J. reine angew. Math. **634** (2010), 141—169
DOI 10.1515/CRELLE.2010.047

Journal für die reine und
angewandte Mathematik
© Walter de Gruyter
Berlin · New York 2010

Classification of the simple factors appearing in composition series of totally disconnected contraction groups

By *Helge Glöckner* at Paderborn and *George A. Willis* at Newcastle

Abstract. Let G be a totally disconnected, locally compact group admitting a contractive automorphism α. We prove a Jordan-Hölder theorem for series of α-stable closed subgroups of G, classify all possible composition factors and deduce consequences for the structure of G.

Introduction

A *contraction group* is a pair (G, α), where G is a topological group and $\alpha : G \to G$ a contractive automorphism, meaning that $\alpha^n(x) \to 1$ as $n \to \infty$, for each $x \in G$. Contraction groups arise in probability theory on locally compact groups (see, e.g., [15]), representation theory ([21], [22], [24], [35]), and the structure theory of locally compact groups initiated in [36] (see [1] and [8]). It is known from the work of Siebert that every locally compact contraction group is a direct product $G = G_e \times D$ of a connected group G_e and an α-stable totally disconnected group D, whence the study of locally compact contraction groups splits into the two extreme cases of connected groups and totally disconnected groups (see [32], Proposition 4.2). Siebert characterized the connected locally compact contraction groups; they are, in particular, simply connected, nilpotent real Lie groups. He also provided some basic information concerning the totally disconnected case. In the present article, we complete the picture by discussing the fine structure of a totally disconnected, locally compact contraction group G. We show the existence of a composition series

$$\mathbf{1} = G_0 \lhd \cdots \lhd G_n = G$$

of α-stable closed subgroups of G, prove a Jordan-Hölder Theorem for the topological factor groups and find the possible composition factors G_j/G_{j-1}. Any such is a *simple* contraction group in the sense that it does not have a non-trivial, proper, α-stable closed normal

---

Research supported by DFG grant 447 AUS-113/22/0-1 and ARC grants LX 0349209 and DP0208137.

subgroup. Our first main result is a classification of the simple, totally disconnected contraction groups.

Theorem A. *Let* (G, α) *be a simple, totally disconnected, locally compact contraction group. Then* G *is a torsion group or torsion-free, and* (G, α) *is of the following form*:

(a) *If* G *is a torsion group, then* (G, α) *is isomorphic to a restricted product* $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ *with the right shift, for a finite simple group* F.

(b) *If* G *is torsion-free, then* (G, α) *is isomorphic to a finite-dimensional* p*-adic vector space, together with a contractive linear automorphism which does not leave any non-trivial, proper vector subspace invariant.*

Conversely, all of the contraction groups described in (a) *and* (b) *are simple.*

The direct power $F^{\mathbb{N}_0}$ is equipped with the (compact) product topology here, and $F^{(-\mathbb{N})} := \{(x_n)_{n \in -\mathbb{N}} \in F^{-\mathbb{N}} : x_n = 1 \text{ for all but finitely many } n\}$ carries the discrete topology.

We remark that the contractive linear automorphisms occurring in (b) can be characterized in terms of their rational normal form (cf. Proposition 6.3). The classification has important consequences for general contraction groups. Our second main result is the following structure theorem.

Theorem B. *Let* (G, α) *be a totally disconnected, locally compact contraction group. Then the set* $\operatorname{tor}(G)$ *of torsion elements and the set* $\operatorname{div}(G)$ *of divisible elements are* α*-stable closed subgroups of* G, *and*

$$G = \operatorname{tor}(G) \times \operatorname{div}(G)$$

(*internally*) *as a topological group. Furthermore*, $\operatorname{div}(G)$ *is a direct product of* α*-stable*, *nilpotent* p*-adic Lie groups* G_p *for certain primes* p,

$$\operatorname{div}(G) = G_{p_1} \times \cdots \times G_{p_n}.$$

Each G_p actually is the group of $\mathbb{Q}_p$-rational points of a unipotent linear algebraic group defined over $\mathbb{Q}_p$ (see [35], Theorem 3.5 (ii), also [25]).

Theorems A and B characterize totally disconnected locally compact contraction groups completely, as was done by Siebert in the connected case. Just as the first step in the treatment of connected groups, carried out by Müller-Römer [25], (1.10), is to use the solution of Hilbert's fifth problem, [23], Theorem 4.6, to reduce to Lie groups, Lazard's analyticity criteria [19] (which solve the analogue of Hilbert's fifth problem for p-adic Lie groups) play a role in the appearance of p-adic Lie groups as the torsion-free factors in Theorems A and B. More is involved however, because the existence of composition series, the splitting into torsion and torsion-free factors and the analysis of the torsion factors all require separate arguments.

Mention should be made of what Theorems A and B do *not* say. Contraction groups arise from arbitrary automorphisms $\alpha : G \to G$ of a topological group G for, if we define

$$U_\alpha := \{x \in G : \alpha^n(x) \to 1 \text{ as } n \to \infty\},$$

then U_α is an α-stable subgroup of G and $(U_\alpha, \alpha|_{U_\alpha})$ with the subspace topology is a contraction group. Returning to a totally disconnected, locally compact group G, the results in this paper apply to U_α if it is closed in G because then U_α is locally compact.[1] Contraction groups are automatically closed if G is a p-adic Lie group, [35], but not in general: consider the shift automorphism on $F^{\mathbb{Z}}$ for F a non-trivial finite group.

The contraction groups U_α are encountered in many contexts.

In representation theory the *Mautner phenomenon* is the fact that, if $\pi : G \to U(\mathscr{H})$ is a continuous unitary representation, and $v \in \mathscr{H}$ is fixed by $\pi(g)$ for some $g \in G$, then v is also fixed under $\pi(U_\alpha)$ for the inner automorphism $\alpha : G \to G$, $x \mapsto gxg^{-1}$ (see [21], Chapter II, Lemma 3.2; cf. [22], [24], [35] for special cases). A wider notion of contractibility is used in [25] in connection with certain questions of abstract harmonic analysis.

In probability theory continuous one-parameter semigroups, $(\mu_t)_{t \geqq 0}$, of probability measures on G with $\mu_0 = \delta_1$ are studied. Such semigroups are *semistable with respect to an automorphism* α of G if $\alpha_*(\mu_t) = \mu_{ct}$ for some $c \in \,]0,1[$, and in this case each μ_t is concentrated on U_α (see [13] or [32]). The embeddability of measures (or translates thereof) into continuous one-parameter semigroups and limit theorems for probability measures have been studied in [31] and [34] with emphasis on totally disconnected groups.

In the structure theory of totally disconnected, locally compact groups, contraction groups are related to the tools introduced in [36]. Indeed, closedness of U_α was characterized in terms of the structure theory in [1] for metrizable, totally disconnected locally compact groups G (also see [8]; the metrizability condition can now be omitted, [20]).

A more general concept that is useful for studying contraction groups is that of groups *contracting modulo a closed subgroup*. For $H \subseteqq G$ closed and α-stable, define $U_{\alpha/H}$ to be the set of all $x \in G$ such that $\alpha^n(x)H \to H$ in G/H as $n \to \infty$. Contraction modulo H arose in probability theory, where it was shown that the measures μ_t are concentrated on $U_{\alpha/H}$ if $(\mu_t)_{t \geqq 0}$ is a semistable semigroup that has μ_0 equal to Haar measure on a compact subgroup H (see [3], Introduction; cf. [14]). This concept also arises in structure theory because α is contractive on $\overline{U_\alpha}$ modulo a certain compact subgroup, [1]. It was shown in [3] that $U_{\alpha/H} = U_\alpha H$ if G is a p-adic Lie group (cf. [14] for the case of real Lie groups) and this was extended to all totally disconnected locally compact groups G and α-stable closed subgroups H in [1] and [20] (see also [8] for remarks about the non-metrizable case). Theorems A and B do not extend to locally compact groups contracting modulo a compact α-stable subgroup H, as the following example shows.

---

[1] More generally, the results apply if the topology induced by G on U_α can be refined to a locally compact group topology $\mathcal{O}$ such that α restricts to a contractive automorphism of $(U_\alpha, \mathcal{O})$. Such refinements (and non-locally compact generalizations) have been studied in [33]. They always exist if α is an analytic automorphism of a Lie group over a local field [11], Corollary 13.4.

Example. Let $\mathscr{T}_{p+1}$ be the homogeneous tree where every vertex has degree $p+1$ and let G be the group of automorphisms of $\mathscr{T}_{p+1}$ that have finite orbits (i.e. are elliptic) and stabilize a given end, ω, of $\mathscr{T}_{p+1}$. Then G is locally compact when equipped with the topology of pointwise convergence on finite sets of vertices. Fix a geodesic path, ℓ, in $\mathscr{T}_{p+1}$ that has ω as one end. Then the elements of G stabilizing all vertices on ℓ form a compact subgroup H of G. Let x be an automorphism of $\mathscr{T}_{p+1}$ that translates ℓ in the direction away from ω. Then the map $\alpha : G \to G$, $g \mapsto xgx^{-1}$ is an automorphism that contracts G to H and $\overline{U_\alpha} = G$. However G has no closed, normal α-stable subgroups, and so is simple as a contraction group modulo H, but nothing like Theorem A and B holds. In fact, the group, D, of unipotent upper triangular matrices over the field $\mathbb{Q}_p$, of p-adic numbers, and, T, of unipotent upper triangular matrices over the field $\mathbb{F}_p(\!(X)\!)$, of Laurent series over the finite field $\mathbb{F}_p$ both act on $\mathscr{T}_{p+1}$ as closed subgroups of G via the action of SL_2, [30], Chapter II. Both are subgroups of U_α but D is divisible while T is a torsion group.

Organization of the article. Sections 1 and 2 are of a preparatory nature. In Section 1, we compile several basic facts concerning contraction groups. In Section 2, we fix our terminology concerning topological groups with operators and formulate a criterion for the validity of a Jordan-Hölder Theorem, which can be verified for the cases of relevance (in Section 3). This is quite remarkable, because composition series can rarely be used with profit in the theory of topological groups (typically they need not exist, and if they do, then uniqueness of the composition factors cannot be insured). Sections 4 and 5 prepare the proof of the classification (given in Section 6). Notably, we show there that every simple totally disconnected contraction group is *pro-discrete*, i.e., every identity neighbourhood contains an open normal subgroup. As a tool for the proof of Theorem B, we explain in Section 7 how a canonical series of α-stable normal subgroups can be associated with a contraction group. The proof of the Structure Theorem is outlined in Section 8 and details are provided in Sections 9–11.

Acknowledgement. We are grateful to Udo Baumgartner for discussions in the early stages of this work. Moreover, we thank the referee for pointing out some simplifications.

