ | $|z_5|$ |
|-------------|-----------|-------------|-------------|-------------|-------------|-------------|-------------|----------|----------|----------|----------|----------|
| 0.3         | 5         | 1.97        | 184°        | 122°        | 76°         | 20°         | 0°          | 2.4°     | 8.9°     | 16.1°    | 29.2°    | 45.7°    |
| 0.2         | 5         | 2.49        | 139°        | 105°        | 56°         | 13°         | 0°          | 3.9°     | 10.1°    | 15.8°    | 31.0°    | 44.2°    |
| 0.2         | 2         | 2.38        | 234°        | 154°        | 98°         | 35°         | 0°          | 2.7°     | 7.3°     | 17.6°    | 27.7°    | 46.3°    |

Table 3.2: Natural oscillation profiles (fliptail locomotor).

Figure 3.2: Natural oscillation snapshots with different stiffness. Above: $\kappa = 0.5$; Below: $\kappa = 5$ (fliptail locomotor).
CHAPTER 3. OSCILLATION PATTERN AND ENTRAINMENT CONTROL

Comparison between optimal and natural oscillation pattern

Figure 3.3: Objective function versus frequency $\omega$: power in W, torque in $(N \cdot m)^2$, shape derivative in $(\text{deg/ms})^2$ (fliptail locomotor).

Natural Oscillation

Figure 3.4: Natural oscillation for snapshots comparison (fliptail locomotor).
3.4. OSCILLATION PATTERNS OF THREE LOCOMOTORS

Figure 3.5: Optimal oscillations for snapshots comparison (fliptail locomotor).
\[ \omega_{\text{cut-off}} = 16.2 \text{ rad/s}, \] since after which the solution to optimization problem (3.7) does not exist.

It can be noted from Fig 3.3 that the optimal oscillation pattern in terms of minimizing shape derivative and input energy in one cycle are relatively close to the natural oscillation pattern in terms of frequency and performance indices. The snapshots of the natural oscillation and three different optimal oscillation patterns are shown in Fig 3.4 and Fig 3.5. Every snapshot contains four images of body shapes at every \( \frac{1}{4} \)-period time step. It is seen that the number of traveling waves for the natural oscillation pattern is between that of optimal oscillation patterns with minimum shape derivative and input energy. It seems that the natural oscillation is more likely to be similar to the optimal oscillation with minimum shape derivative.

### 3.4.2 Snake-like Locomotor

The snake-like locomotor in the example also consists of \( p = 5 \) links, and the dimension of the body shape variable is \( n = 4 \). Table 3.3 shows the profiles for the cases that \( \kappa = 5 \) and \( \kappa = 0.5 \) at velocity \( v = 0.15 \text{ m/s} \). Fig 3.6 shows the natural oscillation snapshots corresponding to different stiffnesses. The same conclusion can be drawn that more traveling waves are exhibited if the body is softer. Also, the oscillation amplitude of the tail tends to be larger when the body is harder. The snapshots of snake-like locomotor are slightly different from fliptail locomotor, as the head is not fixed. One snapshot consists of nine images showing body shapes at every \( \frac{1}{8} \)-period time step, where blue dots represent the joints. For oscillation comparison, we take \( v_o = 0.1 \text{ m/s} \) as the nominal velocity. Let \( \kappa = 5 \), and coefficients be \( \mu_t = 0.03 \) and \( \mu_n = 1.05 \text{ Ns/m} \). Resembling Fig 3.3, Fig 3.7 illustrates the minimum value of each objective function as a function of frequency \( \omega \) for optimal oscillation patterns as well as performance indices of natural oscillation pattern.
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with markers. Snapshots of natural and optimal oscillation patterns are shown in Fig 3.8. From Fig 3.7 and Fig 3.8, it is rather clear to see that the optimal gaits with minimum shape derivative is the one closest to the natural oscillation. Both for fliptail and snake-like locomotor, the natural frequency closely approximates that of the optimal oscillation with minimum shape derivative.

3.4.3 Flapping-wings Locomotor

As in [2], it was not possible to obtain the globally optimal oscillation patterns, as the optimization problem (3.9) is not convex due to increasing system complexity. But we can still find local optimal oscillation pattern by solving optimization problem (3.9) for a given frequency $\omega$. Here we choose $v_o = 0.15$ m/s and $k_o = 1000$ N/m$^2$, and $\mu_t = 0$, $\mu_b = 0.072$ Ns/m. At first, we will calculate the natural oscillation and frequency, and then we will obtain three optimal oscillation patterns at natural frequency. From Lemma 3.2, the natural frequency is calculated $\omega = 3.26$ rad/s, and the mode shape including amplitudes and phases is illustrated in Fig 3.9. Fig 3.9 plots the shape of both wings in $x$-$y$ body frame, and the movement of point masses can be illustrated by the colored wings showing the tendency of the amplitude variation and phase lag along $x$ and $y$ axises. The figure shows that a phase lag (or wave) propagating from head to tail along each wing and the amplitude increase from each wing’s body side to tip both in an almost straight fashion. In comparison, Fig 3.10 shows the optimal oscillation pattern in terms of minimum shape derivative is very similar to the natural oscillation. It can be observed that the increased tendency of the amplitudes along the $x$-axis is similar with a bit larger maximum amplitude. From phase plot, the natural oscillation pattern seemingly has more traveling wave along $y$-axis. Fig 3.11 and Fig 3.12 illustrate the optimal oscillations in terms of input torque and input energy, respectively. The shape of the amplitudes are similar to the previous cases. Interestingly, the phase plots show that the motion of the wings are now anti-symmetric.
CHAPTER 3. OSCILLATION PATTERN AND ENTRAINMENT CONTROL

The result stresses that the natural oscillation closely matches with optimal oscillation in terms of minimum shape derivative.
3.4. OSCILLATION PATTERNS OF THREE LOCOMOTORS

<table>
<thead>
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<th>( v ) (m/s)</th>
<th>( \kappa )</th>
<th>Period (s)</th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
<th>( \gamma_3 )</th>
<th>( \gamma_4 )</th>
<th>( \mathbf{z}_1 )</th>
<th>( \mathbf{z}_2 )</th>
<th>( \mathbf{z}_3 )</th>
<th>( \mathbf{z}_4 )</th>
</tr>
</thead>
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<tr>
<td>0.15</td>
<td>5</td>
<td>1.22</td>
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<td>115°</td>
<td>38°</td>
<td>0°</td>
<td>3.3°</td>
<td>16.2°</td>
<td>31.0°</td>
<td>45.2°</td>
</tr>
<tr>
<td>0.15</td>
<td>0.5</td>
<td>2.74</td>
<td>293°</td>
<td>196°</td>
<td>94°</td>
<td>0°</td>
<td>18.9°</td>
<td>31.6°</td>
<td>35.1°</td>
<td>26.4°</td>
</tr>
</tbody>
</table>

Table 3.3: Natural oscillation profiles (snake-like locomotor).

Figure 3.6: Natural oscillation snapshots with different stiffness. Left: \( \kappa = 5 \); Right: \( \kappa = 0.5 \) (snake-like locomotor).
Comparison between optimal and natural oscillation pattern

Figure 3.7: Objective function versus frequency $\omega$: power in W, torque in $(N \cdot m)^2$, shape derivative in $(deg/ms)^2$ (snake-like locomotor).
3.4. OSCILLATION PATTERNS OF THREE LOCOMOTORS

Figure 3.8: Snapshots comparison between natural oscillation and optimal oscillation patterns (snake-like locomotor).
Fig. 3.9: Amplitude and phase distributions in natural oscillation (flapping-wings locomotor).
Figure 3.10: Amplitude and phase distributions in optimal oscillation in terms of shape derivative (flapping-wings locomotor).
Figure 3.11: Amplitude and phase distributions in optimal oscillation in terms of input torque (flapping-wings locomotor).
Figure 3.12: Amplitude and phase distributions in optimal oscillation in terms of input energy (flapping-wings locomotor).
Chapter 4

Orientation Control

The chapter is intended for the orientation control of mechanical rectifiers. In the previous chapter, the problem of orientation control was temporarily neglected leaving the body shape fully controlled by enough number of torque inputs achieving the natural oscillation pattern. Due to lack of additional control freedom to handle the orientation, the body orientation is not always as expected while the entrainment controller is applied. The chapter thus will propose an orientation controller on top of the entrainment controller, which makes the body orientation to be aligned with locomotion direction though at the price of the accuracy of the oscillation pattern. However the perturbation from the desired oscillation pattern can be tuned arbitrarily small by setting control parameters.

Two versions of orientation control are devoted to snake-like and flapping-wings locomotors, respectively. Orientation control is not applicable to the fliptail locomotor, as it is fully actuated and does not have an orientation equation. The difference between snake-like and flapping-wings locomotors in terms of orientation control is the dimension of the orientation to be manipulated, that is one for snake-like locomotor and two for flapping-wings locomotor. However the control strategy follows from the same idea to use bias of the body oscillation tuning the orientation.
4.1 Introduction

Many research have studied the generation of the oscillation pattern for locomotion, however few are concerning on orientation control of the robots in the closed loop, especially when slipping sideways is inevitable in the locomotion. As to mechanical rectifiers, it will be revealed that the transfer function of the orientation equation has a rigid body mode, which results in a constant offset on the orientation. Therefore, it is necessary to propose some methods to compensate the offset and make the orientation manipulable.

In the snake-like robots, we note that the parameter $\gamma_r$ in serpenoid curve in (1.1) determines the direction of the serpentine locomotion. Based on this, [149] proposed symmetrical line modulation method and amplitude modulation method by changing $\gamma_r$ in different ways for turning the heading. Besides, [149] proposed phase modulation method by adding phase lag in the sinusoidal wave for side motion. [149] implemented the non-model-based controller to fulfill the trajectory tracking for the joints movement with the ability of manipulating heading. CPG-based control (open-loop) approach can also provide a way to control the orientation [35, 150]. Delivering unequal tonic input to two sides of the CPG can directly instigate the turning motion [151, 150], while [35] adopted the similar idea to change the trajectory setpoints and indirectly induces turning motion. These orientation control methods are not applicable to our case, as they are based on kinematics model or using open-loop trajectory tracking approach.

Moreover, deeply investigating the dynamics model reveals that the locomotion system is under-actuated which exaggerates the difficulty of the control problem. The stabilization of under-actuated system has been tackled by methods such as interconnection and damping assignment (IDA) [152], energy-based control [153] and [154]. These methods provide a control framework to achieve the stabilization to the equilibrium point of under-actuated systems. On the contrary, we expect that the under-actuated locomotion system
4.1. INTRODUCTION

can settle down at a stable limit cycle with oscillatory body movement and orientation. Recent work by Shiriaev et al. [114, 115] has developed one of such design theories. They considered stabilization of a periodic orbit for multiple degree-of-freedom, under-actuated Euler-Lagrange systems. The virtual constraint approach was proposed to reduce the problem to that of a planar (two dimensional) system with an integral of motion. A sufficient condition was given for the existence of periodic solutions, but the problem of setting a desired oscillation profile has not been addressed.

The under-actuated locomotion system (2.4) and (2.5) can be thought of as a fully-actuated subsystem subject to a dynamic constraint imposed by the non-actuated subsystem. In our case, the fully-actuated subsystem of body shape has been excited to oscillate by entrainment controller in previous chapter. In comparison to the existing methods, the feedback controller design in the chapter is intended to control the orientation of the locomotor while still achieving given oscillation pattern. The idea adopted here is to render the existing periodic stable motion as well as to add extra control effort. In this chapter, for an arbitrary signal $x(t)$, the $T$-average value is denoted by $\bar{x}(t)$:

$$\bar{x} = \frac{1}{T} \int_{t}^{t+T} x(\tau) d\tau.$$  

If $x(t)$ is $T$-periodic, then $\bar{x}(t)$ is a constant. The chapter is then organized as follows, Section 4.2 will consider the general form of the mechanical rectifier systems and present the general idea of how to control the orientation. Section 4.3 and Section 4.4 extend the idea to construct the feedback controllers for orientation control of snake-like and flapping-wings locomotors. Section 4.5 summarizes the chapter by a short discussion of proposed theorems.
4.2 Orientation Control for General Locomotor

Consider the general form of mechanical rectifiers (2.4)-(2.5). The first control objective proposed in Chapter 2 has been tackled in previous chapter, while the second control objective is to keep the orientation aligned with the direction of locomotion on average: \( \ddot{\varphi}(t) = 0 \). Let us first examine how this objective is addressed by the controller \( u = \zeta(\dot{\varphi}) \) in Theorem 3.3. Recall that this controller asymptotically achieves the entrainment to the natural oscillation pattern \( \phi(t) \to \phi_n(t) \). In the steady state of body oscillation, the orientation equation (2.5) becomes

\[
J_o \ddot{\varphi} + D_o \dot{\varphi} = \psi,
\]

\[
\psi = \left[ J_s \ddot{\phi}_n + D_s \dot{\phi}_n + K_s \phi_n \right].
\]

The equation has the general steady state solution \( \varphi(t) = \chi(t) + c_o \) with any arbitrary constant vector \( c_o \in \mathbb{R}^{p-n} \) and

\[
\chi(t) := \Re \left( \Gamma z e^{j\omega t} \right),
\]

\[
\Gamma := (-\omega^2 J_o + j\omega D_o)^{-1}(-\omega^2 J_s + j\omega D_s + K_s) \in \mathbb{C}^{(p-n)\times n},
\]

where \((\omega, z)\) are the natural oscillation. We call \( \chi(t) \) natural orientation in the sense that it is induced by the natural oscillation. Clearly, the solution has the average \( \bar{\varphi}(t) = c_o \) due to \( \bar{\chi}(t) = 0 \) for \( T := 2\pi/\omega \). Since the transfer function from \( \psi \) to \( \varphi \), i.e., \((-s^2 J_o + sD_o)^{-1}\), has a rigid body mode (the pole at \( s = 0 \)), the average value \( c_o \) depends on the initial condition.

Thus, the controller \( u = \zeta(\dot{\varphi}) \) does not automatically lead to the desired orientation angle \( \bar{\varphi}(t) = 0 \) in general. However, \( \bar{\varphi}(t) = 0 \) is indeed achieved for one of the solutions with the controller, which makes it a good starting point for the design.
In what follows, we will modify the control law \( u = \zeta(\dot{\phi}) \) so that the average value of the orientation angle is guaranteed to vanish asymptotically; \( \bar{\phi}(t) \to 0 \). It turns out that this requirement is hard to meet while achieving the natural oscillation exactly, and we propose to allow a small periodic perturbation of \( \phi \) from the natural oscillation \( \phi_n \), thereby enabling the convergence of the orientation angle to zero on average. To this end, the following technical lemma will be found useful.

**Lemma 4.1** Consider a stable linear system \( \dot{x} = Ax + Bu \) where \( x \in \mathbb{R}^n \) is the state and \( u \in \mathbb{R}^m \) the input. Let \( T \in \mathbb{R} \) be given. Then for any \( u \) and \( x \) satisfying the system equation, we have

\[
\lim_{t \to \infty} \bar{u}(t) = 0 \implies \lim_{t \to \infty} \bar{x}(t) = 0.
\]

**Proof:** This is actually a simple linearity property of a stable linear system. First, we note

\[
T\dot{x}(t) = \frac{d}{dt} \int_t^{t+T} x(\tau)d\tau = \int_t^{t+T} \frac{dx(\tau)}{d\tau} d\tau.
\]

Now, integrating both sides of the system gives

\[
\int_t^{t+T} \frac{dx(\tau)}{d\tau} d\tau = A \int_t^{t+T} x(\tau)d\tau + B \int_t^{t+T} u(\tau)d\tau,
\]

which is \( \dot{x}(t) = A\bar{x}(t) + B\bar{u}(t) \). Because the system is stable, the condition \( \lim_{t \to \infty} \bar{u}(t) = 0 \) implies \( \lim_{t \to \infty} \bar{x}(t) = 0 \).

This lemma indicates that the average value of the response converges to zero whenever the input average does so, provided the system is linear and stable.
The idea of the new controller is based on the fact that the direction of locomotion can be steered by a bias in the body curvature (or joint angles) [11]. Take the snake-like locomotor (2.10)-(2.12) for instance, constant joint angles $\phi_i = b$ for all joints $i$ give a “C-shaped” body. If the body undulates around this nominal shape (i.e., periodic $\phi(t)$ with $\bar{\phi_i} = b$), then the locomotor would move along a circular path. The turning radius is negatively correlated with $|b|$, and the direction of turn (left/right) is determined by the sign of $b$. Thus, an addition of a controlled bias to the joint angle allows for regulation of the body orientation.

Motivated by this idea, we choose the control input $u$ so that

$$\phi = \phi_n + \beta \nu,$$

(4.2)

in the steady state, where $\beta \in \mathbb{R}^{n \times (p-n)}$ in a constant vector and $\nu(t) \in \mathbb{R}^{p-n}$ is an auxiliary bias signal to be used as the control input to stabilize the orientation dynamics. Let us examine how the auxiliary input $\nu(t)$ can be used to stabilize the average orientation $\bar{\phi}$ around the origin. First we define the “error” signal $\xi$ by

$$\xi := J_o^{-1}J_s(\phi - \phi_n) - \varphi.$$

Then, (2.5) becomes

$$J_o \ddot{\xi} + D_o \dot{\xi} + G \dot{\nu} + K_s \beta \nu + h = 0,$$

$$G = D_s - D_o J_o^{-1} J_s, \quad h = J_s \ddot{\phi}_n + D_s \dot{\phi}_n + K_s \phi_n.$$

If we set $\nu = \kappa(s)\xi$ with linear dynamics $\kappa(s) \in \mathbb{C}$ such that the transfer function from $h$ to $\xi$ is stable, i.e.,

$$s^2 J_o + s D_o + s \kappa(s) G \beta - \kappa(s) K_s \beta = 0 \Rightarrow \Re(s) < 0,$$

(4.3)
4.2. ORIENTATION CONTROL FOR GENERAL LOCOMOTOR

then by Lemma 4.1 we have $\bar{\xi}(t) \to 0$ since $\bar{h}(t)$ due to $\bar{\phi}_n(t) = 0$. If in addition $\kappa(s)$ is stable, then we also have $\bar{\nu}(t) \to 0$, which in turn implies $\bar{\phi}(t) = \bar{\phi}_n(t) + \beta \bar{\nu}(t) \to 0$, and hence $\bar{\phi}(t) = J_o^{-1} J_s(\bar{\phi}(t) - \bar{\phi}_n(t)) - \xi(t) \to 0$. Thus, regulation of the orientation can be achieved by choosing the feedback control $u$ so that $\phi = \phi_n + \beta \kappa(s) \xi$ with a stable $\kappa(s)$ satisfying (4.3).

It now remains to show how to choose $u$ such that (4.2) holds in the steady state, i.e., the signal $\phi - \beta \nu$ converges to the oscillation pattern $\phi_n$. By Theorem 3.3, this is the case for (2.4) if

$$J(\dot{\phi} - \beta \ddot{\nu}) + D(\dot{\phi} - \beta \ddot{\nu}) + K(\phi - \beta \nu) = \zeta(\dot{\phi} - \beta \ddot{\nu}).$$  \hspace{1cm} (4.4)

This equation coincides with the closed-loop system if the control input $u$ is given by

$$u = \zeta(\dot{\phi} - \beta \ddot{\nu}) + (J \beta \ddot{\nu} + D \beta \ddot{\nu} + K \beta \nu).$$  \hspace{1cm} (4.5)

Finally, choosing a stable $\kappa(s)$ satisfying (4.3) and setting $\nu = \kappa(s) \xi$, or equivalently

$$\nu = (\kappa(s) J_o^{-1} J_s \beta - I)^{-1} \kappa(s) \varphi,$$

the controller (4.5) achieves desired orientation $\bar{\phi} \to 0$ through the natural oscillation pattern of $\phi$ with error $\beta \nu$. The simplest choice $\kappa(s) = 1$ gives $\nu$ directly proportional to $\varphi$, requiring that $\bar{\phi}$ be available for feedback. To avoid the difficulty of implementation, we use a second order transfer function for $\kappa(s)$ to arrive at the following result.

Theorem 4.1 Consider the system in (2.4) and (2.5). Let $(\omega, z)$ be the natural oscillation with damping factor $\varrho$ as described in (3.17). Define the period $T := 2\pi/\omega$, and $T$-periodic
signal $\phi_n(t)$ and $\chi(t)$ by (3.15) and (4.1), respectively. Define the nonlinear function $\zeta(\dot{\phi})$ as in Theorem 3.3. Consider the following controller

$$u = \zeta(\dot{\phi} - \beta \dot{\nu}) + (J \beta \ddot{\nu} + D \beta \dot{\nu} + K \beta \nu),$$

(4.6)

where $\nu$ is generated from the following dynamics:

$$J_s \ddot{\nu} + D_s \dot{\nu} + K_s \nu + \kappa_1 \dot{\phi} = 0,$$

$$\dot{\phi} + \kappa_2 \dot{\phi} = \varphi.$$  

(4.7)

The parameter $\kappa_1, \kappa_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}^{n \times (p-n)}$ are given such that

$$J_o \ddot{\phi} + D_o \dot{\phi} + \kappa_1 \dot{\phi} = 0,$$

$$\kappa_2 \dot{\phi} + \dot{\phi} - \varphi = 0.$$  

(4.8)

and the first equation of (4.7) are stable dynamics. Then, the closed system composed of (2.4), (2.5), (4.6) and (4.7) has the following property that $\lim_{t \to \infty} \bar{\varphi}(t) = 0$ and $\lim_{t \to \infty} \bar{\dot{\varphi}}(t) = 0$.

**Proof:** The system (2.4) with the controller (2.5) can be put in the form of (4.4). Denote

$$\delta_1(t) := \phi(t) - \left[ \phi_n(t + t_o) + \beta \nu(t) \right], \quad \delta_2(t) := \dot{\phi}(t) - \left[ \dot{\phi}_n(t + t_o) + \beta \dot{\nu}(t) \right],$$

which satisfy $\lim_{t \to \infty} \delta_i(t) = 0, \ i = 1, 2$, by Theorem 3.3. With the new controller (4.6), We note that the system (2.5) is in the following form:

$$J_o \ddot{\phi} + D_o \dot{\phi} = (J_s \ddot{\phi}_n + D_s \dot{\phi}_n + K_s \phi_n) + (J_s \beta \ddot{\nu} + D_s \beta \dot{\nu} + K_s \beta \nu) + \delta(t)$$

$$= ||z||s(t + t_o) - \kappa_1 \dot{\phi} + \delta(t)$$
4.3  SNAKE-LIKE LOCOMOTOR

where \( s(t) = \Re \left[ \left( -\omega^2 J_s + j\omega D_s + K_s \right) ze^{j\omega t} \right], \ z = z/||z|| \) and

\[
\delta(t) = \left[ J_s \left[ J^{-1}(\zeta(\theta_n + \delta_2) - \zeta(\dot{\theta}_n) - D\delta_2 - K\delta_1) \right] + D_s\delta_2(t) + K_s\delta_1(t) \right].
\]

Now, the \( \varphi \)-dynamics and the \( \dot{\varphi} \)-dynamics become

\[
J_o\ddot{\varphi} + D_o\dot{\varphi} + \nu_1\dot{\varphi} = ||z||s(t + t_o) + \delta(t)
\]

\[
\nu_2\dot{\varphi} + \ddot{\varphi} - \varphi = 0.
\]  
(4.9)

In particular, we note that \( s(t) \) is a periodic signal with the period \( T = 2\pi/\omega \) and \( \bar{s}(t + t_o) = 0 \). Also, it is easy to see \( \lim_{t\to\infty} \delta(t) = 0 \). So the total input to the system (4.9) satisfies \( \lim_{t\to\infty} ||z||s(t + t_o) + \bar{\delta}(t) = 0 \). If the linear system on the left hand side of (4.9) is stable, i.e., (4.8) is stable dynamics. Now, by Lemma (4.1), we have \( \lim_{t\to\infty} \bar{\varphi}(t) = 0 \) and \( \lim_{t\to\infty} \bar{\dot{\varphi}}(t) = 0 \). Then the stable dynamics of the first equation of (4.7) leads to a bounded \( \nu \) signal. Thus the proof is complete.

The controller in Theorem 4.1 is more applicable in practice, as the dynamics of (4.7) only requires \( \varphi \) be available. Theorem 4.1 also shows the proposed controller is able to regulate the average of orientation to be zero, but the body oscillation \( \dot{\varphi} \) is achieved around natural oscillation with perturbation \( \beta\nu \). In the following two sections, the capability of the control idea will be further explored to show that the magnitude of the perturbation can be tuned very small. Two variations of Theorem 4.1 will be dedicated to snake-like and flapping-wings locomotors.

4.3  Snake-like Locomotor

The equations of motion for snake-like locomotor (2.10)-(2.11) essentially take the form of the general system (2.4)-(2.5) by direct calculation, although \( u \) appears in the orientation
equation. The controller idea in the previous section can apply to the snake-like locomotor. But we are motivated to clarify the development of the feedback controller from very beginning, as some of the notation to be used is slightly different. For snake-like locomotor, the controller \( u = \zeta(\dot{\phi}) \) in Theorem 3.3 is utilized to asymptotically achieve the entrainment to the natural oscillation: \( \phi(t) \to \phi_n(t) \). In the steady state, the orientation (2.11) becomes

\[
\ddot{\varphi} + \mu \dot{\varphi} = \psi,
\]
\[
\psi := q^T \zeta(\dot{\phi}_n) - p^T \phi_n = \Re((j\omega q^T - p^T)ze^{j\omega t}).
\]

where \( \varphi \) is defined in (3.17) and \((\omega, z)\) is the natural oscillation pattern. The equation has the general steady state solution \( \varphi(t) = \chi(t) + c_o \) with an arbitrary constant \( c_o \in \mathbb{R} \) and

\[
\chi(t) := \Re(\Gamma ze^{j\omega t}), \quad \Gamma^T := \frac{j\omega q^T - p}{j\omega \mu - \omega^2} \in \mathbb{C}^n. \tag{4.10}
\]

The notation of \( \chi(t) \) in (4.10) is similar to that in (4.1) with different \( \Gamma \). Following the general idea of orientation regulation, we choose the control input \( u \) so that \( \phi = \phi_n + \beta \nu \) in the steady state, where \( \beta \in \mathbb{R}^n \) and \( \nu(t) \in \mathbb{R} \). We define the “error” signal \( \xi \) as

\[
\xi := (\phi - \phi_n) - \varphi.
\]

Then, eliminating \( u \) from (2.10) and (2.11) and substituting (4.2), we have

\[
\ddot{\xi} + \mu \dot{\xi} + b^T \beta \nu + h = 0,
\]
\[
b = M^T q - p, \quad h = q^T (\ddot{\phi}_n + \mu \dot{\phi}_n) + b^T \phi_n.
\]
4.3. **SNAKE-LIKE LOCOMOTOR**

If we can also set \( \nu = \kappa(s)\xi \) with linear dynamics such that the transfer function from \( h \) to \( \xi \) is stable, i.e.,

\[
s^2 + \mu s + b^T \beta \kappa(s) = 0 \Rightarrow \Re(s) < 0, \tag{4.11}
\]

then similarly we have \( \bar{h}(t), \bar{\xi}(t), \bar{\phi}(t), \bar{\varphi}(t) \to 0 \). Given the proposed controller in Theorem 3.3 and equation (4.2), we have

\[
(\ddot{\phi} - \beta \dot{\nu}) + \mu(\dot{\phi} - \beta \dot{\nu}) + M(\phi - \beta \nu) = \zeta(\dot{\phi} - \beta \dot{\nu}). \tag{4.12}
\]

This equation coincides with the closed-loop system if the control input \( u \) is given by

\[
u = \zeta(\dot{\phi} - \beta \dot{\nu}) + M \beta \nu + \beta(\dot{\nu} + \mu \dot{\nu}). \tag{4.13}
\]

The design of the \( \nu \)-dynamics and the feedback controller are presented in the following theorem.

**Theorem 4.2** Consider the snake-like locomotor given by (2.10) and (2.11) and define \( b := M^T q - p \in \mathbb{R}^n \). Let \((\omega, z)\) be the natural oscillation pattern as described in Lemma (3.2). Define the period \( T := 2\pi/\omega \), amplitude \( \alpha_z := ||z|| \), and \( T \)-periodic signals \( \phi_n(t) \) and \( \chi(t) \) by (3.15) and (4.10), respectively. Define the nonlinear function \( \zeta(\dot{\phi}) \) as in Theorem 3.3 and let the parameters \( \nu_1, \nu_2 \in \mathbb{R} \) and \( \beta \in \mathbb{R}^n \) be given such that

\[
(\nu_2^{-1} + \mu)\mu > \nu_1 > 0, \quad \nu_2 > 0, \tag{4.14}
\]

\[
(b^T \beta)/(q^T \beta) > 0.
\]

Consider the following controller

\[
u = \zeta(\dot{\phi} - \beta \dot{\nu}) + M \beta \nu + \beta(\dot{\nu} + \mu \dot{\nu}), \tag{4.15}
\]
where \( \nu \) generated from \( \varphi \) through the following dynamics:

\[
(q^T \beta)(\ddot{\nu} + \mu \dot{\nu}) + (b^T \beta)\nu + \kappa_1 \dot{\varphi} = 0,
\]

\[
\dot{\varphi} + \kappa_2 \ddot{\varphi} = \varphi.
\]

\( (4.16) \)

Then the average value of the orientation tends to zero, i.e., \( \lim_{t \to \infty} \bar{\varphi}(t) = 0 \). The closed system composed of (2.10), (2.11), (4.15) and (4.16) has the following properties.

a. The body shape specified by the joint angles \( \phi \) satisfies

\[
\lim_{t \to \infty} ||\phi(t) - [\phi_n(t + t_o) + \alpha_z \beta \nu_1(t)]|| = 0,
\]

for some \( t_o \in \mathbb{R} \) and \( T \)-periodic signal \( \nu_1(t) \) with \( \bar{\nu}_1(t) = 0 \).

b. The body orientation angle \( \varphi \) satisfies

\[
\lim_{t \to \infty} ||\varphi(t) - [\chi(t + t_o) + \alpha_z \nu_2(t)]|| = 0,
\]

for some \( t_o \in \mathbb{R} \) and \( T \)-periodic signal \( \nu_2(t) \) with \( \bar{\nu}_2(t) = 0 \).

c. The perturbation signals \( \nu_i \) for \( i = 1, 2 \) are independent of the amplitude \( \alpha_z \) and satisfy

\[
\max_{t \in [0,T]} |\nu_i(t)| \to 0 \text{ as } \kappa_2 \to \infty.
\]

\( (4.19) \)

Proof: The system (2.10) with the controller (4.15) can be put in the form of (4.12). Denote

\[
\delta_1(t) := \phi(t) - [\phi_n(t + t_o) + \beta \nu(t)],
\]

\[
\delta_2(t) := \dot{\phi}(t) - \left[ \dot{\phi}_n(t + t_o) + \beta \dot{\nu}(t) \right],
\]
which satisfy $\lim_{t \to \infty} \delta_i(t) = 0, \ i = 1, 2$, by Theorem 3.3. To prove (4.17), it suffices to show

$$\lim_{t \to \infty} \nu(t) - \alpha_z t_1(t) = 0. \quad (4.20)$$

To this end, we note that the system (2.11) is in the following form

$$\ddot{\varphi} + \mu \dot{\varphi}$$

$$= -p^T \varphi + q^T u$$

$$= -p^T [\delta_1 + \phi_n(t + t_o) + \beta \nu(t)] + q^T [\delta_2 + \phi_n(t + t_o)] + M \beta \nu + \beta (\dot{\nu} + \mu \dot{\nu})$$

$$= \alpha_z s(t + t_o) + \delta(t) + (q^T M - p^T) \beta \nu + q^T \beta (\dot{\nu} + \mu \dot{\nu})$$

$$= \alpha_z s(t + t_o) + \delta(t) - \kappa \hat{\varphi},$$

where $s(t) := q^T \zeta (\dot{\phi}_n(t)) - p^T \phi_n(t)$ or $s(t) = \Re [(q^T g j \omega - p^T) z e^{j \omega t}]$ with $z := z/|z|$, and $\delta(t) := -p^T \delta_1(t) + q^T [\delta_2(t) + \phi_n(t + t_o)] - \zeta (\dot{\phi}_n(t + t_o))$. Now, the $\varphi$-dynamics and the $\hat{\varphi}$-dynamics become

$$\ddot{\varphi} + \mu \dot{\varphi} + \kappa \hat{\varphi} = \alpha_z s(t + t_o) + \delta(t)$$

$$\kappa \ddot{\hat{\varphi}} + \ddot{\varphi} - \varphi = 0. \quad (4.21)$$

In particular, we note that $s(t)$ is a $T$-periodic signal and $\bar{s}(t + t_o) = 0$. Also, it is easy to see $\lim_{t \to \infty} \delta(t) = 0$. So the total input to the system (4.21) satisfies

$$\lim_{t \to \infty} [\alpha_z s(t + t_o) + \bar{\delta}(t)] = 0.$$ 

The characteristic equation of the system (4.21) is

$$\lambda(\lambda + \mu)(\lambda + \kappa \varphi^{-1}) + \kappa_1 \kappa_2^{-1} = 0,$$
which is stable under the condition (4.14) using Routh-Hurwitz test. Now, by Lemma 4.1, we have \( \lim_{t \to \infty} \tilde{\varphi}(t) = 0 \) and \( \lim_{t \to \infty} \bar{\varphi}(t) = 0 \). More specifically, it can be explicitly solved that

\[
\lim_{t \to \infty} \left\{ \varphi(t) - \Re \left( \frac{[q^T \varrho j\omega - p^T] z e^{j\omega(t+t_0)}}{-\omega^2 + \mu j\omega + \kappa_1/(1 + \kappa_2 j\omega)} \right) \right\} = 0.
\]

Therefore, we have the equation (4.18) satisfied for \( \iota_2(t) \).

Next, consider the system composed of the first equation of (4.16) and (4.21). For \( \alpha_z = 1 \), using Lemma (4.1) again gives

\[
\lim_{t \to \infty} \nu(t) - \iota_1(t) = 0,
\]

for a \( T \)-periodic signal \( \iota_1(t) \) independent of \( \alpha_z \) and satisfying \( \bar{\iota}_1(t) = 0 \). For a general \( \alpha_z \), we have (4.20) due to the linear property, and hence (4.17) satisfied. Now, the property (a) and (b) have been proved.