1. Preliminaries

Let us agree on the following conventions concerning subgroups and automorphisms of a topological group G. All automorphisms α of G are assumed bicontinuous. A subgroup $H \leqq G$ is called *α-stable* if $\alpha(H) = H$ while it is *α-invariant* if $\alpha(H) \subseteq H$. If H is α-stable for each automorphism α of G, then H is called *topologically characteristic*. It is *topologically fully invariant* if it is invariant under each endomorphism of the topological group G. An automorphism α of G is called *contractive* if $\alpha^n(x) \to 1$ for all $x \in G$. The module of an automorphism α of a locally compact group G is defined as $\Delta_G(\alpha) := \dfrac{\lambda(\alpha(U))}{\lambda(U)}$, where $U \subseteq G$ is any non-empty, relatively compact, open subset and λ a left invariant Haar measure on G.

We recall various important facts concerning contractive automorphisms.

Proposition 1.1. *For each totally disconnected, locally compact group G and contractive automorphism $\alpha : G \to G$, the following holds*:

(a) *α is compactly contractive, i.e., for each compact subset $K \subseteq G$ and identity neighbourhood $W \subseteq G$, there is $N \in \mathbb{N}$ such that $\alpha^n(K) \subseteq W$ for all $n \geqq N$ (and hence also $K \subseteq \alpha^{-n}(W)$).*

(b) *G has a compact, open subgroup W such that $\alpha(W) \subseteq W$. If G is pro-discrete, then W can be chosen as a normal subgroup of G.*

(c) *If $G \neq \mathbf{1}$, then G is neither discrete nor compact.*

(d) *If $W \subseteq G$ is a relatively compact, open identity neighbourhood, then $\{\alpha^n(W) : n \in \mathbb{N}_0\}$ is a basis of identity neighbourhoods, and $G = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(W)$. In particular, G is first countable and σ-compact.*

(e) *The module $\Delta_G(\alpha^{-1})$ is an integer greater than or equal to 2 if $G \neq \mathbf{1}$.*

(f) *If G has a compact, open, normal subgroup, then G is pro-discrete.*

Proof. (a) See [35], Proposition 2.1, or [32], Lemma 1.4 (iv).

(b) See [32], Lemma 3.2 (i) and Remark 3.4 (2).

(c) See [32], 3.1.

(d) follows directly from (a); see also [32], 1.8 (a).

(e) For W as in (b), W is an open subgroup of the compact group $\alpha^{-1}(W)$ and thus

$$\Delta_G(\alpha^{-1}) = \frac{\lambda(\alpha^{-1}(W))}{\lambda(W)} = [\alpha^{-1}(W) : W] \in \mathbb{N}.$$

If $[\alpha^{-1}(W) : W]$ was 1, then we would have $W = \alpha^{-1}(W)$ and thus

$$G = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(W) = W,$$

whence G would be compact, contradicting (c).

(f) See [32], Remark 3.4 (2). □

If G is a topological group and $\alpha : G \to G$ a contractive automorphism, it is convenient to call (G, α) a *contraction group*.

In the following, all contraction groups are assumed locally compact and totally disconnected, unless the contrary is stated.

The proof of the classification hinges on the theory of analytic pro-p-groups. We refer to [4] for background information. Generalities concerning p-adic Lie groups can also be found in [2] and [29]. Standard facts from the theory of pro-finite groups and their Sylow subgroups (as provided in [27] or [37]) will be used freely. We shall say that a topological group G is *locally pro-p* if it has an open subgroup which is a pro-p-group.

2. Topological groups with operators

We are interested in series of α-stable closed subgroups of contraction groups, but also in series of α-stable, closed, normal subgroups. Topological groups with operators provide the appropriate language to deal with both cases simultaneously. They also enable us to formulate sufficient conditions for the validity of a Jordan-Hölder Theorem.

Definition 2.1. Let Ω be a set. A *topological Ω-group* is a Hausdorff topological group G, together with a map $\kappa : \Omega \times G \to G$, $(\omega, g) \mapsto \kappa(\omega, g) =: \omega.g$ such that $\kappa(\omega, \bullet) : G \to G$ is a continuous endomorphism of G, for each $\omega \in \Omega$. A subgroup H of a topological Ω-group G is called an *Ω-subgroup* if it is closed and $\Omega.H \subseteq H$. A continuous homomorphism $\phi : G \to H$ between topological Ω-groups is called an *Ω-homomorphism* if $\phi(\omega.g) = \omega.\phi(g)$ for all $\omega \in \Omega$ and $g \in G$.

Each Ω-subgroup $H \leqq G$ of a topological Ω-group G and each quotient G/N by a normal Ω-subgroup is a topological Ω-group in a natural way.

Remark 2.2. If (G, α) is a totally disconnected contraction group, we shall turn G into a topological Ω-group in two ways:

(a) $\Omega = \langle \alpha \rangle \leqq \mathrm{Aut}(G)$. Then $\langle \alpha \rangle$-subgroups are closed α-stable subgroups.

(b) $\Omega = \langle \mathrm{Int}(G) \cup \{\alpha\} \rangle \leqq \mathrm{Aut}(G)$, where $\mathrm{Int}(G)$ is the group of inner automorphisms of G. In this case, the Ω-subgroups of G are the closed α-stable normal subgroups of G.

Let G be a topological Ω-group. As might be expected, a series

$$\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G \tag{1}$$

is called an *Ω-series* if each G_j is an Ω-subgroup of G (and hence closed). It is an *Ω-refinement* of another Ω-series $\mathbf{1} = H_0 \lhd H_1 \lhd \cdots \lhd H_m = G$ if

$$\{H_0, \ldots, H_m\} \subseteq \{G_0, \ldots, G_n\},$$

and a *proper Ω-refinement* if $\{H_0, \ldots, H_m\}$ is a proper subset of $\{G_0, \ldots, G_n\}$. An *Ω-composition series* is an Ω-series (1) without repetitions which does not admit a proper Ω-refinement. Two Ω-series $\boldsymbol{S}$ and $\boldsymbol{T}$ are Ω-isomorphic if there is a bijection from the set of factors of $\boldsymbol{S}$ onto the set of factors of $\boldsymbol{T}$ such that corresponding factors are Ω-isomorphic as topological Ω-groups.

Definition 2.3. A totally disconnected contraction group (G, α) is called *simple* if it is a simple topological $\langle \alpha \rangle$-group, that is, $G \neq \mathbf{1}$ and G has no α-stable closed normal subgroups except for $\mathbf{1}$ and G.

Evidently, an $\langle \alpha \rangle$-series $\mathbf{1} = G_0 \lhd \cdots \lhd G_n = G$ of a contraction group (G, α) is an $\langle \alpha \rangle$-composition series if and only if all factors G_j/G_{j-1} are simple contraction groups.

We now formulate a criterion for the validity of the Jordan-Hölder Theorem.

Proposition 2.4. *Let G be a σ-compact, locally compact group with set of operators Ω. Assume that G satisfies the following "closed product property": For all Ω-subgroups $H_1, H_2 \leqq G$ such that H_2 normalizes H_1, the product $H_1 H_2$ is closed in G (and hence is an Ω-subgroup). Then the following holds:*

(a) (*Schreier Refinement Theorem*) *Any two Ω-series of G have Ω-isomorphic Ω-refinements.*

(b) (*Jordan-Hölder Theorem*) *If $\boldsymbol{S}$ is an Ω-composition series and $\boldsymbol{T}$ is any Ω-series of G, then $\boldsymbol{T}$ has an Ω-refinement which is an Ω-composition series and is Ω-isomorphic with $\boldsymbol{S}$. In particular, any two Ω-composition series of G are Ω-isomorphic.*

Proof. If $S \leqq G$, $H \leqq S$ and $N \lhd S$ are closed subgroups such that HN is closed, then H and HN/N are σ-compact, locally compact groups. Therefore

$$H/(H \cap N) \to HN/N, \quad x(H \cap N) \mapsto xN$$

is a topological isomorphism, by [16], (5.33). Using this information and the closed product property, we find that all subgroups occurring in the standard proofs of the Zassenhaus Lemma and the Schreier Refinement Theorem (as in [28], 3.1.1–3.1.2) are closed and that all relevant abstract isomorphisms are isomorphisms of topological groups. Thus (a) holds. Part (b) is a direct consequence. □

3. The Jordan-Hölder Theorem for contraction groups

We now verify the closed product property (from Proposition 2.4) for α-stable closed subgroups of a totally disconnected contraction group (G, α). As a consequence, a Jordan-Hölder Theorem holds for contraction groups. We shall also see that $\langle \alpha \rangle$-composition series always exist.

The following proposition is one of the main technical tools of this article. It ensures that an α-invariant closed subgroup of a contraction group (G, α) always is an open subgroup of a suitable α-stable closed subgroup of G.

Proposition 3.1. *Let (G, α) be a totally disconnected, locally compact contraction group and $H \leqq G$ be a closed subgroup such that $\alpha(H) \subseteq H$. Then*

(a) $S := \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(H)$ *is a closed α-stable subgroup of G, and H is open in S.*

Furthermore:

(b) *There exists a compact, open subgroup $K \leqq H$ such that $\alpha(K) \subseteq K$.*

(c) *If H is compact, then $[H : \alpha(H)] < \infty$.*

(d) *If H is normal in G, then also S is normal in G.*

Proof. (a) and (d): Since $\alpha(H) \subseteq H$, we have $\alpha^{-j}(H) \subseteq \alpha^{-j-1}(H)$ for all $j \in \mathbb{N}_0$, entailing that S is a subgroup of G (which is normal if so is H). By Proposition 1.1 (b), there exists a compact, open subgroup $W \leqq G$ such that $\alpha(W) \subseteq W$. We claim that

$$S \cap \alpha^{n_0}(W) = H \cap \alpha^{n_0}(W) \tag{2}$$

for some $n_0 \in \mathbb{N}_0$. If this claim is true, then $S \cap \alpha^{n_0}(W)$ is a compact identity neighbourhood in S, whence S is locally closed and hence closed in G (by [16], Theorem 5.9). Furthermore, H is open in S, as it contains the open set $S \cap \alpha^{n_0}(W)$. To prove (a), it therefore only remains to establish (2). We proceed in steps.

Step 1. We first note that the indices

$$\begin{aligned}\ell_n := [H \cap \alpha^n(W) : H \cap \alpha^{n+1}(W)] &= \big[\big(H \cap \alpha^n(W)\big)\alpha^{n+1}(W) : \alpha^{n+1}(W)\big]\\ &\leqq [\alpha^n(W) : \alpha^{n+1}(W)] = [W : \alpha(W)]\end{aligned}$$

are bounded. Furthermore, the sequence $(\ell_n)_{n \in \mathbb{N}_0}$ is monotonically increasing, as

$$\begin{aligned}\big[\big(H \cap \alpha^n(W)\big)\alpha^{n+1}(W) : \alpha^{n+1}(W)\big] &= \big[\big(\alpha(H) \cap \alpha^{n+1}(W)\big)\alpha^{n+2}(W) : \alpha^{n+2}(W)\big]\\ &\leqq \big[\big(H \cap \alpha^{n+1}(W)\big)\alpha^{n+2}(W) : \alpha^{n+2}(W)\big].\end{aligned}$$

Here, we applied the automorphism α to all subgroups, and then used that $\alpha(H) \subseteq H$. As a consequence, $(\ell_n)_{n \in \mathbb{N}}$ becomes stationary; there are $n_0 \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ such that $[H \cap \alpha^n(W) : H \cap \alpha^{n+1}(W)] = \ell$ for all $n \geqq n_0$.