For \( \alpha_z = 1 \) and \( \varphi(t) = \chi(t+t_0) + \iota_2(t) \), we may define the asymptotic solution for \( \hat{\varphi}(t) \) is \( \hat{\chi}(t) \) which is \( T \)-periodic. From the second equation of (4.21), we have

\[
\kappa_2 \dot{\hat{\chi}} + \hat{\chi} = \Re \left( \frac{[q^T \varrho j\omega - p^T] z e^{j\omega(t+t_0)}}{-\omega^2 + \mu j\omega + \kappa_1/(1 + \kappa_2 j\omega)} \right).
\]

It is easy to see \( \max_{t \in [0,T]} |\hat{\chi}(t)| \to 0 \) as \( \kappa_2 \to \infty \) (noting the low pass filter property.) From the definition of \( \iota_1 \), we see that \( \nu(t) = \iota_1(t) \) and \( \hat{\varphi}(t) = \hat{\chi}(t) \) are a solution to the first equation of (4.16). Now, it is ready to conclude that \( \max_{t \in [0,T]} |\iota_1(t)| \to 0 \) as \( \kappa_2 \to 0 \). The property for \( i = 2 \) is straightforward from the definition of \( \iota_2 \). Now the proof for the property (c) is complete.

\[ \blacksquare \]
Remark 4.1 The previous controller (3.21) achieves the natural oscillation exactly. On the other hand, the controller (4.15) is a modified version of (3.21), and achieves the natural oscillation only approximately, with the perturbation term $\iota_1$ as stated in property (a). The benefit gained by this relaxation is the regulation of the body orientation so that the average orientation angle converges to zero in the steady state, as indicated in property (b). It can be shown that these properties are also achieved by a simple controller obtained by removing the low pass filter $1/(1 + \kappa_2 s)$ in the second equation of (4.16) and setting $\dot{\phi} = \varphi$, provided the first two conditions in (4.14) are replaced by $\kappa_1 > 0$. However, the advantage of the additional low pass filter is obvious from property (c) of Theorem 4.2, that is, the periodic perturbation terms $\iota_i(t)$ can be made arbitrarily small by choosing a sufficiently large time constant $\kappa_2$. As a result, we have $\phi(t) \simeq \phi_o(t + t_o)$ and $\varphi(t) \simeq \chi(t + t_o)$ for some to $t_o \in \mathbb{R}$ in the steady state. Thus, the natural oscillation for the body shape is essentially achieved with the body orientation regulated to be aligned, on average, with the direction of locomotion.

Remark 4.2 When $\kappa_2$ is large, the high frequency components of $\varphi$ are suppressed by the low pass filter, which in turn implies that the signals $\nu(t)$ and $\dot{\nu}(t)$ change slowly, i.e., $\dot{\nu} \approx 0$ and $\ddot{\nu} \approx 0$. In this case, we have, approximately,

$$
\nu = -\left(\kappa_1/(b^T \beta)\right)\dot{\phi},
$$

from (4.16). As a result, the controller described by (4.15) and (4.16) reduces to the following equations

$$
u = \zeta(\dot{\phi}) - h\dot{\phi}, \quad h := \kappa_1 M \beta/(b^T \beta),$$

$$\dot{\phi} + \kappa_2 \ddot{\phi} = \varphi.$$
For this simplified controller, the first term $\zeta(\dot{\phi})$ is exactly the one derived in the last for achieving the natural oscillation, while the second term $-h\dot{\phi}$ is the additional component for orientation regulation. The effectiveness of the controller (4.22) will be examined Chapter 6 through numerical simulations.

### 4.4 Flapping-wings Locomotor

The equations of motion for flapping-wings locomotor (2.14)-(2.15) essentially take the exact form of the general system (2.4)-(2.5), therefore the analysis in section 4.2 can be repeated for the flapping-wings locomotor and all notation can be used. Now, let us proceed to the theorem regarding the feedback controller directly.

**Theorem 4.3** Consider the system in (2.14) and (2.15). Let $(\omega, z)$ be the natural oscillation with damping factor $\varrho$ as described in (3.17). Define the period $T := 2\pi/\omega$, amplitude $\alpha_z := ||z||$, and $T$-periodic signal $\phi_n(t)$ and $\chi(t)$ by (3.15) and (4.1), respectively. Define the nonlinear function $\zeta(\dot{\phi})$ as in Theorem 3.3, let the parameter $\kappa_1, \kappa_2 \in \mathbb{R}$ and $\beta \in \mathbb{R}^{n \times 2}$ be given such that

\[
\begin{align*}
H_1 &= \begin{bmatrix} 0 & I \\ -(J_s\beta)^{-1}K_s\ell & -(J_s\beta)^{-1}D_s\beta \end{bmatrix} < 0, \\
H_2 &= \begin{bmatrix} 0 & I & 0 \\ 0 & -J_o^{-1}D_o & -J_o^{-1}\kappa_1 \\ 1/\kappa_2 I & 0 & -1/\kappa_2 I \end{bmatrix} < 0, \quad \kappa_2 > 0,
\end{align*}
\]

(4.23)

where $I \in \mathbb{R}^{2 \times 2}$ is identity matrix. Consider the following controller

\[
u = \zeta(\dot{\phi} - \dot{\beta}\dot{\nu}) + (J\beta\ddot{\nu} + D\beta\dot{\nu} + K\beta\nu),
\]

(4.24)
where \( \nu \) is generated from the following dynamics:

\[
J_s \beta \ddot{\nu} + D_s \beta \dot{\nu} + K_s \beta \nu + \tau_1 \dot{\phi} = 0,
\]

(4.25)

\[
\dot{\phi} + \tau_2 \ddot{\phi} = \phi.
\]

Then the average value of the orientation tends to zero, i.e., \( \lim_{t \to \infty} \phi(t) = 0 \). The closed-loop system composed of (2.14), (2.15), (4.24) and (4.25) has the following properties.

a. The body shape specified by the joint angles \( \phi \) satisfies:

\[
\lim_{t \to \infty} ||\phi(t) - [\phi_n(t + t_o) + \alpha_2 \beta \nu_1(t)]|| = 0,
\]

(4.26)

for some \( t_o \in \mathbb{R} \) and \( T \)-periodic signal \( \nu_1(t) \) with \( \nu_1(t) = 0 \).

b. The body orientation angle \( \varphi \) satisfies

\[
\lim_{t \to \infty} ||\varphi(t) - [\chi(t + t_o) + \alpha_2 \nu_2(t)]|| = 0,
\]

(4.27)

for some \( t_o \in \mathbb{R} \) and \( T \)-periodic signal \( \nu_2(t) \) with \( \nu_2(t) = 0 \).

c. The perturbation signals \( \nu_i \)'s for \( i = 1, 2 \) are independent of the amplitude \( \alpha_2 \) and satisfy

\[
\max_{t \in [0,T]} |\nu_i(t)| \to 0 \text{ as } \alpha_2 \to \infty.
\]

(4.28)

Proof: The system (2.14) with the controller (4.24) can be put in the form of (4.4). Denote

\[
\delta_1(t) := \phi(t) - [\phi_n(t + t_o) + \beta \nu(t)], \quad \delta_2(t) := \dot{\phi}(t) - [\dot{\phi}_n(t + t_o) + \beta \dot{\nu}(t)],
\]
which satisfy \( \lim_{t \to \infty} \delta_i(t) = 0, \ i = 1, 2 \), by Theorem 3.3. To prove (4.26), it suffices to show

\[
\lim_{t \to \infty} \nu(t) - \alpha_z t_1(t) = 0.
\]  

Subtracting (4.4) from (2.14) with \( \phi = \phi_n \) and Substituting \( \delta_1 \) and \( \delta_2 \) gives

\[
\delta_4(t) = \ddot{\phi}(t) - \ddot{\phi}_n(t + t_o) = \delta_3 + \beta \dot{\nu}(t),
\]

where \( \delta_3 = J^{-1}[\zeta(\dot{\phi}_n + \delta_2) - \zeta(\dot{\phi}) - D\delta_2 - K\delta_1] \) with \( \lim_{t \to \infty} \delta_3(t) = 0 \). We note that the system (2.15) is in the following form:

\[
J_o \dddot{\phi} + D_o \ddot{\phi} = (J_s \ddot{\phi}_n + D_s \phi_n + K_s \phi_n) + (J_s \beta \ddot{\nu} + D_s \beta \dot{\nu} + K_s \beta \nu) \\
+ (J_s \delta_3 + D_s \delta_2 + K_s \delta_1) \\
= \alpha_z s(t + t_o) - \kappa_1 \dot{\phi} + \delta(t),
\]

where \( s(t) = (-\omega^2 J_s + j \omega D_s + K_s) \Re(ze^{j\omega t}), \ z = z/||z_0|| \) and \( \delta(t) = (J_s \delta_3(t) + D_s \delta_2(t) + K_s \delta_1(t)) \). Now, the \( \phi \)-dynamics and the \( \dot{\phi} \)-dynamics become

\[
J_o \dddot{\phi} + D_o \ddot{\phi} + \kappa_1 \dot{\phi} = \alpha_z s(t + t_o) + \delta(t) \\
\kappa_2 \dot{\phi} + \dot{\phi} - \phi = 0.
\]  

In particular, we note that \( s(t) \) is a periodic signal with the period \( T = 2\pi/\omega \) and \( \bar{s}(t + t_o) = 0 \). Also, it is easy to see \( \lim_{t \to \infty} \delta(t) = 0 \). So the total input to the system (4.30) satisfies

\[
\lim_{t \to \infty} [\alpha_z \bar{s}(t + t_o) + \bar{\delta}(t)] = 0.
\]

The system (4.30) is stable, as the state matrix \( H_2 \) in (4.23) is guaranteed to be Hurwitz the second condition of (4.23). The stability of the first equation
of (4.25) is guaranteed by the first condition of (4.23). Now, by Lemma 4.1, we have
\( \lim_{t \to \infty} \bar{\varphi}(t) = 0 \) and \( \lim_{t \to \infty} \bar{\varphi}(t) = 0 \). More specifically, it can be noted that

\[
\lim_{t \to \infty} \varphi(t) - \Re(\Gamma_1 s(t + t_o)\alpha_z) = 0,
\]

where

\[
\Gamma_1 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} (j\omega I - H_2)^{-1} \begin{bmatrix} 0 \\ J_o^{-1} \\ 0 \end{bmatrix} = (j\omega + \frac{1}{\kappa_2}) \left[ \frac{\kappa_1}{\kappa_2} I + (j\omega + \frac{1}{\kappa_2})(-\omega^2 J_o + j\omega D_o) \right]^{-1},
\]

and \( \Gamma_1 \in \mathbb{C}^{2 \times 2} \). Therefore, we have (4.27) satisfied for

\[
\iota_2 = \Re(\Gamma_2 z e^{j\omega t + t_o}),
\]

where \( \Gamma_2 = [\Gamma_1 (-\omega^2 J_o + j\omega D_o + K_o) - \Gamma] \alpha_z \in \mathbb{C}^{2 \times 2} \), \( \Gamma \) is defined in (4.1) and \( \Gamma_2 \to 0 \) as \( \kappa_2 \to \infty \). Next, consider the first equation of (4.25) and the system (4.30). Assume \( \alpha_z = 1 \), using Lemma 4.1 again gives

\[
\lim_{t \to \infty} \nu(t) - \iota_1(t) = 0.
\]

for periodic \( \iota_1(t) \) independent of \( \alpha_z \) and satisfying \( \bar{\iota}_1(t) = 0 \). For a general \( \alpha_z \), we have (4.29) due to the linear property, and hence (4.26). Now, the properties (a) and (c) have been proved.
For $\alpha = 1$ and $\varphi(t) = \chi(t + t_0) + \iota_2(t)$, define the asymptotic solution for $\dot{\varphi}$ is $\dot{\chi}(t)$ which is $T$-periodic. From the second equation of (4.30), we have

$$\varkappa_2 \dot{\chi} + \dot{\chi} = \Re \left( (j\omega I - H_2)^{-1} \begin{bmatrix} s(t) \\ 0 \end{bmatrix} \right).$$

It is easy to see $\max_{t \in [0,T]} |\dot{\chi}(t)| \to 0$ as $\varkappa_2 \to \infty$ (noting the low pass filter property). From the definition of $\iota_1$, we see that $\nu(t) = \iota_1(t)$, and $\dot{\phi}(t) = \dot{\chi}(t)$ are a solution to the first equation of (4.25). Now, it is ready to conclude $\max_{t \in [0,T]} |\iota_1(t)| \to 0$ as $\varkappa_2 \to \infty$. The property for $i = 2$ is straightforward from the definition of $\iota_2$. Now, the proof for the property (c) is complete.

**Remark 4.3** When $\varkappa_2$ is large, the high frequency components of $\varphi$ are suppressed after low pass filter, so are those of $\nu$ and $\dot{\nu}$. In other words, we assume the signal $\nu$ and $\dot{\nu}$ change slowly, i.e., $\dot{\nu} \approx 0$ and $\ddot{\nu} \approx 0$. Therefore, we have, approximately,

$$K_s\beta \nu = -\varkappa_1 \dot{\varphi}$$

from (4.25) and (4.24)-(4.25) reduce to

$$u = \zeta(\dot{\varphi}) - \varkappa_1 K\beta(K_s\beta)^{-1} \dot{\varphi},$$

$$\varkappa_2 \dot{\varphi} = -\dot{\varphi} + \varphi. \quad (4.31)$$

In this controller, the first term $\zeta(\dot{\varphi})$ is exactly the one designed in last section, and the second term $-\varkappa_1 K\beta(K_s\beta)^{-1}$ is the additional component for orientation regulation.
4.5 Summary

In conclusion, Theorem 4.2 and Theorem 4.3 take the idea introduced in Section 4.2 solving the orientation regulation problem for snake-like and flapping-wings locomotors, respectively. The theorems introduces the notation \( \iota_i \) for \( i = 1, 2 \), that are related to the deviation (perturbation) of body oscillation and orientation from the natural oscillation and orientation, respectively. \( \iota_i \) could be tuned by varying the controller parameter \( \gamma_2 \), both property (c)’s in the theorems show that they approach zero when \( \gamma_2 \) tends to infinity. Also, the perturbation is linearly proportional to the amplitude of the natural oscillation. The second equations of the controllers (4.16) and (4.25) both act as low-pass filters. The difference lies on the dimensions of the signal \( \nu \), they are determined by the dimension of the orientation that needs to be controlled.
Chapter 5

Locomotion Analysis

The chapter considers locomotion equation of mechanical rectifiers in order to analyze the
locomotion behavior. Chapter 3 introduced the natural oscillation pattern and proposed a
feedback controller to achieve it. Chapter 4 exposed the design method to add extra con-
troller making the orientation manipulable. Both considered the simplified system with
locomotion velocity approximated at a constant $v_o$. Whether locomotion velocity at mass
center can reach $v_o$ is evaluated through the analysis of the locomotion equation, which de-
scribes the dynamics for the mass center. The chapter will enforce the consistency to prove
locomotion velocity $v_o$ can be achieved by the natural oscillation pattern and orientation.

Observation on the locomotion equation reveals that the forces exerted on the mass center
are be categorized into thrust and drag force both resulting from the body oscillation and
orientation. The balance of two forces in steady state will establish the relation between
locomotion velocity, body oscillation and orientation, which will be used to define the
natural velocity. Techniques such as Fourier series will be applied to demonstrate that the
natural velocity can be achieved approximately. In particular, three theorems are devoted to
the analysis of locomotion equation for fliptail, snake-like and flapping-wings locomotors,
respectively. The results will be illustrated in next chapter with further details.
5.1 Introduction

In the previous chapters, the natural oscillation is defined and achieved by feedback controllers for the oscillation equation (2.4) at a constant velocity $v = v_o$. This constant velocity in oscillation and orientation equations is based on the assumption of the model simplification. It should be noted that due to the mechanism of mechanical rectification, the velocity of the mass center is actually generated through the persistent body undulation. Thus it is important to analyze the dynamics (2.6) where the locomotion mechanism is embedded in order to enforce the consistency that the velocity generated by the body movement is equal to $v_o$.

Although the coordinate transformation (2.3) is introduced, the second equation of the system (2.2) is equivalent to (2.6). It is also the locomotion equation and described by variable $\theta$. The equation shows that the periodic movement of $\theta$ leads to the thrust $-\theta^T \Lambda \dot{\theta}$ and drag $(a + \theta^T Q \theta)v$, where we note that $a \geq 0$ and $Q > 0$. The difference between these two gives the acceleration term $m \ddot{v}$ which is the force propelling system forward and generating velocity for mass center. A close look at the thrust term reveals that the essence of rectification is captured by the skew-symmetric part of the linear coefficient matrix in $\Lambda$ [76, 1]. To explain this, the average thrust $\alpha_{\text{thrust}}$ over a cycle of periodic motion is given by

$$\alpha_{\text{thrust}} = -\frac{1}{T} \int_0^T \theta^T \Lambda \dot{\theta} = -\frac{1}{T} \int_0^T \dot{\theta}^T S \theta, \quad S := \frac{\Lambda - \Lambda^T}{2},$$

where $T$ is the period of $\theta(t)$, and we noted that the integral of $\theta^T P \dot{\theta}$ over a cycle is zero for any periodic signal $\theta$ and for an arbitrary symmetric matrix $P$. We now see that the periodic motion $\theta(t)$ is rectified through the bilinear mechanism $\dot{\theta}^T S \theta$ to generate the thrust. If the original equations of motion of (2.1) are linearized in terms of $\theta$, then we have $\Lambda = 0$ and resulting approximation fails to capture the thrust essential for locomotion. This is
the reason that Taylor’s expansion rather than simply linearization was used in the model simplification. The drag force in average can be represented by

\[
\alpha_{\text{drag}} = \frac{1}{T} \int_0^T (a + \theta^T Q \theta) v.
\]

If \( v \) in steady state is quasi-constant signal averaged at \( v_o \) with relatively small ripples, then approximately \( \alpha_{\text{drag}} = v_o \int_0^T (a + \theta^T Q \theta) \). While in the steady state of the locomotion, the average value of the thrust and drag should be in balance, that is

\[
\alpha_{\text{thrust}} = -\alpha_{\text{drag}} = -\frac{v_o}{T} \int_0^T (a + \theta^T Q \theta). 
\] (5.1)

The coordinate transformation in (2.3) indicates the periodic movement of \( \theta \) can be obtained from the natural oscillation and orientation that are defined at velocity \( v = v_o \). Thus, \( \theta \) depends on \( v_o \), and (5.1) becomes

\[
\int_0^T \dot{\theta}^T(v_o)S\theta(v_o) = v_o \int_0^T (a + \theta^T(v_o)Q\theta(v_o)). 
\] (5.2)

Both sides are functions of \( v_o \), and we will call \( v_o \) satisfying (5.2) a natural velocity. The equality establishes the relation between the natural velocity \( v_o \) and \( \theta(v_o) \) that represents the natural oscillation and orientation. This basic analysis will be exploited to specifically inspect the locomotion behaviors for three different locomotors. Before proceeding to the next section, we introduce the following lemma that will be frequently used in the chapter.