Step 2. $\alpha^m\big(H \cap \alpha^n(W)\big) = H \cap \alpha^{n+m}(W)$, *for all* $n \geqq n_0$ *and* $m \in \mathbb{N}_0$. To see this, note first that $\alpha^m\big(H \cap \alpha^n(W)\big) = \alpha^m(H) \cap \alpha^{n+m}(W) \subseteq H \cap \alpha^{n+m}(W)$. Since

$$\big[\big(H \cap \alpha^n(W)\big)\alpha^{n+1}(W) : \alpha^{n+1}(W)\big] = \ell,$$

we find $x_1, \ldots, x_\ell \in H \cap \alpha^n(W)$ such that

$$\big(H \cap \alpha^n(W)\big)\alpha^{n+1}(W)/\alpha^{n+1}(W) = \{x_1\alpha^{n+1}(W), \ldots, x_\ell\alpha^{n+1}(W)\}$$

and thus $x_i^{-1}x_j \notin \alpha^{n+1}(W)$ whenever $i \neq j$. For each $k \in \mathbb{N}_0$, we then have

$$\alpha^k(x_1), \ldots, \alpha^k(x_\ell) \in \alpha^k(H) \cap \alpha^{n+k}(W) \subseteq H \cap \alpha^{n+k}(W)$$

and $\alpha^k(x_i)^{-1}\alpha^k(x_j) \notin \alpha^{n+k+1}(W)$ if $i \neq j$, entailing that

$$\big(H \cap \alpha^{n+k}(W)\big)\alpha^{n+k+1}(W)/\alpha^{n+k+1}(W) = \{\alpha^k(x_i)\alpha^{n+k+1}(W) : i = 1, \ldots, \ell\}$$

for all $k \in \mathbb{N}_0$. Thus, given $y \in H \cap \alpha^{n+m}(W)$, we find $i_0 \in \{1, \ldots, \ell\}$ such that $\alpha^m(x_{i_0})^{-1}y \in H \cap \alpha^{n+m+1}(W)$. Next, we find $i_1 \in \{1, \ldots, \ell\}$ such that

$$\alpha^{m+1}(x_{i_1})^{-1}\alpha^m(x_{i_0})^{-1}y \in H \cap \alpha^{n+m+2}(W).$$

Proceeding in this way, we obtain a sequence $(i_k)_{k \in \mathbb{N}_0}$ in $\{1, \ldots, \ell\}$ such that

$$\alpha^{m+k}(x_{i_k})^{-1}\cdots\alpha^m(x_{i_0})^{-1}y\in H\cap\alpha^{n+m+k+1}(W)$$

for all $k\in\mathbb{N}_0$ and thus $\alpha^m\big(x_{i_0}\alpha(x_{i_1})\cdots\alpha^k(x_{i_k})\big)\in y\alpha^{n+m+k+1}(W)$, whence

$$y=\lim_{k\to\infty}\alpha^m\big(x_{i_0}\alpha(x_{i_1})\cdots\alpha^k(x_{i_k})\big)=\alpha^m(x)$$

with $x:=\lim\limits_{k\to\infty}x_{i_0}\alpha(x_{i_1})\cdots\alpha^k(x_{i_k})\in H\cap\alpha^n(W)$ (as H is closed).

Step 3. By Step 2, we have $\alpha^{-m}(H)\cap\alpha^{n_0}(W)=H\cap\alpha^{n_0}(W)$ for all $m\in\mathbb{N}_0$, whence $S\cap\alpha^{n_0}(W)=H\cap\alpha^{n_0}(W)$. Thus (2) (and hence (a)) holds.

(b) Let $V\subseteqq S$ be a compact, open subgroup such that $\alpha(V)\subseteqq V$. Since H is open in S, we have $K:=\alpha^n(V)\subseteqq H$ for some $n\in\mathbb{N}$, and this is a subgroup with the desired properties.

(c) Since $\alpha|_S$ is an automorphism of S and H is open in S, the image $\alpha(H)$ is open in H and therefore has finite index if H is compact. □

Corollary 3.2. *Let (G,α) be a totally disconnected contraction group. Then the following holds*:

(a) *H_1H_2 is closed in G, for any closed subgroups $H_1,H_2\leqq G$ such that $\alpha(H_1)\subseteqq H_1$, $\alpha(H_2)\subseteqq H_2$, and H_2 normalizes H_1.*

(b) *For both $\Omega=\langle\alpha\rangle\leqq\mathrm{Aut}(G)$ and $\Omega=\langle\mathrm{Int}(G)\cup\{\alpha\}\rangle$, the topological Ω-group G has the closed product property.*

Proof. (a) By Proposition 3.1 (b), there exists a compact, open subgroup $W\subseteqq H_2$ such that $\alpha(W)\subseteqq W$. Then W normalizes H_1, and thus $H:=H_1W$ is a subgroup of G. We have $\alpha(H_1W)=\alpha(H_1)\alpha(W)\subseteqq H_1W$, and furthermore H_1W is closed in G because H_1 is closed and W is compact [16], (4.4). Hence $S:=\bigcup\limits_{n\in\mathbb{N}_0}\alpha^{-n}(H)$ is closed in G and H is open in S, by Proposition 3.1 (a). Since $H_1\subseteqq\alpha^{-1}(H_1)$ and $H_2\subseteqq\bigcup\limits_{n\in\mathbb{N}_0}\alpha^{-n}(W)$, we have $H_1H_2\subseteqq S$. As $H\subseteqq H_1H_2$, the product H_1H_2 is an open subgroup of S and thus closed.

(b) is a special case of (a). □

Having verified the closed product property, we obtain:

Theorem 3.3. *Let G be a totally disconnected, locally compact group, $\alpha\in\mathrm{Aut}(G)$ be contractive, and $\Omega:=\langle\alpha\rangle$ or $\Omega:=\langle\mathrm{Int}(G)\cup\{\alpha\}\rangle$. Then G admits an Ω-composition series, and the Schreier Refinement Theorem and the Jordan-Hölder Theorem hold in the form described in Proposition* 2.4.

Proof. The existence of an Ω-composition series follows from Lemma 3.5 below. The remainder holds by Proposition 2.4 and Corollary 3.2 (b). □

We complete the proof using a well-known fact (cf. [7], Proposition III.13.20):

Lemma 3.4. *Let α be an automorphism of a locally compact group G and N be an α-stable closed normal subgroup of G. Let $\bar{\alpha}$ be the automorphism induced by α on $G/N =: Q$. Then $\Delta_G(\alpha) = \Delta_N(\alpha|_N) \cdot \Delta_Q(\bar{\alpha})$.* □

Lemma 3.5. *Let (G, α) be a totally disconnected contraction group. Then the length of any $\langle\alpha\rangle$-series $\mathbf{1} = G_0 \triangleleft \cdots \triangleleft G_n = G$ without repetitions is bounded by the number of prime factors of $\Delta_G(\alpha^{-1})$, counted with multiplicities.*

Proof. Let $\alpha_j : G_j/G_{j-1} \to G_j/G_{j-1} =: Q_j$ be the contractive automorphism of Q_j induced by α. Then $\Delta_G(\alpha^{-1}) = \Delta_{Q_1}(\alpha_1^{-1})\Delta_{Q_2}(\alpha_2^{-1}) \cdots \Delta_{Q_n}(\alpha_n^{-1})$ by Lemma 3.4, where $1 < \Delta_G(\alpha_j^{-1}) \in \mathbb{N}$ for each $j \in \{1, \ldots, n\}$, by Proposition 1.1 (e). The asserted bound for n is now immediate. □

For later use, we record another important consequence of Proposition 3.1.

Corollary 3.6. *Let (G, α) and (H, β) be totally disconnected, locally compact contraction groups and $\phi : G \to H$ be a continuous homomorphism such that $\beta \circ \phi = \phi \circ \alpha$. Then $\phi(G)$ is β-stable, closed in H, and $\phi|^{\phi(G)} : G \to \phi(G)$ is a quotient map. In particular, if ϕ is injective, then ϕ is a topological isomorphism onto its image.*

Proof. If $V \subseteq G$ is an identity neighbourhood, then there exists a compact open subgroup $W \subseteq G$ such that $\alpha(W) \subseteq W$ and $W \subseteq V$ (cf. Lemma 1.1 (b) and (d)). Because $\phi(W)$ is a compact subgroup of H with $\beta\big(\phi(W)\big) = \phi\big(\alpha(W)\big) \subseteq \phi(W)$, Proposition 3.1 (a) shows that $S := \bigcup_{n \in \mathbb{N}_0} \beta^{-n}\big(\phi(W)\big)$ is a closed subgroup of H which has $\phi(W)$ as a compact, open subgroup. Since $G = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(W)$ and $\phi \circ \alpha = \beta \circ \phi$, we see that $\phi(G) = S$. Hence $\phi(G)$ is closed and $\phi(V)$ is an identity neighbourhood in $\phi(G)$ (as $\phi(W) \subseteq \phi(V)$), entailing that ϕ is open onto its image. □

Remark 3.7. Using Corollary 3.6, all standard facts concerning Remak decompositions of finite groups, as formulated in [28], 3.3.1–3.3.10, can be adapted directly to totally disconnected contraction groups, notably the Krull-Remak-Schmidt Theorem. The corollary ensures that all homomorphisms encountered in the classical proofs are continuous, and all isomorphisms bicontinuous.

4. Simple contraction groups are pro-discrete

In this section, we show that every non-trivial contraction group (G, α) has a non-trivial α-stable closed normal subgroup $S \triangleleft G$ which is pro-discrete. Therefore every simple contraction group is pro-discrete. This information is essential for the proof of the classification.

Proposition 4.1. *Let (G, α) be a totally disconnected contraction group.*

(a) *Then G has a largest closed normal α-stable subgroup $S^\alpha(G) := S$ possessing a compact, open, α-invariant subgroup which is normal in G. If G is non-trivial, then also S is non-trivial.*

(b) *S can be obtained as follows*: *Let* $W \leqq G$ *be a compact*, *open subgroup such that* $\alpha(W) \subseteq W$, *and* $N := \bigcap_{x \in G} xWx^{-1}$ *be its core. Then N is an* α*-invariant closed normal subgroup of G and* $S = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(N)$.

(c) *S is the set of all elements* $x \in G$ *with relatively compact conjugacy class. Hence S is topologically characteristic in G.*

Proof. We may assume without loss of generality that $G \neq \mathbf{1}$.

(a) and (b): Let $W \leqq G$ and N be as in (b); then clearly N is closed, and it is the largest normal subgroup of G contained in W. Furthermore, $\alpha(N) = \bigcap_{x \in G} x\alpha(W)x^{-1} \subseteq N$. Thus $S := \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(N)$ is a closed normal subgroup of G possessing N as an open subgroup, by Proposition 3.1 (a) and (d). If $H \lhd G$ is any α-stable closed normal subgroup of G possessing an α-invariant, compact, open subgroup $K \leqq H$ which is normal in G, then there is $m \in \mathbb{N}$ such that $\alpha^m(K) \subseteq W$ and thus $\alpha^m(K) \subseteq N$ (since $\alpha^m(K) \lhd G$), entailing that $H = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(\alpha^m(K)) \subseteq \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(N) = S$. Thus, it only remains to prove that $S \neq \mathbf{1}$. We proceed in steps.

4.2. For each $n \in \mathbb{N}_0$, the set $V_n := \bigcap_{x \in W} x\alpha^n(W)x^{-1}$ is compact, and it is the largest subgroup of $\alpha^n(W)$ which is normal in W. Now $O := \alpha^n(W)$ is open in W and hence has finite index in it. Thus $V_n = \bigcap_{xO \in W/O} x\alpha^n(W)x^{-1}$ is open in W. For each $k \in \mathbb{N}_0$, we have

$$\alpha^{-k}(V_n) = \bigcap_{x \in \alpha^{-k}(W)} x\alpha^{n-k}(W)x^{-1}. \tag{3}$$

Since $W \subseteq \alpha^{-k}(W)$, we see that $\alpha^{-k}(V_n)$ is normalized by W, for each $k \in \mathbb{N}_0$.

4.3. $\alpha^{-k}(V_n)$ *is a normal subgroup of W*, *for all* $n \in \mathbb{N}_0$ *and* $k \in \{0, 1, \ldots, n\}$. Indeed, we have $\alpha^{n-k}(W) \subseteq W$ and thus $\alpha^{-k}(V_n) \subseteq W$, by (3). As W normalizes $\alpha^{-k}(V_n)$, the assertion follows.

4.4. $\alpha^{-1}(V_n) \nsubseteq V_n$ *holds*, *for each* $n \in \mathbb{N}_0$. Otherwise $\alpha^{-k}(V_n) \subseteq V_n$ and thus

$$V_n \subseteq \alpha^k(V_n) \quad \text{for all } k \in \mathbb{N}_0. \tag{4}$$

Because $V_n \neq \mathbf{1}$, there exists an identity neighbourhood $P \subseteq G$ which is a proper subset of V_n. Then $\alpha^k(V_n) \subseteq P$ for large k, since α is compactly contractive. This contradicts (4).

4.5. *For each* $n \in \mathbb{N}_0$, *we have* $U_n := \alpha^{-n}(V_n) \nsubseteq \alpha(W)$. Indeed, otherwise $\alpha^{-1}(V_n) = \alpha^{n-1}(U_n) \subseteq \alpha^n(W) \subseteq W$. Hence $\alpha^{-1}(V_n)$ is a subgroup of $\alpha^n(W)$ which is normal in W (by 4.2). As V_n is the largest such subgroup (see 4.2), we have $\alpha^{-1}(V_n) \subseteq V_n$. This contradicts 4.4.

4.6. Since $U_n = \bigcap_{x \in \alpha^{-n}(W)} xWx^{-1}$, where $W \subseteq \alpha^{-1}(W) \subseteq \alpha^{-2}(W) \subseteq \cdots$ and

$$G = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(W),$$

we see that $U_1 \supseteqq U_2 \supseteqq \cdots$ and

$$\bigcap_{n \in \mathbb{N}_0} U_n = \bigcap_{x \in G} xWx^{-1} = N. \tag{5}$$

Since $U_n \cap (W \backslash \alpha(W)) \neq \emptyset$ by 4.5, the set $\{U_n \cap (W \backslash \alpha(W)) : n \in \mathbb{N}_0\}$ of compact sets has the finite intersection property, and thus $N \cap (W \backslash \alpha(W)) = \bigcap_{n \in \mathbb{N}_0} (U_n \cap (W \backslash \alpha(W))) \neq \emptyset$, showing that $N \neq \mathbf{1}$ and hence also $S \neq \mathbf{1}$. This completes the proof of (a) and (b).

(c) Since N is a compact normal subgroup of G, each $x \in N$ has a relatively compact conjugacy class. Hence also each $x \in \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(N) = S$ has a relatively compact conjugacy class. Conversely, let $x \in G$ have a relatively compact conjugacy class C. Since α is compactly contractive, $\alpha^m(C) \subseteqq W$ for some $m \in \mathbb{N}$. Then $\alpha^m(C) \subseteqq \bigcap_{x \in G} xWx^{-1} = N$ and hence $x \in \alpha^{-m}(N) \subseteqq S$. □

Note that $S^\alpha(G)$ is pro-discrete, by Proposition 1.1 (f). We readily deduce:

Corollary 4.7. *Every simple, totally disconnected contraction group is pro-discrete.* □

Remark 4.8. Elements of a topological group G with relatively compact conjugacy class are also called $[\mathrm{FC}]^-$-elements (cf. [12] and [26], Chapter 12, for further information). They always form a topologically characteristic subgroup. It is remarkable that this subgroup is closed in the case of a totally disconnected, locally compact contraction group (G, α), by Proposition 4.1.

5. Further technical tools

In this section, we compile two technical lemmas, which will be used to prove the classification. The first of these provides information concerning the closed, normal, α-invariant subgroups of a simple contraction group.

Lemma 5.1. *Let (G, α) be a simple totally disconnected contraction group. If $N \lhd G$ is a closed normal subgroup such that $\alpha(N) \subseteqq N$, then either $N = \mathbf{1}$ or N is open in G.*

Proof. By Proposition 3.1 (a) and (d), $S := \bigcup_{k \in \mathbb{N}_0} \alpha^{-k}(N)$ is an α-stable, closed normal subgroup of G possessing N as an open subgroup. Since (G, α) is simple we either have $S = \mathbf{1}$ or $S = G$. The assertion follows. □

The next lemma will be used later to identify those simple contraction groups which are abelian torsion groups. Here and in the following, two contraction groups (G, α) and (H, β) are called *isomorphic* if there exists an isomorphism of topological groups $\phi : G \to H$ such that $\beta \circ \phi = \phi \circ \alpha$.

Lemma 5.2. *Let (G, α) be a simple totally disconnected contraction group. If there exists a non-trivial, finite, normal subgroup $N \lhd G$, then (G, α) is isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift, for some finite, simple group F.*

Proof. Let $F \subseteq N$ be a minimal non-trivial normal subgroup of G. For $n \in \mathbb{N}_0$, consider the map

$$\phi_n : F^{\{0,1,\dots,n\}} \to G, \quad (x_0, \dots, x_n) \mapsto x_0\alpha(x_1)\cdots\alpha^n(x_n).$$

We show by induction on $n \in \mathbb{N}_0$ that ϕ_n is an injective homomorphism. This is trivial if $n = 0$. If ϕ_n is an injective homomorphism, then $F\alpha(F)\cdots\alpha^n(F)$ is a normal subgroup of G, whence also its image $M := \alpha(F)\cdots\alpha^{n+1}(F)$ is a normal subgroup of G. Hence either $F \cap M = \mathbf{1}$ (entailing that indeed ϕ_{n+1} is an injective homomorphism), or $F \cap M = F$ (and thus $F \subseteq M$), by minimality of F. If $F \subseteq M$, then $\alpha^{-1}(M) = F\alpha(F)\cdots\alpha^n(F) \subseteq M\alpha(F)\cdots\alpha^n(F) = M$ and thus $\alpha(M) = M$, as M is finite. Hence $\alpha^k(M) = M$ for each $k \in \mathbb{N}$, contradicting the fact that $\alpha^k(M) \to \mathbf{1}$ as α is compactly contractive.

There is a compact, open, normal subgroup $W \lhd G$ such that $\alpha(W) \subseteq W$, and a maximal number $k \in \mathbb{Z}$ such that $F \subseteq \alpha^k(W)$. Then $F \cap \alpha^{k+1}(W)$ is a normal subgroup of G and a proper subset of F (by maximality of k), and thus $F \cap \alpha^{k+1}(W) = \mathbf{1}$ by minimality of F. Hence, after replacing W with $\alpha^k(W)$, without loss of generality $F \subseteq W$ and $F \cap \alpha(W) = \mathbf{1}$. We define

$$\phi : F^{\mathbb{N}_0} \to G, \quad \phi\big((x_n)_{n\in\mathbb{N}_0}\big) := \lim_{n\to\infty} \phi_n(x_0, \dots, x_n);$$

the limits exist because $\phi_n(x_0, \dots, x_n)^{-1}\phi_{n+m}(x_0, \dots, x_{n+m}) \in \alpha^{n+1}(W)$ for all $n, m \in \mathbb{N}_0$. Each ϕ_n being a homomorphism, also ϕ is a homomorphism. Given $m \in \mathbb{N}_0$, the set $U_m := \{(x_n)_{n\in\mathbb{N}_0} \in F^{\mathbb{N}_0} : x_n = 1 \text{ for all } n < m\}$ is an identity neighbourhood in $F^{\mathbb{N}_0}$, and $\phi(U_m) \subseteq \alpha^m(W)$. Therefore ϕ is continuous at 1 and hence continuous. Furthermore, ϕ is injective. To see this, let $x = (x_n)_{n\in\mathbb{N}_0} \in F^{\mathbb{N}_0}$ such that $x \neq 1$. There exists a smallest integer $m \in \mathbb{N}_0$ such that $x_m \neq 1$. Then $\phi(x) = \alpha^m(x_m)y$ with

$$y = \lim_{n\to\infty} \alpha^{m+1}(x_{m+1})\cdots\alpha^{m+n}(x_{m+n}) \in \alpha^{m+1}(W).$$

If $\phi(x) = 1$, then $x_m^{-1} = \alpha^{-m}(y) \in F \cap \alpha(W) = \mathbf{1}$ and thus $x_m = 1$, which is a contradiction. Thus $\phi(x) \neq 1$, whence $\ker\phi = \mathbf{1}$ and ϕ is injective. The image K of ϕ is compact, and it is normal in G, being the closure of the normal subgroup $\bigcup_{n\in\mathbb{N}_0} \operatorname{im}\phi_n$. Furthermore, $\alpha(K) \subseteq K$ and $K \neq \mathbf{1}$. Hence K is open in G, by Lemma 5.1. Set $H := F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$, $V := F^{\mathbb{N}_0} \subseteq H$ and let $\sigma : H \to H$ be the right shift. Then $\sigma(V) \subseteq V$ and $\phi \circ \sigma|_V = \alpha \circ \phi$, by construction of ϕ. As α and σ are contractive automorphisms, [35], Proposition 2.2 shows that ϕ extends to a topological isomorphism $H \to G$ that intertwines σ and α. □

6. Proof of the classification

We are now in the position to prove the classification of the simple totally disconnected contraction groups described in Theorem A.

Classification of the abelian simple contraction groups. We first determine a system of representatives for the abelian simple contraction groups. For a discussion of automorphisms of totally disconnected, locally compact abelian groups, see also [6]. In the following theorem, C_p denotes the cyclic group of order p.