**Lemma 5.1** Let \( x(t), y(t) \in \mathbb{R}^n \) be \( T \)-periodic sinusoidal vector-valued signals and \( \omega = 2\pi/T \) be the frequency. Then, \( \frac{1}{T} \int_0^T x^T y dt = \Re[\hat{x}^* \hat{y}] / 2 \) where \( \hat{x} \) and \( \hat{y} \) are phasors of signals \( x(t) \) and \( y(t) \).
Proof: For sinusoidal signals, $x(t)$ and $y(t)$ be rewritten as

$$
x(t) = \Re[\hat{x}e^{j\omega t}] = \frac{1}{2}(\hat{x}e^{j\omega t} + \bar{\hat{x}}e^{-j\omega t}),
$$

$$
y(t) = \Re[\hat{y}e^{j\omega t}] = \frac{1}{2}(\hat{y}e^{j\omega t} + \bar{\hat{y}}e^{-j\omega t}),
$$

where $\hat{x}$ and $\hat{y} \in \mathbb{C}^n$ are phasors of $x$ and $y$, respectively. Note that $\int_0^T e^{2j\omega t} dt = \int_0^T e^{-2j\omega t} dt = 0$, then direction calculation proves the lemma.

The Lemma provides a direct way to calculate the integral of inner product of two $T$-periodic sinusoidal vector-valued signals. The result is expressed in terms of the phasors of signals. The remaining of the chapter is divided into three sections, each of which is devoted to the locomotion analysis for one of three robotic locomotors. Section 5.5 will expose the control design framework by summarizing previous results and introduce steps to design an effective controller for locomotion control.

5.2 Fliptail Locomotor

The natural oscillation of fliptail locomotor is defined in Definition 3.1 and achieved by feedback controllers in Theorem 3.3 for a fixed constant velocity $v = v_o$. This section removes the assumption that $v$ is constant, and considers the situation where body oscillations $\phi(t)$ generate a forward velocity $v(t)$ through the locomotion equation (2.8), where the functions $\Psi$ and $\Xi$ are quadratic and given by (2.9). Assuming that the velocity $v$ is available for feedback, we will modify the controller (3.21) so that the natural oscillation is achieved for the complete closed-loop system.
5.2. FLIPTAIL LOCOMOTOR

Let a desired velocity $v_o$ be given. It is not difficult to see that the modified controller

$$u = \zeta(\dot{\phi}) + K(v)\phi - K(v_o)\phi,$$

(5.3)

achieves the natural oscillation $\phi_n(t)$ for (2.7) defined at the constant velocity $v(t) = v_o$, where $\zeta(\dot{\phi})$ is the entrainment controller. The actual velocity $v(t)$, however, is generated through the locomotion dynamics (2.8) driven by $\phi(t) = \phi_n(t)$. In general, the resulting velocity $v(t)$ is not necessarily constant nor equal to $v_o$. It can readily be verified through the averaging of (2.8) that the average value of $v(t)$, when higher order harmonic terms are neglected, is given by $\bar{h}(v)$ which is defined as follows. Denote the natural oscillation at a given constant velocity $v$ by $(\omega(v), z(v))$, making its dependence on $v$ explicit. Then the force balance introduced in (5.1) and Lemma 5.1 lead to the definition of function $h$ as follow:

$$h(v) := \frac{j\omega(v)z(v)^*Sz(v)}{z(v)^*Pz(v) + 2c},$$

(5.4)

where

$$S := \frac{Q - Q^T}{2}, \quad P = C.$$

and the balance condition in (5.2) then is $v_o = h(v_o)$. Here, $Q$, $C$ and $c$ are model parameters defined in Appendix A.1. Thus, the desired velocity $v_o$ used for the control design can indeed be consistent with the system dynamics and equal to the resulting average velocity if it satisfies $v_o = h(v_o)$. The natural velocity $v_o$ is the velocity that satisfies $v_o = h(v_o)$. When $v_o$ is chosen so, one can make a rigorous statement about the closed-loop behavior as follows. Note that the orientation does not need to be handled for fliptail locomotor.
Theorem 5.1  Consider the system described by (2.7) and (2.8), where $\Psi(\phi)$ and $\Xi(\phi)$ are given by (2.9) with $c \geq 0$ and $C > 0$. Let a nonzero natural velocity $v_0$ be given and suppose it satisfies the consistency condition $v_0 = h(v_0)$ and Assumption 2.1 about the system holds for $v = v_0$. Let $(\omega, z)$ be the natural oscillation at velocity $v_0$, and $\mathbb{O}$ be the corresponding orbit defined in (3.18). For the closed-loop system with the controller (3.21), suppose the initial value $x(0)$ for $x := (\phi, \dot{\phi})$ is sufficiently close to the natural oscillation orbit $\mathbb{O}$. Then the trajectory $x(t)$ converges to $\mathbb{O}$, and there exists a $T$-periodic trajectory $\varphi(t)$ with $T := 2\pi/\omega$ such that

$$\lim_{t \to \infty} |v(t) - \varphi(t)| = 0.$$  \hspace{1cm} (5.5)

Moreover, if $c = 0$, then we have

$$\max_{t \in [0,T]} |\varphi(t) - v_0| \to 0 \text{ as } ||z|| \to 0.$$  \hspace{1cm} (5.6)

**Proof:** Following Theorem 3.3, it suffices to examine the locomotion dynamics (2.8) with $\phi(t) = \phi_n(t)$ and $\dot{\phi}(t) = \dot{\phi}_n(t)$ (we ignore the time shift $t_0$ for brevity without loss of generality.) Let us denote, with a slight abuse of notation, $\Psi(\phi_n(t))$ and $\Xi(\phi_n(t), \dot{\phi}_n(t), \ddot{\phi}_n(t))$ by $\Psi(t)$ and $\Xi(t)$, respectively, which are both periodic with period $T := 2\pi/\omega$. Noting that $\Psi(t) > 0$ holds for all $t$, it can be verified that the trajectory $v(t)$ of the system (2.8) is always bounded. Since the system (2.8) is a linear $T$-periodic system with a uniformly bounded solution (since $C > 0$ within $\Psi(\phi)$), and it admits a $T$-periodic solution $\varphi(t)$ (see Theorem 20.3, [155])). It is easy to see that the error $v(t) - \varphi(t)$ satisfies

$$(\ddot{v} - \ddot{\varphi}) + \Psi(v - \varphi) = 0,$$
5.2. FLIPTAIL LOCOMOTOR

which is a stable system due to $\Psi(t) > 0$. As a result, we have (5.5). What is left is to show the property (5.6) for $\varphi(t)$. Denote the Fourier series of the $T$-periodic signal $\varphi(t)$ as follows:

$$\varphi(t) = \bar{\varphi} + \sum_{k=1}^{\infty} C_k \cos \Omega_k, \quad \Omega_k := 2k\omega t + \Upsilon_k,$$

for some $\bar{\varphi}, C_k, \Upsilon_k \in \mathbb{R}$. Substituting this $\varphi$ into (2.8) as a solution $v = \varphi$, we have

$$\sum_{k=1}^{\infty} (2k\omega)C_k \sin \Omega_k = \Psi \left( \bar{\varphi} + \sum_{k=1}^{\infty} C_k \cos \Omega_k \right) + \Xi.$$

Note that the magnitudes of $\Psi(t)$ and $\Xi(t)$ approach zero as $||z|| \to 0$ when $c = 0$. Hence, in this limit, we have $C_k \to 0$ for all $k$. Consequently, the constant term in the above equation gives $\Psi \bar{\varphi} + \bar{\Xi} \to 0$, where and $\bar{\Psi}$ and $\bar{\Xi}$ the average values of $\Psi$ and $\Xi$ over one cycle, respectively. Now, it suffices to show $-\bar{\Psi}^{-1}\bar{\Xi} = v_o$, which is true by noting $-\bar{\Psi}^{-1}\bar{\Xi} = h(v_o) = v_o$. The proof is thus complete.

Theorem 5.1 shows that, when the controller (5.3) is used to drive the system described by (2.7) and (2.8), the natural oscillation for velocity $v_o$ is achieved exactly, and the actual velocity $v(t)$ converges to a $T$-periodic signal in the steady state. Moreover, the average velocity approaches the nonzero constant value $v_o$ when the amplitude $||z||$ of the natural oscillation approaches zero, provided $c = 0$ (or $\mu_t = 0$ see Appendix A.1). In reality, however, the locomotion velocity $v_o$ would approach zero when the amplitude approaches zero and the locomotor body becomes motionless. The gap is caused by the assumption $c = 0$, or $\Psi(0) = 0$ in (2.9), that idealizes the model (2.8). If the assumption is removed and $c$ is nonzero, then the function $h(v)$, and hence $v_o$ satisfying $v_o = h(v_o)$ as well, approach zero when $||z||$ goes to zero, correctly reflecting the reality.

The term $\Psi(\phi)v$ in (2.8) represents the drag resulting from the environment force. The assumption $\Psi(0) = 0$ means that the environmental drag experienced by the body at its
nominal posture $\phi = 0$, is ignored. For instance, $\Psi(0)v$ would be the fluid drag experienced by a swimming eel with its body straight, moving at velocity $v$. This drag is usually small in comparison with $\Psi(\phi)v$ under a nominal swimming condition (see e.g. [137]), and hence the idealized model with $\Psi(0) = 0$ can capture the reality reasonably well. In the case $c > 0$, Theorem 5.1 establishes the convergence of the velocity to a periodic signal $\bar{\omega}$, and hence the average value $\bar{\omega}$ is well defined and may be considered as a function of $v_o$ that defines the natural oscillation. Redefining $\bar{h}(v_o)$ to be this function, and choosing $v_o$ to satisfy $v_o = \bar{h}(v_o)$, one can achieve the steady locomotion at average velocity $v_o$ through the natural oscillation. Thus, the idea underlying Theorem 5.1 works for the realistic case $c > 0$ as well. The benefit of considering the ideal case $c = 0$ is that one can gain insights through the analytical formula for $\bar{h}(v)$ in (5.4).

5.3 Snake-like Locomotor

In the previous section, the natural oscillation of the body shape and regulation of the body orientation have been achieved by the nonlinear feedback controller (4.15) for the locomotor model (2.10) and (2.11). Recall that the model has been developed under a nominal locomotion condition in which the locomotion velocity is nearly constant: $\bar{\omega}(t) \cong \bar{\omega}_o = \begin{bmatrix} v_o & 0 \end{bmatrix}^T$. In particular, the coefficients $M$ and $p$ depend on the velocity $v$ in general, but are chosen to take fixed values $M(v_o)$ and $p(v_o)$ for a selected nominal velocity $v_o$ (see Appendix A). Hence, the control design would be valid only if the actual velocity $\bar{\omega}$, achieved by the controller (4.15) through the dynamics (2.12) for the mass center, turns out to be equal to the nominal velocity $\bar{\omega}_o$. The objective is to check and enforce the consistency. Specifically, we will give a condition under which the controller (4.15) designed for $M = M(v_o)$ and $p = p(v_o)$ yields the actual velocity $\bar{\omega}(t) \to \bar{\omega}_o$. 
For a given constant velocity $v$, let $(\omega(v), z(v))$ be the natural oscillation of (2.10) with $M = M(v)$. Suppose the velocity $\varpi(t) \equiv \varpi_o$ is consistent with the dynamics (2.12), i.e., satisfies (2.12) approximately, when the natural oscillation $\phi = \phi_n$ in (3.15) and the associated natural orientation $\varphi = \chi$ in (4.10) are achieved. In this case, averaging of the first row of (2.12) over one cycle ($T := 2\pi/\omega$) and Lemma 5.1 give the force balance equation:

$$(z(v)^* P z(v))v = j\omega(v)(z(v)^* S z(v)),$$

where the left/right hand sides are the drag/thrust, and

$$P := P + \bar{\Gamma}^T \Gamma, \ S := (Q - Q^*)/2, \ Q := Q + c\Gamma.$$ 

Here, $P$, $Q$, and $c$ are model parameters defined in Appendix A.2, $\Gamma$ is defined in (4.10), and $\bar{\Gamma}$ is the complex conjugate of $\Gamma$.

Let us define

$$h(v) := j\omega(v)(z(v)^* S z(v))/(z(v)^* P z(v)),$$

where $h$ is a function of $v$ through the dependence of $(\omega(v), z(v))$ on $v$. The force balance equation in (5.2) can also be written as $v_o = h(v_o)$. Recall that the definition of the natural oscillation does not specify the amplitude of oscillation $||z_o||$. However, since $h(v_o)$ is independent of $||z_o||$, there is no ambiguity in defining the consistent velocity $v_o$ by the condition $v_o = h(v_o)$. Then $v_o$ satisfying $v_o = h(v_o)$ is a natural velocity. Plotting the curve $y = h(x)$ and straight line $y = x$ and searching for the intersection gives a numerical method to find a natural velocity. Such method will be illustrated in next chapter.

When the controller (4.15) is designed for the natural oscillation associated with $v_o$ satisfying $v_o = h(v_o)$, one can prove the following rigorous statement.
**Theorem 5.2** Consider the system described by (2.10)-(2.12), where $\Psi(\phi, \varphi)$ and $\Xi(\phi, \dot{\phi}, \dot{\varphi})$ are given by (2.13) with $P > 0$. Let $v_o$ be a natural velocity satisfying $v_o = h(v_o)$ where $h(v)$ is defined in (5.7), and $(\omega, z)$ be the corresponding natural oscillation. Define $T := 2\pi/\omega$, $\varpi_o := \begin{bmatrix} v_o & 0 \end{bmatrix}^T$, $\alpha_z := ||z||$, and $T$-periodic signals $\phi_n(t)$ and $\chi(t)$ by (4.15) and (4.10), respectively. Consider the locomotion system composed of (2.10)-(2.12), and the feedback controller defined by (4.15) and (4.16). Then, there exist a $T$-periodic signal $\varphi(t) \in \mathbb{R}^2$ and a constant $\iota_3 \in \mathbb{R}$ such that

$$\lim_{t \to \infty} ||\varpi(t) - \varphi(t)|| = 0,$$

$$\varphi(t) \to \begin{bmatrix} v_o + \iota_3 & 0 \end{bmatrix}^T \text{ as } \alpha_z \to 0,$$

$$\iota_3 \to 0 \text{ as } \kappa_2 \to \infty,$$

where the perturbation $\iota_3$ is independent of $\alpha_z$.

**Proof:** Following Theorem 4.2, it suffices to examine the locomotion dynamics (2.12) with $\phi(t) = \phi_n(t) + \alpha \beta \iota_1(t)$ and $\varphi(t) = \chi(t) + \alpha \iota_2(t)$. Obviously, $\phi(t)$ and $\varphi(t)$ are $T$-periodic signals. Let $z := z/\alpha_z$. As a result, one has

$$\int_0^T \left[ \phi^T(t)P\phi(t) + \varphi^2(t) \right] dt = \alpha_z^2(z^*Pz) + \alpha_z^2 \iota_a,$$

$$\int_0^T \left[ \phi^T(t)Q\phi(t) + \phi(t)\dot{\phi}(t) \right] dt = j\omega \alpha_z^2(z^*Sz) + \alpha_z^2 \iota_b,$$

where $\iota_a$ and $\iota_b$ are caused by $\iota_1(t)$ and $\iota_2(t)$. It is easy to see $\iota_a, \iota_b \to 0$ as $\max_{t \in [0,T]} |\iota_1(t)| \to 0$ and $\max_{t \in [0,T]} |\iota_2(t)| \to 0$. Now, the dynamics (2.12) can be put in the following form

$$\ddot{\varpi} + [\tilde{\Psi} + \tilde{\Psi}(t)]\varpi = \tilde{\Xi} + \tilde{\Xi}(t),$$

(5.11)
5.3. SNAKE-LIKE LOCOMOTOR

where

\[ \bar{\Psi} = \mu \begin{bmatrix} \alpha_z^2(z^*Pz) + \alpha_x^2\iota_a & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ \bar{\Xi} = \mu \begin{bmatrix} j\omega \alpha_z^2(z^*S\dot{z}) + \alpha_x^2\iota_b \\ 0 \end{bmatrix}, \]

and \( \bar{\Psi}(t) \) and \( \bar{\Xi} \) are \( T \)-periodic signals with zero average values. The system (5.11) is a linear stable \( T \)-periodic system (since \( P > 0 \) within \( \Psi(\phi, \varphi) \)), and it admits a \( T \)-periodic solution \( \varphi(t) \) (see Theorem 20.3, [155]). Thus, it is easy to see \( \lim_{t \to \infty} ||\varpi(t) - \varphi(t)|| = 0 \).