Theorem 6.1. *Let (G, α) be a simple totally disconnected contraction group which is abelian. Then G is locally pro-p for some prime p and (G, α) is either isomorphic to $C_p^{(-\mathbb{N})} \times C_p^{\mathbb{N}_0}$ with the right shift, or isomorphic to $(\mathbb{Q}_p^d, \beta)$ for some $d \in \mathbb{N}$ and a contractive $\mathbb{Q}_p$-linear automorphism $\beta : \mathbb{Q}_p^d \to \mathbb{Q}_p^d$ which does not leave any proper non-trivial vector subspace invariant.*

Proof. Let $W \leqq G$ be a compact, open subgroup such that $\alpha(W) \subseteqq W$. Then W has a non-trivial p-Sylow subgroup W_p for some prime p. Since W is abelian, W_p is unique (cf. [37], Proposition 2.2.2(d)). Since $\alpha(W_p) \leqq W$ is a pro-p-group, we have $\alpha(W_p) \subseteqq W_p$ (cf. [37], Proposition 2.2.2(b)). By Lemma 5.1, W_p is open in G. Hence G is locally pro-p, and we may assume that $W = W_p$. Then $[W : \alpha(W)] = p^m$ for some $m \in \mathbb{N}$. For each $n \in \mathbb{N}$, we write $W^{\{n\}} := \{x^n : x \in W\}$. Since $W^{\{p\}}$ is a closed, normal, α-invariant subgroup of G, either $W^{\{p\}} = \mathbf{1}$ or $W^{\{p\}}$ is open in G, by Lemma 5.1. If $W^{\{p\}} = \mathbf{1}$, then W (and hence G) is a torsion group of exponent p, whence (G, α) is isomorphic to $C_p^{(-\mathbb{N})} \times C_p^{\mathbb{N}_0}$, as a consequence of Lemma 5.2. If $W^{\{p\}}$ is an open subgroup of G, then $\alpha^n(W) \subseteqq W^{\{p\}}$ for some $n \in \mathbb{N}$. Thus

$$W^{\{p^k\}} \supseteqq \alpha^{nk}(W) \quad \text{for all } k \in \mathbb{N}, \tag{6}$$

by induction: we have

$$W^{\{p^{k+1}\}} = (W^{\{p^k\}})^{\{p\}} \supseteqq \big(\alpha^{nk}(W)\big)^{\{p\}} = \alpha^{nk}(W^{\{p\}}) \supseteqq \alpha^{nk}\big(\alpha^n(W)\big) = \alpha^{n(k+1)}(W).$$

Thus

$$\begin{aligned}[W : W^{\{p^k\}}] \leqq [W : \alpha^{nk}(W)] &= [W : \alpha(W)] \cdots [\alpha^{nk-1}(W) : \alpha^{nk}(W)] \\ &= [W : \alpha(W)]^{nk} = p^{nmk}\end{aligned}$$

and so

$$[W : W^{\{p^k\}}] \leqq p^{nmk} \quad \text{for each } k \in \mathbb{N}. \tag{7}$$

Using [4], Theorem 3.16, we deduce from (7) that the pro-p-group W has finite rank. Therefore G is a p-adic Lie group (see [4], Corollary 8.33). Since G is an abelian Lie group, its Lie algebra $L(G)$ is an abelian Lie algebra. Therefore the Campbell-Hausdorff multiplication coincides with the addition map $L(G) \times L(G) \to L(G)$, and we find an exponential map $\phi : P \to Q$ which is an isomorphism of topological groups from a compact, open additive subgroup $P \leqq L(G)$ onto a compact, open subgroup $Q \leqq G$. Set $\beta := L(\alpha) : L(G) \to L(G)$. After shrinking P and Q, we may assume that

$$\phi \circ \beta|_U = \alpha \circ \phi|_U \tag{8}$$

for some open subgroup $U \subseteqq P$ such that $\beta(U) \subseteqq P$. After shrinking U, we may assume that $V := \phi(U) = \alpha^N(W)$ for some $N \in \mathbb{N}$. Since $\alpha(V) \subseteqq V$, we deduce from (8) that $\beta(U) \subseteqq U$. Hence $\phi \circ \beta^n|_U = \alpha^n \circ \phi|_U$ for each $n \in \mathbb{N}$, entailing that β is a contractive automorphism of $L(G)$. Now (G, α) is isomorphic to $\big(L(G), \beta\big)$ by [35], Proposition 2.2. □

It remains to describe normal forms in the p-adic case.

Definition 6.2. Given a prime p, let $R_p \subseteq \mathbb{Q}_p[X]$ be the set of all irreducible monic[2] polynomials f whose roots in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ have absolute value < 1. For $f = X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in R_p$, we set $E_f := \mathbb{Q}^d$, let $e_1, \ldots, e_d \in \mathbb{Q}^d$ be the standard basis vectors and define α_f as the linear automorphism of E_f determined by $\alpha_f(e_j) = e_{j+1}$ for $j \in \{1, \ldots, d-1\}$ and $\alpha_f(e_d) = -\sum_{i=1}^{d} a_{i-1}e_i$. Thus f is the minimal polynomial of α_f (and its characteristic polynomial).

Note that R_p has continuum cardinality, as $\{X - a : a \in \mathbb{Q}_p, |a| < 1\} \subseteq R_p$.

Proposition 6.3. *The family* $(E_f, \alpha_f)_{f \in R_p}$ *is a system of representatives for the isomorphism classes of the simple totally disconnected contraction groups* (G, α) *such that* G *is abelian, torsion-free, and locally pro-p.*

Proof. Abbreviate $\mathbb{K} := \mathbb{Q}_p$ and let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{Q}_p$. Given (G, α) as described in the proposition, we may assume that $G = E$ is a finite-dimensional $\mathbb{K}$-vector space and α a continuous linear map, by Proposition 6.1. We consider E as a $\mathbb{K}[X]$-module via $X.v := \alpha(v)$ for $v \in E$. Then E is irreducible and thus $E \cong \mathbb{K}[X]/f\mathbb{K}[X]$ for a unique monic irreducible polynomial $f \in \mathbb{K}[X]$ (cf. [17], §3.9, Exercise 2). Given $r > 0$, let $\overline{E}_r$ be the sum of all generalized eigenspaces of $\alpha \otimes \overline{\mathbb{K}}$ in $E \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ to eigenvalues $\lambda \in \overline{\mathbb{K}}$ such that $|\lambda| = r$. Then $E = \bigoplus_{r>0} E_r$, where $E_r := \overline{E}_r \cap E$ is α-stable (see [21], p. 81), and thus $E = E_r$ for some $r > 0$, as (E, α) was assumed simple. There exists an ultrametric norm $\|.\|$ on $E = E_r$ such that $\|\alpha(x)\| = r\|x\|$ for each $x \in E$ (see [10], Proposition 4.3). Since α is contractive, it follows that $r < 1$ and thus $f \in R_p$. Let d be the degree of f. With respect to the basis corresponding to $X^0, \ldots, X^{d-1}$, the automorphism α has the same matrix as α_f with respect to $e_1, \ldots, e_d$, and thus (G, α) is isomorphic to (E_f, α_f).

Conversely, for each $f \in R_p$, the $\mathbb{K}[X]$-module E_f (with $X.v := \alpha_f(v)$) is irreducible and uniquely determines f (cf. [17], §3.9). By irreducibility, $E_f = (E_f)_r$ for some $r > 0$ (with notation as before), where $r < 1$ by definition of R_p. As there exists an ultrametric norm $\|.\|$ on $E_f = (E_f)_r$ such that $\|\alpha_f(v)\| = r\|v\|$ for each $v \in E_f$, we see that α_f is a contractive automorphism. To complete the proof, let $N \subseteq E_f$ be a non-trivial, α_f-stable closed additive subgroup. Then $\operatorname{span}_{\mathbb{Q}_p}(N)$ is an α_f-stable, non-trivial vector subspace of E_f and hence $E_f = \operatorname{span}_{\mathbb{Q}_p}(N)$, by irreducibility. Since N is open in $\operatorname{span}_{\mathbb{Q}_p}(N) = E_f$ and α_f is contractive, we deduce that $E_f = \bigcup_{n \in \mathbb{N}_0} \alpha_f^{-n}(N) = N$. Thus (E_f, α_f) is a simple contraction group. $\square$

By the preceding proof, for each $f \in R_p$ all eigenvalues λ of α_f in $\overline{\mathbb{Q}_p}$ have the same absolute value $r := |\lambda|$.

Classification of the non-abelian simple contraction groups. To classify the non-abelian simple contraction groups, we use a folklore lemma from group theory (which follows from Remak's Theorem, [28], 3.3.12).

---

[2] That is, with leading coefficient 1.

Lemma 6.4. *Let G be a group and $N_1, \ldots, N_n$ be pairwise distinct normal subgroups of G such that G/N_k is a non-abelian simple group, for each $k \in \{1, \ldots, n\}$. Abbreviate $D := N_1 \cap \cdots \cap N_n$ and $D_k := \bigcap_{j \neq k} N_j$. Then*

$$\theta : G/D \to G/N_1 \times \cdots \times G/N_n, \quad xD \mapsto (xN_1, \ldots, xN_n)$$

is an isomorphism of groups which takes D_k/D isomorphically onto G/N_k, for each $k \in \{1, \ldots, n\}$. □

Theorem 6.5. *Let (G, α) be a simple totally disconnected contraction group. If G is non-abelian, then (G, α) is isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift for a non-abelian, finite simple group F.*

Proof. We let $W \lhd G$ be a compact, open, normal subgroup such that $\alpha(W) \subseteqq W$. As G is non-abelian, there are $g, h \in G$ such that $ghg^{-1}h^{-1} \neq 1$. After applying a suitable power of α to both elements, we may assume that $h \in W$. There is $m \in \mathbb{N}$ such that $ghg^{-1}h^{-1} \notin \alpha^m(W)$. As a consequence, $g \notin \ker \phi$ for the homomorphism

$$\phi : G \to \mathrm{Aut}\big(W/\alpha^m(W)\big), \quad \phi(x)\big(w\alpha^m(W)\big) := xwx^{-1}\alpha^m(W).$$

Since $\alpha^m(W) \subseteqq \ker \phi$, we deduce that $N := \ker \phi$ is a proper, open, normal subgroup of G. The group $\mathrm{Aut}\big(W/\alpha^m(W)\big)$ being finite, N has finite index in G. Consequently, there exists a maximal proper normal subgroup M of G such that $N \subseteqq M$. Then M is open, and $F := G/M$ is a finite simple group. The closure $\overline{[G, G]}$ of the commutator subgroup is a non-trivial, α-stable closed normal subgroup of G and thus $\overline{[G, G]} = G$, by simplicity. Hence $[F, F] = F$ and thus F is non-abelian. Next, we observe that

$$\bigcap_{k \in \mathbb{Z}} \alpha^k(M) = \mathbf{1}, \tag{9}$$

because this intersection is an α-stable, closed, normal, proper subgroup of G and (G, α) is simple. Here $\alpha^k(M)$ is a normal subgroup of G such that $G/\alpha^k(M) \cong G/M = F$ is a non-abelian, simple group. If $k_1 \neq k_2$, say $k_2 > k_1$, then $\alpha^{k_1}(M) \neq \alpha^{k_2}(M)$ because otherwise $\alpha^{k_2 - k_1}(M) = M$, entailing that $\bigcap_{k \in \mathbb{Z}} \alpha^k(M) = \bigcap_{k=0}^{k_2 - k_1 - 1} \alpha^k(M)$ is open as a finite intersection of open sets, which is absurd. Let $\psi : G \to G/M = F$ be the quotient homomorphism and set $\psi_k := \psi \circ \alpha^{-k}$ for $k \in \mathbb{Z}$. Since $\ker \psi_k = \alpha^k(M)$, in view of the properties just established Lemma 6.4 shows that the map

$$\psi_{n,m} := (\psi_k)_{k=n}^{m} : G \to F^{\{n, \ldots, m\}}, \quad x \mapsto \big(\psi_n(x), \ldots, \psi_m(x)\big)$$

is surjective, for all $n, m \in \mathbb{Z}$ such that $n \leqq m$. Given $x \in G$, there is $k_0 \in \mathbb{Z}$ such that $x \in \alpha^k(M)$ for all $k \leqq k_0$. We can therefore define a homomorphism

$$\eta := (\psi_k)_{k \in \mathbb{Z}} : G \to F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}, \quad \eta(x) := \big(\psi_k(x)\big)_{k \in \mathbb{Z}}.$$

We let σ be the right shift on $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0} =: H$. Then $\sigma \circ \eta = \eta \circ \alpha$ by construction of η. To complete the proof, we show that η is an isomorphism of topological groups. First, η is injective, because $\ker \eta = \mathbf{1}$ by (9). Since $G = \bigcup_{n \in \mathbb{Z}} \alpha^{-n}(W)$ as an ascending union, there is

$n \in \mathbb{Z}$ such that $\alpha^{-n}(W) \nsubseteq M$. On the other hand, $\alpha^k(W) \subseteq M$ for large k since α is compactly contractive. Hence n can be chosen minimal. As a consequence, $W \subseteq \ker \psi_k$ for all $k < n$ while $W \nsubseteq \ker \psi_k$ for all $k \geqq n$. Thus $\eta(W) \subseteq F^{\{n,n+1,\ldots\}}$. Since H induces the product topology on $F^{\{n,n+1,\ldots\}}$, we see that $\theta := \eta|_W$ is continuous and hence also η. Let $m \geqq n$. Because W is normal in G and $\psi_{n,m} : G \to F^{\{n,\ldots,m\}}$ is surjective, the image $\psi_{n,m}(W)$ is a normal subgroup of the product $F^{\{n,\ldots,m\}}$ of non-abelian simple groups. By Remak's Theorem [28], 3.3.12, $\psi_{n,m}(W) = F^J$ for a subset $J \subseteq \{n,\ldots,m\}$. Since $\psi_k(W) \neq \mathbf{1}$ for each $k \in \{n,\ldots,m\}$, we see that $J = \{n,\ldots,m\}$ and thus $\psi_{n,m}(W) = F^{\{n,\ldots,m\}}$. As a consequence, θ has dense image. Now $\theta(W)$ being also compact and thus closed, we deduce that $\eta(W) = \theta(W) = F^{\{n,n+1,\ldots\}}$. Hence η has open image, and since $\psi_{n,m}$ is surjective for all $n,m \in \mathbb{Z}$ such that $n \leqq m$, we see that $\eta(G)$ is dense in H and hence equal to H. Because $\eta|_W$ is a homeomorphism onto its open image, η is an isomorphism of topological groups. □

Remark 6.6. The finite group F in Theorem 6.5 is uniquely determined up to isomorphism. To see this, note that every compact, open, normal subgroup $W \subseteq F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ such that $\sigma(W) \subseteq W$ is of the form $W = F^{\{n,n+1,\ldots\}}$ for some $n \in \mathbb{Z}$, and $W/\sigma(W) \cong F$.

7. Canonical α-stable series

We now describe how a series $\mathbf{1} = S_0^\alpha(G) \triangleleft S_1^\alpha(G) \triangleleft \cdots \triangleleft S_n^\alpha(G) = G$ of topologically characteristic (hence α-stable and normal) closed subgroups can be associated to each totally disconnected contraction group (G,α) in a canonical way. This series will serve as a technical tool in the proof of the Structure Theorem (Theorem B). It can also be used to show that $\langle \mathrm{Int}(G) \cup \{\alpha\}\rangle$-composition factors are pro-discrete (Proposition 7.3).

Definition 7.1. Let G be a totally disconnected, locally compact group and $\alpha : G \to G$ be a contractive automorphism. We define $S_0^\alpha(G) := \mathbf{1}$ and $S_1^\alpha(G) := S^\alpha(G)$ (as in Proposition 4.1 (a)), which is a topologically characteristic, closed subgroup of G (see Proposition 4.1). Inductively, having defined the topologically characteristic, closed subgroup $S_{j-1}^\alpha(G) \triangleleft G$, we set $Q_j := G/S_{j-1}^\alpha(G)$, let $q_j : G \to Q_j$ be the quotient map and $\alpha_j : Q_j \to Q_j$ be the contractive automorphism determined by $\alpha_j \circ q_j = q_j \circ \alpha$. Since $S^{\alpha_j}(Q_j)$ is topologically characteristic and closed in Q_j and $S_{j-1}^\alpha(G)$ is topologically characteristic in G, it follows that $S_j^\alpha(G) := q_j^{-1}(S^{\alpha_j}(Q_j))$ is a topologically characteristic, closed subgroup of G. By Lemma 3.5, the series $S_0^\alpha(G) \triangleleft S_1^\alpha(G) \triangleleft \cdots$ becomes stationary. Thus, there is a smallest $n \in \mathbb{N}_0$ such that $S_{n+1}^\alpha(G) = S_n^\alpha(G)$. We call $\mathbf{1} = S_0^\alpha(G) \triangleleft S_1^\alpha(G) \triangleleft \cdots \triangleleft S_n^\alpha(G) = G$ the *canonical α-stable series of G*.

Let us call an $\langle \mathrm{Int}(G) \cup \{\alpha\}\rangle$-series $\mathbf{1} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ *special* if G_i/G_{i-1} has a compact, open subgroup which is normal in G/G_{i-1} and invariant under the automorphism of G/G_{i-1} induced by α, for each $i \in \{1,\ldots,n\}$. The following proposition compiles various useful properties of the canonical α-stable series. In particular, it is special and ascends faster than any other special $\langle \mathrm{Int}(G) \cup \{\alpha\}\rangle$-series.

Proposition 7.2. *Let (G,α) be a totally disconnected contraction group and $\Omega := \langle \mathrm{Int}(G) \cup \{\alpha\}\rangle$. Then the following holds*:

(a) *The canonical α-stable series $\mathbf{1} = S_0^\alpha(G) \lhd S_1^\alpha(G) \lhd \cdots \lhd S_n^\alpha(G) = G$ is a special Ω-series without repetitions.*

(b) *$S_j^\alpha(G)/S_{j-1}^\alpha(G)$ is a pro-discrete, closed normal subgroup of $Q_j := G/S_{j-1}^\alpha(G)$, for each $j \in \{1, \ldots, n\}$.*

(c) *If $\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$ is any special Ω-series, then $m \geqq n$ and $G_j \subseteqq S_j^\alpha(G)$ for each $j \in \{0, \ldots, n\}$.*

Proof. (a) and (b): By construction, each $S_j^\alpha(G)$ is an α-stable, closed normal subgroup of G and hence an Ω-subgroup. If $G \neq \mathbf{1}$, then $S_1^\alpha(G) \neq \mathbf{1}$ by Proposition 4.1 (a) and thus $\mathbf{1} = S_0^\alpha(G) \subset S_1^\alpha(G)$. Likewise $S^{\alpha_j}(Q_j) \neq \mathbf{1}$ for $j \in \{1, \ldots, n\}$ and hence $S_{j-1}^\alpha(G) \subset q_j^{-1}\big(S^{\alpha_j}(Q_j)\big) = S_j^\alpha(G)$, using the notation of the preceding definition. Therefore no repetitions occur. By Proposition 4.1 (a), $S_j^\alpha(G)/S_{j-1}^\alpha(G) = S^{\alpha_j}(Q_j)$ has an α_j-invariant, compact, open subgroup W_j which is normal in Q_j. Hence the canonical α-stable series is a special Ω-series. In particular, W_j is a compact, open, normal subgroup of $S_j^\alpha(G)/S_{j-1}^\alpha(G) = S^{\alpha_j}(Q_j)$, and hence $S_j^\alpha(G)/S_{j-1}^\alpha(G)$ is pro-discrete by Proposition 1.1 (f).

(c) We show by induction on $i \in \{0, \ldots, n\}$ that $m \geqq i$ and $G_i \subseteqq S_i^\alpha(G)$. For $i = 0$, this is clear. Now assume that $i \in \{0, \ldots, n\}$, $m \geqq i - 1$, and $G_{i-1} \subseteqq S_{i-1}^\alpha(G)$. Since $S_{i-1}^\alpha(G)$ is a proper subset of G, by the preceding so is G_{i-1} and hence $m \geqq i$. By hypothesis, there exists a compact, open subgroup $K \subseteqq G_i/G_{i-1}$ which is normal in G/G_{i-1} and invariant under the automorphism of G/G_{i-1} induced by α. The continuous homomorphism $q : G/G_{i-1} \to G/S_{i-1}^\alpha(G)$, $xG_{i-1} \mapsto xS_{i-1}^\alpha(G)$ intertwines α and the contractive automorphism α' of $G/S_{i-1}^\alpha(G)$ induced by α. As a consequence of Corollary 3.6, $q(G_i/G_{i-1})$ is a closed, α'-stable normal subgroup of $G/S_{i-1}^\alpha(G)$ which has $q(K)$ as an open subgroup. Since $q(K)$ is normal in $G/S_{i-1}^\alpha(G)$ and α'-invariant, Proposition 4.1 (a) shows that $q(G_i/G_{i-1}) \subseteqq S^{\alpha'}\big(G/S_{i-1}^\alpha(G)\big) = S_i^\alpha(G)/S_{i-1}^\alpha(G)$. Therefore $G_i S_{i-1}^\alpha(G) \subseteqq S_i^\alpha(G)$ and thus $G_i \subseteqq S_i^\alpha(G)$. □

We know from Corollary 4.7 that composition factors of $\langle\alpha\rangle$-composition series are pro-discrete. As a first application of the canonical α-stable series, we now show that also $\langle \mathrm{Int}(G) \cup \{\alpha\}\rangle$-composition factors are pro-discrete.