What is left is to show the property (5.9) for \( \varphi(t) \). Denote the Fourier series of the \( T \)-periodic signal \( \varphi(t) \) as follows:

\[ \varphi(t) = \bar{\varphi} + \sum_{k=1}^{\infty} C_k \cos(k\omega t + \Upsilon_k), \]

where \( C_k \in \mathbb{R}^{2 \times 2} \) is a diagonal matrix, and \( \Upsilon_k \in \mathbb{R}^2 \) is a column vector. Substituting it into the following equation

\[ \dot{\varphi} + [\bar{\Psi} + \tilde{\Psi}(t)] \varphi - \bar{\Xi} - \tilde{\Xi}(t) = 0, \]

and balancing the coefficients of the constant term and the term \( \sin(k\omega t + \Upsilon_k) \) give

\[ \bar{\Psi}\bar{\varphi} + (1/T) \int_0^T \sum_{i=1}^{\infty} \tilde{\Psi}(t) C_i \cos(i\omega t + \Upsilon_i) dt - \bar{\Xi} = 0, \quad (5.12) \]

and

\[ S_k \left\{ \tilde{\Psi}(t)\bar{\varphi} + \sum_{i=1}^{\infty} \tilde{\Psi}(t) C_i \cos(i\omega t + \Upsilon_i) - \tilde{\Xi}(t) \right\} - (k\omega)C_k = 0, \quad (5.13) \]
where $S_k$ is the coefficient of $\sin(k\omega t + \Upsilon_k)$. Since the terms $\tilde{\Psi}(t), \tilde{\Xi}(t) \to 0$ as $\alpha_z \to 0$, we have $C_k \to 0$ as $\alpha_z \to 0$ in (5.13). As a result, (5.12) implies $\bar{\Psi} \tilde{\varphi} \to \tilde{\Xi}$, i.e., $\tilde{\varphi} \to \bar{\Psi}^{-1} \tilde{\Xi}$. In particular,

$$\bar{\Psi}^{-1} \tilde{\Xi} = \begin{bmatrix} [j\omega(z^*Sz) + \iota_b]/[(z^*Pz) + \iota_a] \\ 0 \end{bmatrix} = \begin{bmatrix} [j\omega(z^*Sz)]/[(z^*Pz)] + \iota_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{h}(v_o) + \iota_3 \\ 0 \end{bmatrix},$$

for $\iota_3 := [j\omega(z^*Sz) + \iota_b]/[(z^*Pz) + \iota_a] - [j\omega(z^*Sz)]/[(z^*Pz)]$. The property (5.9) is thus proved by noting $\bar{h}(v_o) = v_o$ from the definition of natural velocity.

Finally, as $\kappa_2 \to 0$, one has $\max_{t \in [0,T]} |\iota_1(t)| \to 0$ and $\max_{t \in [0,T]} |\iota_2(t)| \to 0$, then $\iota_a, \iota_b \to 0$, and hence $\iota_3 \to 0$. The property (5.10) is thus proved.

**Remark 5.1** When the natural oscillation $\phi = \phi_n$ of the body-environment dynamics is exploited for locomotion, there is also a particular velocity $v_o$ chosen by the dynamics, characterized by $v_a = \bar{h}(v_o)$ in (5.7). Theorem 4.2 has shown that controller (4.15) is able to achieve the natural oscillation $\phi = \phi_n$ with the body orientation regulated at $\varphi = \chi$, with arbitrarily small perturbation terms $\iota_1$ and $\iota_2$. Theorem 5.1 now shows that the same controller (4.15) also achieves regulation of the locomotion velocity at $\varpi = \varpi_o$, with a perturbation that vanishes when the time constant $\kappa_2$ of the low pass filter in (4.16) approaches infinity and the amplitude of body undulation $|z|$ approaches zero. Thus, the performance of the controller (4.15) is precisely characterized by these theorems. For practical purposes, the results suggest to consider the limiting case $\kappa_2 \to \infty$ and use the controller in (4.22).
5.4 FLAPPING-WINGS LOCOMOTOR

Remark 5.2  The locomotor model equations for the control design have been derived by assuming a small oscillation amplitude $||z||$. Also, as shown in (5.8) and (5.9), our controller achieves locomotion at a velocity closer to the natural velocity when the oscillation amplitude is smaller. However, the oscillation amplitude $||z||$ is a design parameter that has to be chosen to have a reasonable magnitude to overcome the tangential drag $\mu_t$ (caused by $c_t$) ignored in the design model. In the next chapter, we show through design examples that the natural oscillation is still effectively achieved when $||z||$ is reasonably large and comparable to the amplitudes observed in animal locomotion.

5.4 Flapping-wings Locomotor

For flapping-wings locomotor, the natural oscillation $\phi_n$ and the associated natural orientation $\chi$ have been achieved based on the dynamics (2.14) and (2.15), which are developed at the neighborhood of locomotion velocity $v_y = v_o$. So, an interesting question is whether the natural oscillation/orientation in turn generates the desired forward velocity $v_y = v_o$ through the model (2.16). Following the same idea adopted for fliptail and snake-like locomotors, the balance function for the natural oscillation will be established. The natural velocity is calculated below under the ideal situation with $\mu_t = 0$, $\mu_b = 0$, and hence $a = 0$ in (2.16). Denote

$$T = \begin{bmatrix} J & \Gamma^T \end{bmatrix}^T, \quad Q_T = T^*QT, \quad \Lambda_T = T^*\Lambda T,$$

where $\Gamma$ is defined in (4.1), and $Q$ and $\Lambda$ in (2.16).

For a given constant velocity $v$, let $(\omega(v), z(v))$ be the natural oscillation of (2.14). Suppose the velocity $v_y \equiv v_o$ is consistent with the dynamics (2.16), that is, satisfies (2.16) approximately, when the natural oscillation $\phi = \phi_n$ and the associated orientation $\phi = \chi$
in (4.1) are achieved. In this case, averaging (2.16) over one cycle \((T_o := 2\pi/\omega_o)\) and Lemma 5.1 give the force balance equation

\[
(z_o^*Qz_o)v_o = j\omega_o(z_o^*S_zo),
\]

where the left/right hand sides are the drag/thrust, and

\[
Q = \frac{Q_T + Q_T^*}{2}, \quad S = \frac{\Lambda_T - \Lambda_T^*}{2}.
\]

(5.14)

Let us define

\[
h(v) := j\omega(v)(z(v)^*Sz(v))/(z(v)^*Qz(v)),
\]

(5.15)

The force balance equation then is \(v_o = h(v_o)\), and any \(v_o\) satisfying (5.15) is called a natural velocity for flapping-wing locomotor.

The following theorem states that the natural velocity can be effectively achieved by the controller designed in the previous sections.

**Theorem 5.3** Consider the system described by (2.14)-(2.16), where \(Q > 0\) in (2.16). Let \(v_o\) be a natural velocity satisfying \(v_o = h(v_o)\) with \(h(v)\) defined in (5.15), \((\omega_o, z_o)\) be the corresponding oscillation, and \(\phi(t) = \phi_n(t)\) and \(\varphi(t) = \chi(t)\) the corresponding natural orbit and natural oscillation. Consider the complete system composed of (2.14), (2.15) and (2.16), and the controller composed of (4.24) and (4.25). Assume the oscillation amplitude \(\alpha_z := \|z_o\|\) is sufficiently small. Then, there exists a \(T\)-periodic trajectory \(\varsigma(t)\), for \(T = 2\pi/\omega_o\), such that

\[
\lim_{t \to \infty} v_y(t) - \varsigma(t) = 0,
\]

(5.16)
and
\[ \zeta(t) \to v_o + \iota_3 \text{ as } \alpha_z \to 0. \quad (5.17) \]

Moreover, the perturbation \( \iota_3 \) is independent of \( \alpha_z \) and it can be arbitrarily small, i.e., \( \iota_3 \to 0 \) as \( \gamma_2 \to \infty \).

**Proof:** Following Theorem (4.3) with \( \phi(t) = \phi_n(t) + \alpha_z \beta \iota_1(t), \varphi(t) = \chi(t) + \alpha_z \iota_2(t) \),

We note that \( \chi(t) = \Re(\Gamma z e^{j\omega t}) \) from (4.1), therefore

\[ \Theta = \vartheta = \Re \left( \begin{bmatrix} I & \iota_2 \\ \Gamma & \iota_1 \end{bmatrix} z e^{j\omega t} \right) \alpha_z. \]

Now, the dynamics (2.16) can be put in the following form

\[ m \ddot{v}_y + \Theta^T Q \dot{\vartheta} v_y + \Theta^T \Lambda^T \dot{\vartheta} = 0, \quad (5.18) \]

with \( a_y = 0 \). Obviously, \( \phi_n(t) \) and \( \chi(t) \) are \( T \)-periodic signals. Let \( z := z_0 / \alpha_z \), according to Lemma 5.1 one has

\[ \int_0^T \Theta^T(t) Q \dot{\vartheta}(t) dt = \alpha_z^2 (z^* Q z) + \alpha_z^2 \iota_\alpha, \]
\[ \int_0^T \Theta^T(t) \Lambda^T \dot{\vartheta}(t) dt = -j \omega \alpha_z^2 (z^* S z) - \alpha_z^2 \iota_\beta, \]

where \( Q \) and \( S \) are defined in (5.14), and \( \iota_\alpha \) and \( \iota_\beta \) are caused by \( \iota_i(t) \) for \( i = 1, 2 \). It is easy to see \( \iota_\alpha, \iota_\beta \to 0 \) as \( \max_{t \in [0, T]} |\iota_i(t)| \) for \( i = 1, 2 \).

Now, the dynamics (5.18) can be put in the following form:

\[ m \dot{v}_y + \begin{bmatrix} \bar{\Psi} + \tilde{\Psi}(t) \end{bmatrix} v_y = \bar{\xi} + \tilde{\xi}(t), \quad (5.19) \]
where

\[ \bar{\Psi} = \alpha_z^2 (z^* Q z) + \alpha_z^2 t_\alpha, \]

\[ \bar{\xi} = j\omega \alpha_z^2 (z^* S z) + \alpha_z^2 t_b, \]

and \( \bar{\Psi}(t) \) and \( \bar{\xi}(t) \) are \( T \)-periodic signals with zero average values. The system (5.19) is a linear stable \( T \)-periodic system (since \( Q > 0 \) in (2.16)), and it admits a \( T \)-periodic solution \( \varsigma(t) \) (see Theorem 20.3, [155]). Thus it is easy to check

\[ \lim_{t \to \infty} \nu(t) - \varsigma(t) = 0. \]

What is left is to show the property (5.17) for \( \varsigma(t) \). Denote the Fourier series of the \( T \)-periodic signal \( \varsigma(t) \) as follows:

\[ \varsigma(t) = \bar{\varsigma} + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t + \Upsilon_k), \]

where \( c_k, \beta_k \in \mathbb{R} \). Substituting it into the following equation

\[ m_t \dot{\varsigma} + (\bar{\Psi} + \bar{\Psi}(t)) \varsigma - \bar{\xi} - \bar{\xi}(t) = 0, \]

and balancing the coefficients of the constant term and term \( \sin(k\omega_0 + \Upsilon_k) \) give

\[ \bar{\Psi} \bar{\varsigma} + (1/T) \int_0^T \sum_{k=1}^{\infty} \bar{\Psi}(t)c_k \cos(k\omega_0 t + \Upsilon_k) dt - \bar{\xi} = 0, \]

and

\[ -(k\omega_0)m_t c_k + \mathcal{G}_k \left\{ \bar{\Psi}(t) \bar{\varsigma} + \sum_{i=1}^{\infty} \zeta c_k \cos(i\omega_0 t + \Upsilon_i) - \bar{\xi}(t) \right\} = 0, \]
where $S_k$ is the coefficient of $\sin(k\omega_o t + \Upsilon_k)$. Since the terms $\tilde{\Psi}(t), \tilde{\xi}(t) \to 0$ as $\alpha_z \to 0$, we have $c_k \to 0$ as $\alpha_z \to 0$. As a result, $\tilde{\Psi} \zeta - \tilde{\xi} \to 0$, i.e., $\zeta \to \tilde{\xi}/\tilde{\Psi}$. Now it suffices to show $\tilde{\xi}/\tilde{\Psi} = \nu_o + \iota_3$, which is true by noting

$$\frac{\tilde{\xi}}{\tilde{\Psi}} = \frac{j\omega \alpha_2^2(z^*Sz) + \alpha_z^2t_b}{\alpha_z^2(z^*Qz) + \alpha_z^2t_\alpha} = \beta(v_o) + t_3 = \nu_o,$$

for

$$t_3 = \frac{j\omega(z^*Sz) + \|z\|t_b}{(z^*Qz) + \|z\|t_\alpha} - \frac{j\omega(z^*Sz)}{(z^*Qz)}.$$

The property (5.16) is thus proved by noting $h(\nu_o) = \nu_o$ from the definition of natural velocity.

Finally, as $\zeta_2 \to \infty$, one have $\max_{t \in [0,T]} |t_i| \to 0$ for $i = 1, 2$, then $t_\alpha, t_\beta \to 0$, and hence $t_3 \to 0$. The property is (5.17) is thus proved.

### 5.5 Summary

The consistency of the locomotion velocity is evaluated by the proposed three theorems for different robotic locomotors, respectively. Flipped locomotor, as is fully actuated, can achieve the natural oscillation exactly using the modified controller (5.3), and the natural velocity is also exactly achieved when amplitude of the natural oscillation tends to zero. As they are under-actuated, the snake-like and flapping-wings locomotors achieve the natural oscillation approximately with small deviation specified by $\iota_3$. The deviation can be tuned as small as possible by increasing the controller parameter $\zeta_2$ in orientation control according to the statement of Theorem 5.2 and Theorem 5.3. For three locomotors, the analysis of the locomotion equation takes similar procedures. The equality $\nu_o = h(\nu_o)$ resulting from thrust and drag force balance is used to calculate natural velocity.
CHAPTER 5. LOCOMOTION ANALYSIS

Through three previous chapters, problems raised in Chapter 2 including how to design bio-inspired entrainment controller, how to control orientation, as well as how to enforce that the locomotion speed is correctly achieved have been fully tackled. The framework of designing an effective locomotion controller has been gradually clear. In summary, it is composed of the following five steps:

1) apply \( h(v) \) functions to identify the natural speed \( v_0 \);

2) use Lemma 3.2 to calculate the natural oscillation \( \phi_n \) or \((\omega, z)\);

3) following the theorem 3.3, design the entrainment controller for the given velocity \( v_0 \) to achieve the natural oscillation;

4) utilize (4.1) or (4.10) to calculate the natural orientation \( \chi(t) \);

5) following the theorem 4.1, design the additional controller on top of entrainment controller to fulfill the orientation control.

As entrainment controller and orientation controller are dependent on a constant velocity \( v_0 \), the determination of the natural velocity is thus placed in the first step. In next chapter, the numerical simulation will apply these five steps to design the controller for three different locomotors to achieve locomotion behavior.
Chapter 6

Numerical Simulation

The previous chapters exposed the unified framework giving a method to design an effective feedback controller in order to achieve the desired locomotion behavior. Theoretical results were supported by detailed analysis and rigorous proof. On other hand, this chapter aims at verifying effectiveness of the control design framework using numerical simulation. The framework will be applied to the three robotic locomotors, whose models were presented in Chapter 2. Following steps described in previous chapter, the nonlinear feedback controller will be designed and applied in numerical simulation. It will show that the desired locomotion behavior is effectively achieved.

The simulation results will be graphically presented in order for clear demonstration. Several aspects that were or were not explicitly introduced in previous chapters will be demonstrated with description. First, the numerical calculation of the natural velocity will be graphically illustrated and it will reveal that varying system stiffness will result in different natural velocity. Second, some light will be shed on the role of the parameters that play on the convergence to the natural oscillation pattern. Third, the comparison between theoretical calculation and simulation will be given to indicate the effectiveness of the controller. At last, the turning motion will be demonstrated for snake-like and flapping-wings locomotors.
6.1 Introduction

Fliptail, snake-like and flapping-wings locomotors are three typical representatives of the mechanical rectifiers. In this chapter, we would like to design three corresponding numerical examples with dynamical models introduced in Chapter 2. Not restricted to the three specific locomotors, the control design framework is also applicable to the mechanical rectifiers that can be described by (2.4)-(2.6). The procedure of control design for locomotion will follow steps proposed in last chapter. It will be shown that the control design framework is able to derive the controller and achieve the locomotion behavior. Besides, simulation examples will be used to address the following issues that are not explicitly mentioned in previous chapters.

- The design examples will reveal that system stiffness will affect the natural velocity calculated according to $v_o = \dot{h}(v_o)$.

- The design examples will reveal how the controller parameters will affect the performance of closed-loop systems.

- The control design theories presented in previous chapters show that the natural oscillation, orientation regulation, and locomotion are achieved approximately, with some small perturbation terms $\iota_i(t)$. The design example will provide some idea about how large the perturbation terms are.