Proposition 7.3. *Let (G, α) be a totally disconnected contraction group and $\Omega := \langle \mathrm{Int}(G) \cup \{\alpha\}\rangle$. Then the factors of each Ω-composition series of (G, α) are pro-discrete. Furthermore, (G, α) has an Ω-composition series which is a special Ω-series.*

Proof. Since all Ω-composition series are equivalent by the Jordan-Hölder Theorem (Theorem 3.3), to prove the first assertion it suffices to consider an Ω-composition series $\mathbf{1} = H_0 \lhd H_1 \lhd \cdots \lhd H_m = G$ which has been obtained by refining the canonical α-stable series of G (this is possible by the Schreier Refinement Theorem). Let $i \in \{1, \ldots, m\}$. Then $i \in \{k+1, \ldots, k+\ell\}$ for some $k \in \{0, \ldots, m-1\}$ and $\ell \in \{1, \ldots, m-k\}$ such that, for some $j \in \{1, \ldots, n\}$, $H_k = S_{j-1}^\alpha(G)$ and $H_{k+\ell} = S_j^\alpha(G)$. By Proposition 7.2 (a), $H_{k+\ell}/H_k = S_j^\alpha(G)/S_{j-1}^\alpha(G)$ has a compact, open subgroup W/H_k which is normal in G/H_k and invariant under the automorphism of G/H_k induced by α. Now standard arguments show that $(W \cap H_i)/H_{i-1}$ is a compact, open subgroup of H_i/H_{i-1} which is normal in G/H_{i-1} and invariant under the automorphism of H_i/H_{i-1} induced by α. □

8. Proof of the Structure Theorem

We now outline the main steps of the proof of the Structure Theorem (Theorem B from the Introduction). The details of Steps 2 to 4 will be given in Sections 9 to 11.

Throughout the following, (G, α) is a totally disconnected, locally compact contraction group (unless we state the contrary). Furthermore,

$$\mathbf{1} = G_0 \lhd \cdots \lhd G_n = G \tag{10}$$

is an $\langle\alpha\rangle$-composition series for G. By the classification, each factor G_j/G_{j-1} is pro-discrete and is isomorphic to either (a) $(\mathbb{Q}_p^d, +)$ for some prime p and some $d \in \mathbb{N}$; or to (b) a restricted product over $\mathbb{Z}$ of copies of a finite simple group. In case (a), G_j/G_{j-1} is divisible and torsion-free. In case (b), G_j/G_{j-1} is a torsion group of finite exponent.

It is useful to consider the special cases first where either all $\langle\alpha\rangle$-composition factors are torsion groups, or all of them are torsion-free.

Step 1. The case when all composition factors are torsion groups. If each of the factors G_j/G_{j-1} is a torsion group of finite exponent, then also G is a torsion group of finite exponent, as a special case of the following lemma (the proof of which is based on obvious inductive arguments):

Lemma 8.1. *Let* $\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ *be a series of groups.*

(a) *If* G_j/G_{j-1} *is a torsion group for each* $j \in \{1, \ldots, n\}$, *then* G *is a torsion group. If* G_j/G_{j-1} *is a torsion group of exponent* m_j *for each* $j \in \{1, \ldots, n\}$, *then* G *is a torsion group of finite exponent which divides* $m_1 \cdot \ldots \cdot m_n$.

(b) *If* G_j/G_{j-1} *is torsion-free for* $j = k, \ldots, n$, *then* $\operatorname{tor}(G) = \operatorname{tor}(G_{k-1})$. □

The next lemma implies that the exponent of a torsion factor G_j/G_{j-1} divides the module of the automorphism induced by α^{-1} on G_j/G_{j-1}. This information will be useful later.

Lemma 8.2. *Let* F *be a finite group,* $H := F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ *and* σ *be the right shift on* H. *Then* $\Delta_H(\sigma^{-1}) = |F|$.

Proof. For the compact open subgroup $W := F^{\mathbb{N}}$ of H, we have $\sigma^{-1}(W) = F^{\mathbb{N}_0}$ and $\Delta_H(\sigma^{-1}) = \lambda(\sigma^{-1}(W))/\lambda(W) = [\alpha^{-1}(W) : W] = |F|$. □

Step 2. The special case of torsion-free composition factors. The following proposition (proved in Section 9) describes the structure of contraction groups all of whose composition factors are torsion-free.

Proposition 8.3. *Let* (G, α) *be a totally disconnected contraction group possessing an* $\langle\alpha\rangle$*-composition series* $\mathbf{1} = G_0 \lhd \cdots \lhd G_n = G$ *such that* G_j/G_{j-1} *is torsion-free, for each* $j \in \{1, \ldots, n\}$. *Then* G *is an internal direct product* $G = G_{p_1} \times \cdots \times G_{p_r}$ *of certain nilpotent* p*-adic Lie groups* G_p. *Each* G_p *is topologically fully invariant in* G (*and hence* α*-stable*).

In the situation of Proposition 8.3, G is divisible and torsion-free, as a consequence of the next lemma.

Lemma 8.4. *Let G be a p-adic Lie group admitting a contractive automorphism α. Then G is divisible and torsion-free.*

Proof. Let $\exp : V \to U$ be an exponential map of G, which is a diffeomorphism from an open $\mathbb{Z}_p$-submodule $V \subseteqq L(G)$ onto an open subgroup $U \leqq G$. Then each $x \neq 1$ in U has the form $x = \exp(X)$ for some $X \neq 0$ in V. For each $n \in \mathbb{N}$, we then have $x^n = \exp(nX) \neq 1$, showing that U is torsion-free and hence also $G = \bigcup_{k \in \mathbb{N}_0} \alpha^{-k}(U)$. Furthermore, $\{x^n : x \in U\} = \exp(nV)$ is an identity neighbourhood in G consisting of elements possessing an n-th root. Hence every element of $G = \bigcup_{k \in \mathbb{N}_0} \alpha^{-k}\big(\exp(nV)\big)$ has an n-th root. □

Step 3. The set of torsion elements is a subgroup. If one of the composition factors in (10) is torsion, then it may be assumed that G has torsion elements and that G_1 is torsion (see Section 10). As a consequence, it may always be supposed that torsion factors appear first in the composition series:

Lemma 8.5. *Each totally disconnected contraction group (G, α) admits an $\langle\alpha\rangle$-composition series $\mathbf{1} = G_0 \lhd \cdots \lhd G_n = G$ such that, for suitable $k \in \{0, \dots, n\}$, the factors G_j/G_{j-1} are torsion groups for $j \in \{1, \dots, k\}$ and all other factors are torsion-free. In particular,* $\operatorname{tor}(G)$ *is equal to G_k and is a subgroup of G.*

The proof of Lemma 8.5 uses the following result.

Lemma 8.6. *If $\mathbf{1} = G_0 \lhd \cdots \lhd G_k$ is an $\langle\alpha\rangle$-composition series for G_k with G_i/G_{i-1} a torsion group for $i \in \{1, \dots, j\}$ and G_i/G_{i-1} a torsion-free group for $i \in \{j+1, \dots, k\}$, then $G_j = \operatorname{tor}(G_k)$ is a characteristic subgroup of G_k and G_k/G_j is torsion-free.*

Proof. Lemma 8.1(a) and (b) show that G_j is a torsion group and $\operatorname{tor}(G_k) = \operatorname{tor}(G_j) = G_j$. Thus G_j is a characteristic subgroup of G_k. We can apply Lemma 8.1(b) to $\mathbf{1} = G_j/G_j \lhd \cdots \lhd G_k/G_j$ since $(G_i/G_j)/(G_{i-1}/G_j) \cong G_i/G_{i-1}$ is torsion-free for $i = j+1, \dots, k$. Thus $\operatorname{tor}(G_k/G_j) = G_j/G_j = \mathbf{1}$. □

Now Lemma 8.5 readily follows: If $n = 0$ or if all factors G_j/G_{j-1} are torsion-free, or if all factors are torsion, there is nothing to show. Now assume that n is arbitrary and that G has torsion elements but is not a torsion group. By Lemma 10.1, we may assume that G_1/G_0 is torsion, whence there exist $k, m \in \{1, \dots, n\}$ with $m > k$ such that G_j/G_{j-1} is torsion for all $j \in \{1, \dots, k\}$ while G_j/G_{j-1} is torsion-free for $j \in \{k+1, \dots, m\}$ and m cannot be increased. We assume that the $\langle\alpha\rangle$-composition series has been chosen such that k is maximal. Then $G_k = \operatorname{tor}(G_m)$ by Lemma 8.1. If $m = n$, there is nothing more to show. Otherwise, G_{m+1}/G_m is a torsion group and since $G_k = \operatorname{tor}(G_m)$ is characteristic in G_m, we deduce that G_k is normal in G_{m+1}. Now the torsion factor in the composition series of G_{m+1}/G_k can be moved to the bottom, and hence G_{k+1}/G_k can be replaced by a torsion factor, contradicting the maximality of k. □

Step 4. Definition of a complementary subgroup *D*. We now choose the $\langle\alpha\rangle$-composition series (10) as described in Lemma 8.5. Thus G_j/G_{j-1} is torsion for

$j \in \{1, \dots, k\}$ while G_j/G_{j-1} is torsion-free and divisible for $j \in \{k+1, \dots, n\}$. Then $T := \operatorname{tor}(G) = G_k$ is a characteristic (and hence α-stable) subgroup of G.

Lemma 8.7. *Put* $t_\alpha := \Delta_T(\alpha^{-1}|_T)$. *Then* $x^{t_\alpha} = 1$ *for all* $x \in T$.

Proof. Consider the $\langle\alpha\rangle$-series $\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_k = T$. For $j \in \{1, \dots, k\}$, let α_j be the automorphism induced by α on $Q_j := G_j/G_{j-1}$ and put $t_j := \Delta_{Q_j}(\alpha_j^{-1})$. Then $t_\alpha = t_1 \cdots t_k$ (see proof of Lemma 3.5). Furthermore, Q_j is a torsion group of exponent dividing t_j (cf. Lemma 8.2). Thus T is a torsion group of finite exponent that divides t_α, by Lemma 8.1 (a). $\square$

Define $D := \overline{\langle x^{t_\alpha} \mid x \in G \rangle}$. Then D is a closed, topologically characteristic (and hence α-stable) subgroup of G. We record an essential property of D:

Lemma 8.8. *The map* $\phi : D \to G/T$, $\phi(x) := xT$ *is surjective.*

Proof. Let yT be in G/T, where $y \in G$. Since, by Lemma 8.4, G/T is divisible, there is $xT \in G/T$ such that $yT = (xT)^{t_\alpha}$. Then $yT = x^{t_\alpha}T$ belongs to the range of ϕ. $\square$

The following lemma (established in Section 11) completes the proof of the first half of the Structure Theorem:

Lemma 8.9. *D is a divisible group,* $D = \operatorname{div}(G)$, *and* $G = T \times D$.

Now also the second half of the Structure Theorem readily follows: Since $D \cap T = \{1\}$, the group D is torsion-free. Hence all composition factors of D are torsion-free (see Section 10) and therefore $D = G_{p_1} \times \cdots \times G_{p_r}$ is an internal direct product of α-stable p-adic Lie groups G_p, by Proposition 8.3.

9. The case of torsion-free factors

In this section, we prove Proposition 8.3, thus completing Step 2 of Section 8. The proof is based on the following lemma.