- The model we used for developing the design method is obtained by simplifying a fully nonlinear model under the assumption that the amplitude of body oscillation is small. However, we expect that the method works reasonably well for practical situations because the qualitative behavior of the full dynamics is captured by the simplified model. To examine whether this expectation is met, we will design a controller
to achieve the natural oscillation with an amplitude comparable to those observed in fish swimming, and test the design through simulations of the fully nonlinear model.

- In general, a feedback control system designed for setpoints regulation can be used to “steer” the system by adjusting the set point if the design is sufficiently robust (with a large domain of attraction and fast convergence rate). Analogously, our design method may be useful for steering the direction of locomotion by rotating the inertial frame so that locomotion orientation is aligned with the desired direction.

Note that in equations (2.4) and (2.5), the terms $K(v_o)$ and $K_s(v_o)$ are constant, as the model results from approximation at a constant velocity $v_o$. We would like to show that the proposed controller can deal with time-varying $K(v)$ and $K_s(v)$. Therefore, $v_o$ in these terms will be replaced by $v$, when the simulation runs for the whole locomotion system (2.4)-(2.6). If the simulation results are obtained to show natural oscillation and orientation by only considering the oscillation equation and/or orientation equation, then $v_o$ is used instead. The rest of the chapters are divided into three sections, each of which is devoted to a simulation example for one of three robotic locomotors. Each section will start with a short introduction of the example.

### 6.2 Fliptail Locomotor

Consider the fliptail locomotor (2.7)-(2.8), we firstly would like to propose the nonlinear entrainment controller satisfying Theorem 3.3, where two controller parameters can be tuned. Two groups of the controller parameters will be selected to show their effects on the performance of the closed-loop system (2.7). The profile of the natural oscillation by theoretical calculation and numerical simulation will be compared. Then the simulation of the
closed-loop system (2.7)-(2.8) with feedback controller as in equation (5.3) will be illustrated, with natural velocity found using $v_o = \hat{h}(v_o)$. A modified version of controller (5.3) is used here also for demonstration of velocity setpoints tracking. At last the non-idealized case with $c \neq 0$ in (2.8) or $\mu_t \neq 0$ will be explored, followed by applying the controller to the original nonlinear model.

Let us firstly ignore the locomotion equation and only consider the oscillation equation of fliptail locomotor (2.7) with stiffness $\kappa = 5$ under constant velocity $v_o = 0.3$, for which the critical damping factor in (3.17) is $\varrho = 2.7$. A controller is designed using (3.21) with

$$\kappa(x) := \epsilon + (\varrho - \epsilon)\varepsilon(x)/\varepsilon(\omega), \quad \varepsilon(x) := \eta/(1 + \eta x),$$

the resulting oscillation profiles of the closed-loop system are calculated from simulations for two cases $(\eta, \epsilon) = (2, 2.3)$ and $(\eta, \epsilon) = (0.3, 2.3)$, where the period is $T = 2\pi/\omega$, the $i$th phase of $\phi_n$ is $\gamma_i := \angle z_i$ (with $\gamma_5 = 0^\circ$ being the reference phase). As theoretically guaranteed, the profiles turned out to exactly match the natural oscillation profile (the first row of Table 3.2) up to the number of digits shown. The waveforms of the oscillations are shown in Fig 6.1, where the limit cycle trajectory achieved by the controller is purely sinusoidal. Fig 6.2 shows the asymptotic convergence of $\phi_1(t)$ for the two cases. It is seen that the parameters $(\eta, \epsilon)$ can be tuned for faster convergence.

Next, consider the fliptail locomotor system described by (2.7) and (2.8) with $\mu_t = 0, \mu_n = 0.5$ Ns/m, and $\kappa = 4$. In this case, the critical damping factor $\varrho = 5.2$, and the nonzero solution to $v_o = \hat{h}(v_o)$ in (5.4) is uniquely given by $v_o = 0.345$ m/s. The controller (5.3) is designed using function $\kappa$ in (6.1) with $(\eta, \epsilon) = (3, 0)$. The closed-loop simulation result is given in Fig 6.3, showing that this same velocity $v_o$ can be approximately achieved with different oscillation amplitudes ($\|z\| = 0.5$ and $\|z\| = 1$) under the idealized condition where $\mu_t = 0$. By Theorem 5.1, the ripple in $v(t)$ should get smaller as the oscillation amplitude approaches zero. This phenomenon is demonstrated in Fig 6.3 where magnitude
Figure 6.1: The closed-loop oscillations are sinusoidal (fliptail locomotor).

Figure 6.2: Asymptotic convergence of $\theta_1(t)$ for different $\eta$ and $\epsilon$ (fliptail locomotor).
### Chapter 6. Numerical Simulation

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Table 6.1: Oscillation profiles with different amplitude where $\gamma_5 = 0°$ (fliptail locomotor.)

![Figure 6.3: Self-generated velocity $v$. The expected velocity $v_o = 0.345$ is effectively achieved with different amplitudes (fliptail locomotor).](image-url)
Figure 6.4: Solution to $\alpha(x) = x$. The solid curves represent $y = \alpha(x)$ for $\kappa = 1, 2, 3, 4, 5$, along the arrow, and the dash line is $y = x$ (fliptail locomotor).

Figure 6.5: Set point tracking of the velocity through varying $\kappa$, where $\|z\| = 0.5$ (fliptail locomotor).
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Figure 6.6: Solution to $\dot{h}(x) = x$ with $\mu_t \geq 0$ (fliptail locomotor).

Figure 6.7: Self-generated velocity $v$ with $\mu_t = 0.01$. The expected velocities $v_o = 0.097$ and $v_o = 0.242$ are effectively achieved (dropped from $v_o = 0.345$ with $\mu_t = 0$) (fliptail locomotor).
A desired locomotion velocity \( v_o \) can be achieved by adjusting the stiffness, exploiting the fact that the nonzero solution \( x = v_o \) to \( h(x) = x \) depends on the value of \( \varkappa \). Fig 6.4 shows plots of \( y = h(x) \) for several \( \varkappa \) values, where \( v_o \) is identified from the intersection of a curve \( y = h(x) \) and the straight line \( y = x \). The intersection \( h(x) = x \) occurs at \( x = 0.185, 0.257, 0.309, 0.345, 0.372 \) m/s respectively. We see that a larger stiffness leads to a larger velocity \( v_o \). For all the cases, the slope of \( h(x) \) at \( x = v_o \) is less than one, which in fact indicates stability of (or convergence to) the velocity \( v_o \). On the other hand, the curves also intersect with \( y = x \) at the origin with slopes larger than one (not shown), indicating instability of the zero velocity.

The controller (5.3) can be modified by an additional term \( k_c B B^T \phi \) to adjust the closed-loop stiffness from \( k_o \) to \( k_o + k_c \). The additional term can be used to tune the locomotion velocity of the system for set point tracking. This tracking scenario is demonstrated in Fig 6.5 where \( \varkappa \) is changed stepwise at \( t = 100 \) and \( 200 \) s to take values 3, 5, 1 for successive durations. The closed-loop system approximately achieves the corresponding velocities calculated in Fig 6.4.

Next, we examine how the proposed controller performs when \( \Psi(0) \neq 0 \) in (2.8), which indicates that \( \mu_t \neq 0 \) (see Appendix A.1). Consider the system with \( \varkappa = 4, \mu_n = 0.5 \), and \( \mu_t = 0.01 \). When \( \mu_t \) is nonzero, the velocity \( v_o \) satisfying \( v_o = h(v_o) \) takes different values for different oscillation amplitude \( \|z\| \) as seen in Fig 6.6, which can be verified by equation (5.4) with \( c \neq 0 \). The bold curves is with \( \mu_t = 0 \) copied from Fig 6.4. The other two curves are for \( \|z\| = 0.5 \) and \( \|z\| = 1 \) with \( \mu_t = 0.01 \) which intersect \( y = x \) at \( x = 0.129 \) and \( x = 0.259 \), respectively. For all cases, \( \varkappa = 4 \) is used. The closed-loop simulation for system (2.7)-(2.8), Fig 6.7 shows that the controller (5.3) still...
achieves the velocity $v_o$ approximately when $\Psi(0)$ is small although the assumption $c = 0$ in Theorem 5.1 is violated.

Finally, we examine whether the control design based on the approximate model is effective in the more realistic situation. For this purpose, we designed four controllers with four different values of the oscillation amplitude $||z||$, and applied them to the fully nonlinear equations of motion (without simplification or bi-linearization). Here we use the relative amplitude of $|z_i|$ with $z_i := z_i / ||z||$. When the amplitude is small, the simplified model captures the original nonlinear dynamics more accurately, and hence the controller is expected to perform well in achieving the desired locomotion. This study reveals how much performance degradation occurs when the amplitude becomes larger. The simulation results are summarized in Table 6.2. As expected, the natural oscillation with the natural velocity is still effectively achieved when the oscillation angles are in a biologically reasonable range, e.g., $||z|| = 1.07$ rad in the table. When the oscillation amplitudes are further increased, one can observe significant distortion in oscillation profile. If the error creates problems, an additional velocity feedback loop may be necessary.

| Period | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ | $||z||$ |
|--------|-------------|-------------|-------------|-------------|-------------|-----------|
| Nat.Osc. | 1.12 | 332° | 232° | 141° | 64° | 0° | 0.1/0.5/1.0/1.5 |
| Simu. | 1.12 | 333° | 235° | 143° | 65° | 0° | 0.1 |
| | 1.10 | 333° | 249° | 139° | 62° | 0° | 0.52 |
| | 1.10 | 344° | 258° | 128° | 61° | 0° | 1.07 |
| | 1.23 | 327° | 209° | 116° | 53° | 0° | 1.44 |

| Period | $|z_1|$ | $|z_2|$ | $|z_3|$ | $|z_4|$ | $|z_5|$ | $||z||$ | $\bar{v}$ |
|--------|--------|--------|--------|--------|--------|--------|--------|
| Nat.Osc. | 1.80° | 7.06° | 17.12° | 29.54° | 45.44° | 0.1/0.5/1.0/1.5 | 0.345 |
| Simu. | 1.79° | 7.03° | 17.12° | 29.54° | 45.43° | 0.1 | 0.345 |
| | 1.35° | 6.82° | 17.50° | 29.61° | 45.28° | 0.52 | 0.336 |
| | **1.21°** | **8.38°** | **19.4°** | **30.8°** | **43.39°** | **1.07** | **0.314** |
| | 4.74° | 11.23° | 24.17° | 31.8° | 39.24° | 1.44 | 0.307 |

Table 6.2: Oscillation profile of the oscillation for nonlinear system (fliptail locomotor).
6.3 Snake-like Locomotor

Consider the snake-like locomotor (2.10)-(2.12), firstly we calculate the natural velocity through numerical solution to $v_0 = \hat{h}(v_0)$ and derive the nonlinear entrainment controller satisfying Theorem 3.3. The impact of the controller parameters is neglected for this case. Then, the simulation of the closed-loop system (2.10)-(2.12) with feedback controller in (4.22) will be illustrated. At last, the turning motion of the system will be described and shown, followed by application of the controller to the original nonlinear model.

Let us first find the natural oscillation. Fig 6.8 shows the plots of the curve $y = \hat{h}(x)$ and the line $y = x$, the intersection of which gives the theoretical natural velocity $v_0 = 0.35 \text{ m/s}$ satisfying $\hat{h}(v_0) = v_0$. For $v_0 = 0.35 \text{ m/s}$, one has $\varrho = 11.0 \text{ Ns/(m \cdot kg)}$ and $\omega = 7.0 \text{ rad/s}$, and the profile parameters for the natural oscillation are listed in the first row of Table 6.3, where the period is $T = 2\pi/\omega$, the phase is $\gamma_i := \angle z_i$ (with $\gamma_4 = 0^\circ$ being the reference phase), the relative amplitude of $|z_i|$ with $z_i := z_i/\|z\|$, the overall amplitude is $\|z\|$, and the average speed is $\bar{v} = v_0$. The decreasing value of the phase indicates that the body waves travel from head to tail. The increasing value of the relative amplitude indicates that the tail oscillates with a larger amplitude than the head. The snapshots of the body under the natural oscillation within a period are shown in Fig 6.9. The amplitudes growing toward the tail appear similar to swimming carangiform fish or crawling snakes. We see that the body undulation is consistent with the observations from Table 6.3.

To achieve the natural oscillation, the feedback controller (3.21) is designed with the following details. The function $\kappa(s) = \varrho + 300 \left[ (1 + \sqrt{s})^{-1} - (1 + \sqrt{\omega})^{-1} \right], \forall s > 0$, is selected to satisfy the conditions in Theorem 3.3. On top of that, we use the simplified controller (4.22) for orientation control. The vector $\beta = M^{-1}e$ with $e := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$ simply makes $h$ a uniform gain vector in (4.22). The other parameters are selected as $\kappa_1 = 0.38$ and $\kappa_2 = 10$ to satisfy (4.14). The closed-loop system is then simulated with the
CHAPTER 6. NUMERICAL SIMULATION

initial condition $\phi(0) = \begin{bmatrix} 0.0265 & -0.0805 & 0.0443 & 0.0094 \end{bmatrix}^T \text{ rad} \ (\text{randomly selected}),$

$v(0) = 0.1 \text{ m/s}, \text{ and zeros for all the remaining states.} \ \text{Since the oscillation amplitude } ||z|| = 1.2 \text{ rad is not zero (or small), the natural oscillation is not guaranteed to be achieved exactly according to Theorem 5.2.} \ \text{The profile parameters for the closed-loop oscillation are computed from the simulation and are listed in the second row of Table 6.3.} \ \text{We see that the simulated oscillation is reasonably close to the natural oscillation in terms of the period, relative phases, and relative amplitudes.} \ \text{The time courses of the simulated joint angles } \phi_i \ \text{are plotted in Fig 6.10.} \ \text{The oscillations are essentially sinusoidal and the perturbation term } \nu_1(t) \ \text{is not detectable by eyes.}

\text{Next, we will steer the direction of locomotion by resetting the inertial frame within the controller.} \ \text{To see how this works, consider coordinate } X-Y \text{ fixed to the inertial frame, and coordinate } x-y \text{ for the temporary inertial frame used by the controller so that the } x-\text{axis is aligned with the current desired direction of locomotion.} \ \text{The idea is to successively rotate the } x-y \text{ frame with respect to the } X-Y \text{ frame and to make the locomotor turn to the direction of the } x-\text{axis.} \ \text{In the simulation, the two frames initially coincide each other and, starting at } t = 150 \text{ s, the } x-y \text{ frame rotates by } 60^\circ. \ \text{Fig 6.11 shows the time courses of the orientation angle measured in the } X-Y \text{ frame (on the left) and the forward and lateral components } v \text{ and } w \text{ of the locomotion velocity measured in the } x-y \text{ frame (on the right).} \ \text{The body orientation follows the rotation of the } x-\text{axis, and the desired forward velocity } v_o = 0.35 \text{ m/s is effectively achieved while keeping the average value of the side velocity } \bar{w} = 0 \text{ m/s.} \ \text{As shown in the trajectory of the center of gravity in Fig 6.12, the left plot shows that the locomotor initially moves along the } X-\text{axis, and then turns left by } 60^\circ \text{ as desired. The right plot illustrates the body snapshots (taken for every } 1.35 \text{ periods) showing how the system turns.} \ \text{The snapshots of the undulating body (not shown) are very similar to those for the natural oscillation in Fig 6.9.}
### 6.4. FLAPPING-WINGS LOCOMOTOR

| Theo. | 0.90 | 277.2° | 180.1° | 84.6° | 0° |
| Simu. | 0.90 | 279.2° | 183.3° | 85.4° | 0° |

| Theo. | 8.5° | 21.7° | 34.8° | 39.1° | 1.2 | 0.35 |
| Simu. | 9.2° | 22.6° | 35.3° | 38.0° | 1.22 | 0.38 |

Table 6.3: Oscillation profiles (snake-like locomotor).

| Theo. | 0.90 | 277.2° | 180.1° | 84.6° | 0° |
| Simu. | 0.90 | 281.7° | 182.8° | 85.6° | 0° |
| Simu. | 0.90 | 288.0° | 183.0° | 84.5° | 0° |
| Simu. | 0.90 | 290.4° | 181.8° | 83.4° | 0° |
| Simu. | 0.90 | 289.2° | 173.5° | 79.8° | 0° |

| Theo. | 8.5° | 21.7° | 34.8° | 39.1° | 0.5/1.2/1.5/2.0 | 0.35 |
| Simu. | 9.2° | 22.8° | 35.1° | 38.0° | 0.52 | 0.37 |
| Simu. | 9.2° | 23.6° | 34.9° | 37.7° | 1.22 | 0.33 |
| Simu. | 9.7° | 24.4° | 34.9° | 37.1° | 1.53 | 0.30 |
| Simu. | 11.5° | 26.6° | 34.5° | 35.4° | 2.08 | 0.26 |

Table 6.4: Oscillation profiles on exact model (snake-like locomotor).
Figure 6.8: Solution to $h(x) = x$. The dashed line is $y = x$, the bold curve is $y = h(x)$ (snake-like locomotor).

Figure 6.9: Body oscillation snapshots in one period taken for every $1/8$ period (snake-like locomotor).
6.4. FLAPPING-WINGS LOCOMOTOR

Figure 6.10: The time courses of the body joint angles $\phi_i$, obtained from simulation of the closed-loop system (snake-like locomotor).

Figure 6.11: Simulated locomotion with a turn at $t = 150$ s (snake-like locomotor).
Finally, as we did for the fliptail locomotor, the control design based on the approximate bilinear model will be applied to the original nonlinear system. For this purpose, we designed four controllers with four different values of the oscillation amplitude \( \|z\| \), and applied them to the fully nonlinear equations of motion without simplification. When the amplitude is small, the controller is expected to perform well. However with the increase in the amplitude, the performance of the controller would degrade. The simulation results are summarized in Table 6.4. As expected, the natural oscillation with the natural velocity is still effectively achieved when the oscillation angles are biologically reasonable (\( \|z\| = 1.22 \) rad in the table corresponds to absolute (rather than relative) oscillation amplitudes \(|z_1| = 11.2^\circ\), \(|z_2| = 28.8^\circ\), \(|z_3| = 42.6^\circ\) and \(|z_4| = 46.0^\circ\).) When the oscillation amplitudes are further increased, one can observe significant distortion in oscillation profiles. The average velocity starts to deviate more rapidly than the oscillation profile, but accuracy of the velocity regulation may not be important for applications to autonomous robot [156].

Figure 6.12: Turning behavior in the inertial frame X-Y (snake-like locomotor).
6.4 Flapping-wings Locomotor

Consider the flapping-wings locomotor (2.14)-(2.16), firstly we calculate the natural velocity through numerical solution to \( v_o = h(v_0) \) and derive the nonlinear entrainment controller satisfying Theorem 3.3. The feedback controller (4.31) will be used in the simulation of closed-loop system. The snapshot extracted from the simulation data will be illustrated to give an idea of the flapping-wings motion. At last, the turning motion of the system will be described and shown. Due to lack of the original nonlinear model in the literature, we cannot show the performance of the controller when applied to the real model.

We use the same model parameters given in the Section 3.4.3 and in [2]. The first step to realize a locomotion is to find the natural velocity and the corresponding natural oscillation. Fig. 6.13 shows the plots of the curve \( y = h(x) \) and the line \( y = x \), the intersection of which gives the theoretical natural velocity. The stiffness coefficient \( k \) ranging from 1000 to 3000 N/m\(^2\) to explore the effect of stiffness on locomotion behavior. It is observed that the system with harder stiffness in the wing structure tends to move faster (i.e., \( v_o \) is larger).

In particular, we choose \( k_o = 1000 \) N/m\(^2\) and find \( v_o = 0.24 \) m/s as the natural velocity to design the controller for the entrainment to the natural oscillation and the natural orientation according to Theorem 3.3 and Theorem 5.3. For the orientation controller (4.31), \( \varkappa_1 = 0.1 \) and \( \varkappa_2 = 1 \) are selected and \( \beta \) is chosen to satisfy the condition (4.23). The initial velocity \( v_y(0) \) is 0.1 m/s.

Table 6.5 demonstrates the comparison between the theoretical natural oscillation and the actual oscillation achieved by the proposed controller in simulation in terms of frequency, phases, and amplitudes. In the table, N.O. and Sim. stand for the natural oscillation in theoretical calculation and in simulation, respectively. \( \gamma_i \) and \( \|z_i\| \) represent the phase and amplitude of the \( i \)th point mass. The comparison indicates that the oscillation achieved by the entrainment controller is approximately close to the theoretical natural oscillation.
<table>
<thead>
<tr>
<th></th>
<th>$\omega$ (rad/s)</th>
<th>$\gamma_{15}$</th>
<th>$\gamma_{30}$</th>
<th>$\gamma_{45}$</th>
<th>$\gamma_{60}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N.O.</td>
<td>3.032</td>
<td>13.20°</td>
<td>10.00°</td>
<td>26.49°</td>
<td>48.26°</td>
</tr>
<tr>
<td>Sim.</td>
<td>3.032</td>
<td>13.43°</td>
<td>10.76°</td>
<td>28.95°</td>
<td>50.89°</td>
</tr>
<tr>
<td>$\times 10^{-1}$</td>
<td>$|z_1|$</td>
<td>$|z_{15}|$</td>
<td>$|z_{30}|$</td>
<td>$|z_{45}|$</td>
<td>$|z_{60}|$</td>
</tr>
<tr>
<td>N.O.</td>
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<td>0.155</td>
<td>1.178</td>
<td>0.354</td>
<td>0.064</td>
</tr>
<tr>
<td>Sim.</td>
<td>0.023</td>
<td>0.162</td>
<td>1.237</td>
<td>0.371</td>
<td>0.066</td>
</tr>
</tbody>
</table>

Table 6.5: Oscillation profiles (flapping-wings locomotor).

Figure 6.13: Solution to $h(x) = x$ with $k_\omega = 1000, 1500, 2000, 2500, 3000$ along the arrow. The dashed line is $y = x$, the bold curve is $y = h(x)$ (flapping-wings locomotor).
6.4. FLAPPING-WINGS LOCOMOTOR

Figure 6.14: Performance of the velocity $v_y$ in time course.

Figure 6.15: Natural Oscillation gait snapshot (flapping-wings locomotor).
Next, we extract one cycle of the periodic motion from the simulation and plot the snapshots of the wings’ shape. In Fig. 6.15, the subplots (a) to (d) correspond to the wing shape at equally spaced time instants from \( t = t_0 \) to \( t_0 + 3T/4 \). It indicates that the flapping gait in the natural oscillation looks close to the natural flapping motion observed in a bird or a ray fish. In Fig. 6.14, the system starts from an initial velocity other than the natural velocity but it converges to the natural value in around 7s with small ripples. Also, Fig. 6.16 shows that the average orientation \([\bar{\beta}_b \bar{\gamma}_b]\) is regulated to zero by the proposed controller. In particular, the amplitude of the orientation \( \gamma_b \) varies with \( v_y \) until \( v_y \) reaches the natural velocity and then the amplitude of \( \gamma_b(t) \) is 2.8° rad and that of \( \beta_b(t) \) is 0.85°.

Recall that the flapping-wings locomotor is in a three dimensional space equipped with the coordinate frame \( \Sigma = (x, y, z) \). It has been demonstrated that the model moves with an essential forward velocity in the y-direction with roll and pitch orientation \( \bar{\beta}_b = 0 \) and \( \bar{\gamma}_b = 0 \). Furthermore, we define the earth coordinate frame by \( \Sigma_e = (x_e, y_e, z_e) \). The coordinate frame \( \Sigma \) can be simply selected as \( \Sigma = \Sigma_e \). Also, \( \Sigma \) can be reset to tune the orientation with respect to the earth frame \( \Sigma_e \). In particular, an interesting scenario is demonstrated as follows. In this scenario, the frame \( \Sigma \) is initially overlapped with \( \Sigma_e \) and then successively rotated with respect to \( \Sigma_e \) about \( x_e \)-axis. As a result, the locomotor initially moves horizontally along the \( y_e \)-direction and then turns (up or down) together with the \( \Sigma \) frame. The simulation is given in Fig. 6.17-6.19. In the simulation, the two frames \( \Sigma \) and \( \Sigma_e \) initially coincide with each other and, starting at \( t = 100 \) s, the \( \Sigma \) frame rotates by 20°. Figure 6.17 shows the locomotion velocity \( v_y \) measured in the \((x, y, z)\) frame. The controller (4.31) effectively tunes the average velocity of \( v_y \) toward the natural velocity \( v_y = 0.24 \) m/s in y-axis. Fig 6.18 shows the convergence of orientation \( \gamma_b \) in the \( \Sigma \) frame. The average of \( \gamma_b \) is initially zero and then regulated to zero again after the turning. The turning is clearly demonstrated in Fig 6.19 which shows the time course of the pitch.
angle in the $\Sigma_e$ frame, denoted by $\gamma_e$. In particular, the locomotor moves horizontally and then turns up by $20^\circ$.

![Graph](image)

**Figure 6.16:** Performance of the orientation $[\beta_b \, \gamma_b]$ in time course (flapping-wings locomotor).

![Graph](image)

**Figure 6.17:** The forward velocity $v_y$ in the $\Sigma$ frame (flapping-wings locomotor).
Figure 6.18: The body orientation angle $\gamma_b$ in the $\Sigma$ frame (flapping-wings locomotor).

Figure 6.19: The body orientation angle $\gamma_e$ in the $\Sigma_e$ frame. (flapping-wings locomotor).
Chapter 7

Conclusion

This thesis is focused on the development of control design framework for a class of mechanical rectifier systems. Three robotic locomotors were presented as examples to demonstrate how to design an feedback controller that effectively achieves locomotion behaviors. Three basic problems regarding the control design were firstly proposed and well solved at the end. The effectiveness of control design framework was supported theoretically by rigorous proofs and practically validated by numerical simulation. In what follows, a brief summary and final conclusions on the overall achievements are presented so as to complete this thesis.

7.1 Summary

We have considered a class of mechanical rectifier systems arising from dynamics of animal locomotion. The robotic design could benefit from the conception of mimicking morphology and motion of animals, while its implementation encounters several challenges. Two fundamental tasks closely related to the locomotion control are how to choose efficient gaits (optimization) and how to achieve the movement using actuators (control). A concise reviews of these two aspects were provided in the introduction, followed by critical remarks on current control design methods. In addition, the analysis and design method of nonlinear oscillators including CPG were reviewed. It was revealed that proposing a
controller that achieves the efficient movement of biologically inspired robotic system was quite challenging, due to complexity of mechanical design, nonlinearities in dynamics and hyper-redundant degrees of freedom. The work that derives a dynamics-model-based feedback controller to achieve a limit cycle as the oscillatory trajectory in closed loop was even rare. Thus, the thesis could be regarded as an attempt dedicated to filling the gap and providing feedback control design framework for mechanical rectifiers.

Thesis proceeded in Chapter 2 by presenting the modeling of mechanical rectifiers, which provides a starting point for gait optimization and control design. Using Euler-Lagrange method or Newton’s law, mechanical rectifiers can be modeled mathematically as a group of complex nonlinear differential equations, that fully captures the essence of the mechanical rectification. However, due to the complexity of the highly nonlinear dynamics model, the intuition of the locomotion mechanism is hidden and how to design a feasible controller is unclear and remains difficult. Thus, a model simplification technique based on Taylor’s series was conducted to reduce the complexity, which was followed by coordinate transformation. The simplified dynamical model thus derived revealed that the dynamical structure of the system consists of three important parts: oscillation, orientation and locomotion dynamics. In general, oscillation dynamics was fully controlled, while orientation dynamics was not under direct control.

Chapter 2 went further and introduced three typical robotic implementations of mechanical rectifiers, namely fliptail, snake-like and flapping-wings locomotors. They were used to represent a broad class of locomotion systems mimicking animals such as jelly fish, snake, leach, ray, bird etc. Their equations of motion can be obtained using the general modeling method. Their dynamical structure were basically the same, whereas they had different mechanical structures and they were constrained to move in the different environment. Any idea that was derived for the general system can also apply to these typical locomotors. Given this thought, the following chapters firstly gave the analysis based on the general
7.1. SUMMARY

system, and then introduce slight variation when difference existed. At last, the breakdown of dynamical structure clearly exposed three basic problems that needed to be tackled in order to arrive at the control design framework.

In order to solve *Entrainment to Oscillation Pattern Problem*, Chapter 3 first presented the general definition of oscillation pattern into which two specific patterns can fit, and they were optimal and natural oscillation patterns. The optimal oscillation pattern defined a set of oscillations that can explicitly optimized the locomotion performance, while natural oscillation pattern exploited the idea of mechanical resonance. The comparison of the two patterns for three locomotors were given at the end of the chapter, revealing that natural oscillation closely matched with one of the optimal oscillation patterns. This part of work can be also regarded as the task of gait optimization.

Given the definition of the natural oscillation, a nonlinear controller was required to achieve the pattern with certain oscillation profiles. Thus, Chapter 3 introduced the CPG control structure motivated by biological control mechanisms and then gave a controller based on this structure. However the controller lacked of rigorous proof for the existence of limit cycle and it could only achieve oscillation approximately. Therefore, a new controller that was also based on CPG structure and can achieve the entrainment to natural oscillation pattern exactly was hence proposed with a sound mathematical analysis on the stability analysis. Since, three locomotors had exactly the same body oscillation equation, the proposed entrainment controller could apply to all of three.

Chapter 4 proceeded by exposing the core idea of orientation control for mechanical rectifiers. The general idea was to use the bias in the body oscillation to steer the orientation of the body. The controller that was used to tune the orientation was built on the top of entrainment controller proposed in Chapter 3. The body orientation was achieved to be aligned with locomotion direction although with sacrifice of losing a bit accuracy of the oscillation
pattern for body shape. However theorems indicated that the perturbation away from the desired oscillation pattern could be tuned arbitrarily small by setting control parameters. Then, the general idea of orientation tuning was extended and applied to snake-like and flapping-wings locomotors. The chapter concluded that these two closed-loop systems had very similar property, although they differed on the dimension of the orientation needed to be controlled.

The previous two chapters both considered the simplified system with locomotion velocity of the mass center approximated at a constant, while Chapter 5 was devoted to proving that the locomotion velocity could be generated through the dynamics of the locomotion equation. The chapter first analyzed two forces exerted on the mass center, namely thrust and drag force, which resulted from the body oscillation and orientation. It was observed that the balance of two forces in steady state led to the relation between locomotion velocity and the system variables, which also defined the natural velocity. Then the relation was utilized to construct the theorems verifying the generation of the natural velocity. Three variations of theorems were developed for the typical locomotors. It was shown that the natural velocity of fliptail locomotor tended to be exactly achieved as amplitude of natural oscillation decreased to zero. For the other two locomotors, locomotion velocity is achieved with perturbation, whose magnitude decreased with increase in one of the controller parameters.

Prior to Chapter 6, an unified framework giving a method to design an effective feedback controller had been exposed by solving three problems in Chapter 2. Theoretical results were supported by detailed analysis and rigorous proof. Chapter 6 verified effectiveness of the design framework using the numerical simulation. Examples were designed for those three locomotors and showed several potential aspects of the controller and the closed-loop system. It was also observed that the natural velocity varied with system stiffness and the performance of the controller relied on the corresponding controller parameters. For fliptail and snake-like locomotors, the proposed controller demonstrated its effectiveness
on the original nonlinear systems justifying the validity of the small amplitude assumption and showing that the model simplification retained system essence.

The theoretical work has inspired developing the real-world robotic systems including Prototype Mechanical Rectifier (PMR) system, robotic snakes and 2-D flapping-wings systems in [40, 157]. Those systems applied this control design framework to construct a biologically inspired nonlinear feedback controller in order to fulfill the different locomotion behaviors. The variety of the robotic locomotion systems and behaviors demonstrate the generality and effectiveness of the control design method. In those work, Euler-Lagrange method and Newton’s law are used for the system modeling followed by the similar simplification. It is also shown that the model simplification is not only necessary for the controller derivation but also it does not affect the controller performance very much.

To our knowledge, it is the first work that uses biological CPG network as a building block to design a feedback controller that tunes the mechanical system into its mechanical resonance. [110, 142] adopted a CPG architecture called reciprocal inhibition oscillator to achieve a prescribed mode of mechanical resonance for the closed-loop system, but it dealt with simple linear mechanical systems and does not find any applications yet. In particular, the proposed control method achieves the desired periodic behavior as a stable autonomous motion through feedback control, rather than as a forced response to a fixed trajectory command. Thus, the design would be more robust against disturbances, noises, and neglected higher order nonlinear dynamics. Therefore, the work tends to extend the effort of proposing a feedback controller for the locomotion system, which is rare compared to an open loop controller in [10, 35, 89]. In comparison to the control method based on kinematic model, the use of the dynamical model instead facilitates the analysis of natural oscillation and help exploit the mechanical resonance to reduce the energy consumption. In general, the thesis focuses on the generation of the oscillation pattern and orientation control, while the trajectory tracking for the mass center is rarely considered except for velocity tracking.
for the fliptail locomotors. Trajectory tracking for the mass center in particular with ability of obstacle avoidance would be a natural extension to the current work, but it is very challenging due to the complexity of the dynamical models. More possible future directions are summarized in next section.

7.2 Future Work

Given the presentation of achievement, the thesis will close by listing some work that author considers to be relevant and important. The first one is to propose a biologically inspired controller that can achieve the optimal oscillation pattern and more broadly general oscillation pattern defined in Lemma 3.1. Preliminary (and also unpublished) work has been conducted by author, and it revealed that the controller should be partially constructed as a compensator to modify the system. An easy compensator is to make the natural oscillation of the compensated system coincide with the prescribed oscillation pattern, and then apply the control design method in the thesis to achieve it. To do this, the remaining part of the controller should be designed following Theorem 3.3. Author went further to derive a different way of the compensation, where author took advantage of the oscillation pattern essentially as a planar oscillator (in Lemma 3.1) and designed a nonlinear controller. Rather than these control design, we should explore that if a controller that directly follows CPG structure can achieve any oscillation patterns that fit into the definition in Lemma 3.1.

The other future direction of the work is more ambitious and challenging, that is analyzing and building a typical life-like robot-robotic snake that highly resembles the natural snake. A biological locomotor system interacting with environment typically consists of three components, neuronal control circuit, body and environment. The autonomous locomotion behavior results from the coupling of the three components, where CPG neuronal network plays central role in generating coordinated oscillation. However, the CPG-inspired
controller proposed in the thesis might not reflect the realistic CPG in natural snakes. In literature, the frequently used model for controlling snake-like robots was proposed by Matsuoka [158]. The neural model was a half-center oscillator which consisted of two (extensor and flexor) neurons, having mutual inhibitory interactions. In fact, there is few biological literature about snake’s CPG model, alternatively the CPG model of lamprey could be found as an analogue in [159]. In animals, the CPG neural model actuates the contraction of the muscle resulting in the rhythmic body movement to generate the locomotion. Without a convincing muscle model, the dynamics model is lack of credibility to represent the real snake model. Nevertheless, the muscle model is missing in the current research. In summary, the future work would consider more realistic CPG and muscle actuation model to construct a dynamical system that closely matches with biological finding. With this model, it can be used to testify the biological hypothesis about the interaction between the CPG networks and muscular actuation as well as enlighten the engineering design.
Bibliography


Appendix A

Derivation of Locomotion Models

A.1 Equations of Motion for Flitpail Locomotor

The detailed derivation of the locomotor model (2.7)-(2.8) is given below. Let us consider lower chain of the links and define

\[
B := \begin{bmatrix}
1 & -1 \\
\vdots & \ddots \\
1 & -1 \\
1 & \text{1}
\end{bmatrix}, \quad F := \begin{bmatrix}
1 & 2 & \cdots & 2 \\
1 & \ddots & \vdots \\
& & \ddots & 2 \\
& & & \text{1}
\end{bmatrix}
\]

\[
e^T := \begin{bmatrix}
1 & 1 & \cdots & 1
\end{bmatrix}
\]

and \(S_\phi\) and \(s_\phi\) are the diagonal matrix and the column vector with entries \(\sin \phi_i\) for \(i = 1, \cdots, 2, n\), similarly \(C_\phi\) and \(c_\phi\) are defined with \(\cos \phi_i\). Every link has length \(2l_o\) and evenly distributed mass \(m_o\). Let \(g^x_i\) and \(g^y_i\) be the internal force at joint \(i\), and \(\chi_{n,i}\) and \(\chi_{t,i}\) be environmental force at the mass center of \(i\)th link at normal and tangential direction. Let us denote column vectors \(g^x := \begin{bmatrix} g^x_1 & \cdots & g^x_n \end{bmatrix}^T\), \(g^y := \begin{bmatrix} g^y_1 & \cdots & g^y_n \end{bmatrix}^T\), \(\chi_t := \begin{bmatrix} \chi_{t,1} & \cdots & \chi_{t,n} \end{bmatrix}^T\) and \(\chi_n := \begin{bmatrix} \chi_{n,1} & \cdots & \chi_{n,n} \end{bmatrix}^T\). \((x_i, y_i)\) denotes the position of the mass center of \(i\)th link, for which we define column vectors \(x := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T\) and \(y := \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T\). Then the rotation and translation equations about the mass center
of every link can be described by

\[
\begin{align*}
    m_o \ddot{x} &= Bg^x + C \dot{x}_t - S \dot{\chi}_n, & m_o \ddot{y} &= Bg^y + S \dot{\chi}_t + C \dot{\chi}_n.
\end{align*}
\]

where \( f^t = -\mu n l_o \ddot{\phi} / 3 \) is the torque generated by the environmental forces and

\[
\chi_t = -\mu_t (C \dot{x} + S \dot{y}), \quad \chi_n = -\mu_n (-S \dot{x} + C \dot{y}).
\]

The following gives the geometrical constraint of the system

\[
\begin{align*}
x &= x_0 e - l_o F^t C \phi, & y &= x_0 e - l_o F^t S \phi + c_o,
\end{align*}
\]

where \( c_o \) is the distance from center of the head link to the head joint. Taking derivative twice to above equation leads us to

\[
\begin{align*}
    \dot{x} &= \dot{x}_o e + l_o F^t S \phi \dot{\phi}, & \dot{y} &= \dot{y}_o - l_o F^t C \phi \dot{\phi},
    \\
    \ddot{x} &= \ddot{x}_o e + l_o F^t C \phi \ddot{\phi}^2 + l_o F^t S \phi \dddot{\phi}, & \ddot{y} &= \ddot{y}_o e + l_o F^t S \phi \dddot{\phi}^2 - l_o F^t C \phi \dddot{\phi}.
\end{align*}
\]

Substitute \( \ddot{x}, \ddot{y} \) into the translation equation of the link chain, and the internal forces can be put in the form

\[
\begin{align*}
    Bg^x &= m_o \ddot{x}_o e + m_o l_o F^t C \phi \dddot{\phi}^2 + m_o l_o F^t S \phi \dddot{\phi} - C \dot{\phi}_t + S \dot{\phi}_n, \\
    Bg^y &= m_o \ddot{y}_o e + m_o l_o F^t S \phi \dddot{\phi}^2 - m_o l_o F^t C \phi \dddot{\phi} - S \dot{\phi}_t - C \dot{\phi}_n.
\end{align*}
\]

Let us assume the mass of the head link can be ignored compared to the mass of the tail links, and the head link does not contact with environment. Therefore, the head link is
only subject to internal forces from the first links of lower and upper chains. Due to the symmetric movement of the two chains about $x$-axis, the projection of internal forces on $y$-axis are cancelled out implying $\dot{y}_o = \ddot{y}_o = 0$, while the force on $x$-axis is $2g^x_1$. Noting that $g^x_1 = e^TBg^x$, the dynamics for head link is $2g^x_1 = 0$ or

$$nm_o \ddot{x}_o + m_o l_o e^T C_\phi \dot{\phi}^2 + m_o l_o e^T S_\phi \dot{\phi} - e^T C_\phi \chi_t + e^T S_\phi \chi_n = 0.$$  

Substitution of $\chi_t$ and $\chi_n$ gives the dynamics for the head link

$$a_n \ddot{x}_o + b_n \dot{x}_o + c_n = 0 \quad \text{(A.2)}$$

where

$$a_n = nm_o, \quad b_n = \mu_t e^T C_\phi C_\phi e + \mu_n e^T S_\phi S_\phi e,$$

$$c_n = m_o l_o e^T F^T C_\phi \dot{\phi}^2 + m_o l_o e^T F^T S_\phi \dot{\phi} + e^T (\mu_t C_\phi C_\phi + \mu_n S_\phi S_\phi) l_o F^T S_\phi \dot{\phi},$$

$$-e^T (\mu_t C_\phi S_\phi - \mu_n S_\phi C_\phi) l_o F^T C_\phi \dot{\phi}.$$

Substitution of $Bg^x$, $Bg^y$ and $f^t$ into (A.1) leads to the following rotation dynamics

$$m_o l_o^2 [I/3 + C_\phi F^T F C_\phi + S_\phi F^T S_\phi] \ddot{\phi}$$

$$= m_o l_o^2 (C_\phi F^T S_\phi - S_\phi F^T C_\phi) \dot{\phi}^2 - \mu_n l_o^2 \dot{\phi} / 3 - \mu_t l_o^2 (S_\phi FC_\phi - C_\phi FS_\phi) (C_\phi F^T S_\phi - S_\phi F^T C_\phi) \dot{\phi}$$

$$- \mu_n l_o^2 (C_\phi FC_\phi + S_\phi FS_\phi) (C_\phi F^T C_\phi + S_\phi F^T S_\phi) \dot{\phi}$$

$$- \mu_t l_o (S_\phi FC_\phi - C_\phi FS_\phi) C_\phi e x_o - \mu_n l_o (C_\phi FC_\phi + S_\phi FS_\phi) S_\phi e x_o$$

$$+ m_o l_o S_\phi F e x_o - k_o BB^T \phi + B\tau. \quad \text{(A.3)}$$
A.1. EUQATIONS OF MOTION FOR FLIPTAIL LOCOMOTOR

The nonlinear equation (A.2) and (A.3) can be approximated at \((\phi, \dot{\phi}) = (0, 0)\) as follow:

\[
\begin{align*}
J \ddot{\phi} + D \dot{\phi} + K(v_o)\phi &= u \\
\dot{v} + \Psi(\phi)v + \Xi(\phi, \dot{\phi}, \ddot{\phi}) &= 0
\end{align*}
\]

with

\[
\begin{align*}
J &= m_o l_o^2 (FF^T + I/3), \quad D = (\mu_n/m_o)J, \quad K = v_o \Lambda + k_o BB^T \\
\dot{u} &= B \tau, \quad \Lambda = l_o(\mu_n - \mu_t)F
\end{align*}
\]

and

\[
\begin{align*}
\Psi(\phi) &= c + \phi^T C \phi, \quad \Xi(\phi, \dot{\phi}, \ddot{\phi}) = \dot{\phi}^T S \dot{\phi} + \theta^T S \theta + \dot{\theta}^T Q \theta
\end{align*}
\]

where

\[
\begin{align*}
c &= c_t, \quad C = c_n I/n, \quad c_t := \mu_t/m_o, \quad c_n := \mu_n/m_o \\
S &= (l_o/n) \text{diag}(Fe), \quad Q = (l_o/n) \left( (c_n - c_t) F + c_t S \right)
\end{align*}
\]

The default values are given as follow, unless they are specified in the thesis. The number of links for each chain in the numerical simulation is \(n = 5\), and each link has mass \(m_o = 0.04\) kg and length \(2l_o = 0.1\) m. The environmental force coefficients are \(\mu_t = 0\) and \(\mu_n = 0.2\) Ns/m, and each joint has stiffness \(k_o = 2.5 \times 10^{-4} \kappa\) Nm/rad, where the parameter \(\kappa\) is used to examine the effect of stiffness perturbation.
A.2 Equations of Motion for Snake-like Locomotor

The detailed derivation of the locomotor model (2.10)-(2.12) is given below. We start with the fully nonlinear equations of motion derived in [11, 1]. When the links of the snake-like locomotor system are identical with mass $m_o$, length $2l_o$, and moment of inertia $m_o l_o^2 / 3$, the system is described by

\[
J \ddot{\theta} + G \dot{\theta}^2 + k BB^T \dot{\theta} + D \dot{\theta} + \Lambda \varpi = B \tau,
\]

\[
m \ddot{\varpi} + \Lambda' \dot{\theta} + Q \varpi = 0,
\]

with coefficients defined by

\[
A := \begin{bmatrix} I & o \\ o & I \end{bmatrix},
\]

\[
B^T := \begin{bmatrix} o & I \\ I & o \end{bmatrix},
\]

\[
H := (m_o l_o^2) A^T (B^T B)^{-1} A, \quad F := l_o B (B^T B)^{-1} A,
\]

\[
m := (n + 1) m_o \quad N_\theta := \begin{bmatrix} s_\theta & -c_\theta \end{bmatrix},
\]

\[
J_\theta := S_\theta H S_\theta + C_\theta H C_\theta + (m_o l_o^2 / 3) I,
\]

\[
G_\theta := S_\theta H C_\theta - C_\theta H S_\theta, \quad R_\theta := S_\theta F S_\theta + C_\theta F C_\theta,
\]

\[
\begin{bmatrix} D_\theta & \Lambda_\theta \\ \Lambda_\theta^T & Q_\theta \end{bmatrix} := \begin{bmatrix} (\mu_n l_o^2 / 3) I & 0 \\ 0 & 0 \end{bmatrix} + \mu_n \begin{bmatrix} R_\theta^T \\ N_\theta \end{bmatrix} \begin{bmatrix} R_\theta & N_\theta \end{bmatrix},
\]

where $S_\theta$ and $s_\theta$ are the diagonal matrix and the column vector with entries $\sin \theta_i$ for $i = 1, \cdots, 2, n + 1$, similarly $C_\theta$ and $c_\theta$ are defined with $\cos \theta_i$, and notation $o$ means the $n$
A.2. EQUATIONS OF MOTION FOR SNAKE-LIKE LOCOMOTOR

dimensional zero vector. Assuming small \(|\theta|\), the approximate equation of motion are give by [1]

\[
\begin{bmatrix}
J\ddot{\theta} + kBB^T\theta - B\tau \\
m\dot{v} \\
m\dot{w}
\end{bmatrix}
+ 
\begin{bmatrix}
\mu J & \Lambda \theta & 0 \\
\theta^T\Lambda^T & \mu_n||\theta||^2 & -\mu_n\theta^Te \\
0 & -\mu_n e^T\theta & (n+1)\mu_n
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
v \\
w
\end{bmatrix}
= 0
\]

(A.4)

with

\[
J := m_o(l_0^2 I/3 + F^T F), \quad \mu := \mu_n/m_o, \\
\Lambda := \mu_n F^T, \quad e := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T
\]

We now introduce the joint angles \(\phi \in \mathbb{R}^n\) and the orientation angle \(\varphi \in \mathbb{R}\) as follows:

\[
\begin{bmatrix}
\phi \\
\varphi
\end{bmatrix}
:= 
\begin{bmatrix}
B^r \\
e^T/n
\end{bmatrix}\theta,
W := 
\begin{bmatrix}
T \\
e^T/n
\end{bmatrix}
= 
\begin{bmatrix}
B^r \\
e^T/n
\end{bmatrix}^{-1}
\]

where \(T \in \mathbb{R}^{(n+1)\times n}\) is defined by the second equation. Let

\[
W^T JW :=
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\]

and we define submatrices of \(W^T DW\) and \(W^T \Lambda W\) similarly. Noting that \(\Lambda_{12} = 0\) and \(\Lambda_{22} = 0\) hold under the identical link assumption, the first row of (A.4) becomes

\[
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi} \\
\ddot{\varphi}
\end{bmatrix}
+ 
\begin{bmatrix}
k\phi - \tau \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{\phi} \\
\dot{\varphi}
\end{bmatrix}
+ v
\begin{bmatrix}
\Lambda_{11} \\
\Lambda_{21}
\end{bmatrix}
\phi
= 0
\]

(A.5)

Consider the matrix

\[
U :=
\begin{bmatrix}
I & -J_{12}J_{22}^{-1} \\
-J_{21}J_{11}^{-1} & I
\end{bmatrix},
\]
which is nonsingular since \( J \) is positive definite. Multiplying (A.5) by \( U \) from the left, the equation of motion reduces to

\[
\ddot{\phi} + \mu \dot{\phi} + M(v) \phi = u \ M(v) = \mathcal{J}(v\Lambda + kI),
\]

\[
\ddot{\phi} + \mu \dot{\phi} + p(v)^T \phi = q^T u, \ p(v)^T := \mathcal{J}^{-1}(v\Lambda_o + kL),
\]

\[
u := \mathcal{J}^{-1} \tau, \ q^T = \mathcal{J}_o^{-1} L \mathcal{J}
\]

where

\[
\mathcal{J} := J_{11} - J_{12} J_{22}^{-1} J_{21}, \ \Lambda := \Lambda_{11} - J_{12} J_{22}^{-1} \Lambda_{21},
\]

\[
\mathcal{J}_o := J_{22} - J_{21} J_{11}^{-1} J_{12}, \ \Lambda_o := \Lambda_{21} - J_{21} J_{11}^{-1} \Lambda_{11},
\]

\[
L := -J_{21} J_{11}^{-1}.
\]

Thus, considering the case \( v \simeq v_o \) and introducing the simplified notation \( M := M(v_o) \) and \( p := p(v_o) \), we obtain (2.10) and (2.11). Finally, (2.12) follows directly from the second and third rows of (A.4) by noting that \( \theta = T\phi + e\varphi, \ e^T T = 0, \) and \( e^T F = 0, \) and defining

\[
P := (T^T T)/(n + 1), \ Q := (T^T FT)/(n + 1), \ c := (T^T Fe)/(n + 1). \quad (A.6)
\]

The matrix \( P \) is symmetric positive definite, capturing the drag due to body oscillation, while \( Q \) is a square matrix whose asymmetry is essential for thrust generation.

We consider the system of 5 links (i.e., \( p = 5 \)) in a chain with 4 joints (i.e., \( n = 4 \)) with the total length \( 2(n + 1)l_o = 0.5 \) m and the total mass \( (n + 1)m_o = 0.2 \) kg. The normal drag coefficient for the environmental force is \( \mu_n = 1.0 \) Ns/m, and each joint has torsional stiffness \( k = 1.25 \times 10^{-3} \) Nm/rad. As a result, \( \mu = 25 \) Ns/(mkg).
A.3 Equations of Motion for Flapping-wings Locomotor

The equation of motion for flapping-wings locomotor can be derived by same techniques used for flitptail and snake-like locomotors. Here we do not give elaborated derivation, please refer to [2] for more details. Using the Euler-Lagrange method and the Taylor series expansion of the model equations around $\theta = 0$ and $v = v_o$ for a constant $v_o$, followed by truncation of higher order terms, gives

$$J\ddot{\theta} + D\dot{\theta} + (k_o K_o + v_o \Lambda) \theta = B\tau,$$

$$m\ddot{v} + (a + \theta^T Q \theta) v + \theta^T N^T \dot{\theta} = 0,$$

where $a = \mu b$, $k_o \in \mathbb{R}$ is the stiffness coefficient and $K_o$ is in the form of

$$K_o = \begin{bmatrix} K_c & 0 \\ 0 & 0 \end{bmatrix}.$$

To facilitate the analysis, we introduce a new state vector

$$\Theta = \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} K_c & 0 \\ 0 & I \end{bmatrix} \theta$$

with $\phi = K_c \z b$ and $\varphi = [\beta_b, \gamma_b]^T$. Then the dynamics (A.7) can be rewritten into (2.14)-(2.16).

The natural oscillation will be calculated for the flapping-wings model with all parameters taken from [2]. In particular, the number of the point masses on each wing is $n = 70$. The wing and body density is approximated as the density of water, which roughly simulates a fish swimming model. The wing and body dimensions are arbitrarily chosen to be reasonable for a small unmanned robot. We have $\mu_n = 0.06$ Ns/m and for the stiffness matrix $K(v_0)$ in (2.17), we choose $k_o = 1000$ N/m$^2$ and $v_o = 0.24$ m/s.