Lemma 9.1. *Let N be a closed normal subgroup of a topological group G and* $Q := G/N$.

(a) *If N and Q are p-adic Lie groups, then G is a p-adic Lie group.*

(b) *If Q is a q-adic Lie group and N an internal direct product* $N = \prod_{p \in \mathfrak{p}} N_p$ *of p-adic Lie groups N_p, where $\mathfrak{p}$ is a finite set of primes, then G has an open subgroup U that is an internal direct product* $U = \prod_{p \in \mathfrak{p} \cup \{q\}} U_p$ *of p-adic Lie groups U_p.*

(c) *If N is α-stable for a contractive automorphism α of G in the situation of* (b), *then G is an internal direct product* $G = \prod_{p \in \mathfrak{p} \cup \{q\}} G_p$ *of α-stable p-adic Lie groups G_p.*

Proof. (a) G is locally compact by [16], (5.25), and totally disconnected, whence it has a compact open subgroup U. Then $N \cap U$ and $U/(N \cap U) \cong UN/N \subseteq G/N$ are p-adic Lie groups, and after shrinking U both of these groups are pro-p-groups of finite rank (by [4], Corollary 8.33, and [37], Proposition 8.1.1 (a)). By [4], Proposition 1.11 (ii), and [37], Proposition 8.1.1 (b), also U is a pro-p-group of finite rank and thus G is a p-adic Lie group by [4], Corollary 8.33.

(b) We may assume that $q \in \mathfrak{p}$ (otherwise, define $N_q := \mathbf{1}$). For each $p \in \mathfrak{p}$, we let $V_p \subseteq N_p$ be an open subgroup which is a pro-p-group (see [4], Corollary 8.33). Let $U \subseteq G$ be a compact, open subgroup such that $U \cap N \subseteq \prod_{p \in \mathfrak{p}} V_p$ and such that $\pi(U)$ is a pro-q-group, where $\pi : G \to Q$ is the quotient map. It readily follows from [37], Proposition 2.4.3, that $U \cap N = \prod_{p \in \mathfrak{p}} (U \cap V_p)$ (see [9], Proposition 2.2); hence $U \cap N = \prod_{p \in \mathfrak{p}} V_p$ without loss of generality. Being the unique p-Sylow subgroup, V_p is topologically characteristic in the normal subgroup $U \cap N$ of U, and hence V_p is a normal subgroup of U. Given $p \in \mathfrak{p} \setminus \{q\}$, let U_p be a p-Sylow subgroup of U. Then $\pi(U_p) = \mathbf{1}$ and thus $U_p \subseteq U \cap N$, entailing that $U_p = V_p$ is a p-adic Lie group and a normal subgroup of U. Hence $M := \prod_{p \neq q} U_p$ is a normal subgroup of U. Let U_q be a q-Sylow subgroup of U containing V_q (see [37], Proposition 2.2.2 (c)). Then $M \cap U_q = \mathbf{1}$ and MU_q is a subgroup of U. By [37], Proposition 2.2.3 (b), $\pi(MU_q) = \pi(U_q)$ is a q-Sylow subgroup of $\pi(U)$ and thus $\pi(MU_q) = \pi(U)$. As MU_q is saturated under $U \cap N = MV_q$, we deduce that $U = MU_q$. Thus $U = M \rtimes U_q$. For $p \neq q$, the conjugation action of U_q on U_p gives rise to a continuous homomorphism $\phi_p : U_q \to \operatorname{Aut}(U_p)$, where $\operatorname{Aut}(U_p)$ is equipped with the compact-open topology (cf. [5], Theorem 3.4.1). Like every compact p-adic Lie group, U_p is topologically finitely generated. Hence $\operatorname{Aut}(U_p)$ has an open subgroup O_p which is a pro-p-group (see [4], Theorem 5.6, and [27], Theorem 4.4.2). Since every continuous homomorphism from a pro-q-group to a pro-p-group is trivial, we deduce that $\phi_p^{-1}(O_p) \subseteq \ker \phi_p$. Hence $W_q := \bigcap_{p \neq q} \phi_p^{-1}(O_p)$ is an open subgroup of U_q. After replacing U with its open subgroup MW_q and U_q with W_q, we may assume that U_q centralizes U_p for each $p \neq q$. Thus $U = M \times U_q = \prod_{p \in \mathfrak{p}} U_p$ as an internal direct product. It only remains to observe that U_q is a q-adic Lie group by (a), because $\pi(U_q) = \pi(U)$ and $N \cap U_q = V_q$ are q-adic Lie groups.

(c) Without loss of generality $q \in \mathfrak{p}$. By (b), G has a compact open subgroup U which is a direct product $U = \prod_{p \in \mathfrak{p}} U_p$ of p-adic Lie groups. After shrinking U, we may assume that each U_p is a pro-p-group and hence topologically fully invariant in U (being its unique p-Sylow subgroup). There exists a compact, open subgroup $W \leq G$ such that $W \subseteq U$ and $\alpha(W) \subseteq W$. Then $W = \prod_{p \in \mathfrak{p}} (W \cap U_p)$; after replacing U with W, we may assume that $\alpha(U) \subseteq U$ and thus $\alpha(U_p) \subseteq U_p$ for each $p \in \mathfrak{p}$. By Proposition 3.1 (a), $G_p := \bigcup_{k \in \mathbb{N}_0} \alpha^{-k}(U_p)$ is an α-stable closed subgroup of G which has U_p as an open subgroup. Hence G_p is a p-adic Lie group. Let $p_1, \ldots, p_r$ be the distinct elements of $\mathfrak{p}$ and consider the product map $\psi : \prod_{p \in \mathfrak{p}} G_p \to G$, $(x_{p_1}, \ldots, x_{p_r}) \mapsto x_{p_1} \cdots x_{p_r}$. Then $\beta := \alpha|_{G_{p_1}} \times \cdots \times \alpha|_{G_{p_r}}$ is a contractive automorphism of $\prod_{p \in \mathfrak{p}} G_p$ and ψ intertwines β and α,

$$\alpha \circ \psi = \psi \circ \beta. \tag{11}$$

We now use that ψ induces a bijection from $\prod_{p \in \mathfrak{p}} U_p$ onto U. Since ψ is an injective homomorphism on $\prod_{p \in \mathfrak{p}} U_p$ and $\bigcup_{k \in \mathbb{N}_0} \beta^{-k}\Big(\prod_{p \in \mathfrak{p}} U_p\Big) = \prod_{p \in \mathfrak{p}} G_p$, we deduce from (11) that ψ is injective and a homomorphism. Hence $G_p \lhd G$ for each p. Since $U \subseteqq \operatorname{im}(\psi)$ and $\bigcup_{k \in \mathbb{N}_0} \alpha^{-k}(U) = G$, using (11) we see that ψ is also surjective. Hence ψ is an isomorphism. □

Proof of Proposition 8.3. Each composition factor G_j/G_{j-1} being a p-adic Lie group for some prime p by Theorem A, a straightforward induction on n based on Part (c) of Lemma 9.1 shows that G is an internal direct product $G = G_{p_1} \times \cdots \times G_{p_r}$ of certain α-stable subgroups G_p which are p-adic Lie groups. Each G_p has an open pro-p subgroup U_p; then $\alpha^{-k}(U_p)$ also is a pro-p-group for each $k \in \mathbb{N}$, and $G_p = \bigcup_{k \in \mathbb{N}_0} \alpha^{-k}(U_p)$. Since, for $p \neq q$, each continuous homomorphism from a pro-p-group to a pro-q-group is trivial, we deduce that each endomorphism of the topological group G takes G_p to G_p. Thus G_p is topologically fully invariant. To complete the proof, we recall from [34], Theorem 3.5 (ii), that every p-adic contraction group is a unipotent p-adic algebraic group and hence nilpotent. □

10. Shifting torsion factors to the bottom

In this section, we prove the following lemma, which completes the details of Step 3 in Section 8.

Lemma 10.1. *Let* (G, α) *be a totally disconnected contraction group such that at least one* $\langle\alpha\rangle$*-composition factor of* G *is a torsion group. Then* G *has an* $\langle\alpha\rangle$*-composition series* (10) *such that* G_1 *is a torsion group. In particular*, G *has non-trivial torsion elements.*

Proof. If the lemma was false, we could find a counterexample with an $\langle\alpha\rangle$-composition series of minimal length $n \geqq 2$. By minimality, for each $\langle\alpha\rangle$-composition series $\mathbf{1} = G_0 \lhd \cdots \lhd G_n = G$, the factors G_j/G_{j-1} have to be torsion-free for all $j \in \{1, \ldots, n-1\}$, while G_n/G_{n-1} is a torsion group. By Proposition 8.3, $G_{n-1} = H_{p_1} \times \cdots \times H_{p_r}$ is a direct product of certain topologically characteristic (and hence α-stable) nilpotent p-adic Lie groups $H_p \neq \{1\}$. If $r \geqq 2$, then $K := H_{p_2} \times \cdots \times H_{p_r}$ is topologically characteristic in G_{n-1} and hence normal in G_n. We may assume that $K = G_j$ for some $j \in \{1, \ldots, n-2\}$. Because $Q := G/K$ has a properly shorter $\langle\alpha\rangle$-composition series than G, it has an $\langle\alpha\rangle$-composition series starting in a torsion factor and thus also G_{j+1}/G_j can be chosen as a torsion group, which is a contradiction. Thus $r = 1$ and G_{n-1} is a nilpotent p-adic Lie group for some p. The closed commutator subgroup C of G_{n-1} being topologically characteristic in G_{n-1}, arguing as before we reach a contradiction unless $C = \mathbf{1}$. Hence G_{n-1} is an abelian p-adic Lie group. The next two lemmas will help us to reach a final contradiction.

Lemma 10.2. *Let* (G, α) *be a totally disconnected contraction group and* $N \lhd G$ *be an* α*-stable closed normal subgroup. If* G/N *is a torsion group and* N *an abelian* p*-adic Lie group, then* N *is contained in the centre of* G.

Proof. As in the proof of Proposition 6.1, the fact that the abelian p-adic Lie group N is a contraction group implies that $N \cong \mathbb{Q}_p^d$ for some $d \in \mathbb{N}_0$. Hence $\operatorname{Aut}(N) \cong \mathrm{GL}_d(\mathbb{Q}_p)$, which we equip with its usual topology (of pointwise convergence). We now choose a torsion-free open subgroup $W \subseteq \mathrm{Aut}(N)$ (for example, we can take any open subgroup W isomorphic to a ball in $\mathfrak{gl}_d(\mathbb{Q}_p)$, equipped with the Campbell-Hausdorff multiplication). Since N is abelian, a homomorphism of groups can be defined via

$$\phi : G/N \to \mathrm{Aut}(N), \quad \phi(xN)(y) := xyx^{-1}.$$

Let $q : G \to G/N$ be the quotient map. Since $G \to N$, $x \mapsto \phi\big(q(x)\big)(y) = xyx^{-1}$ is continuous for each $y \in N$, it follows that $\phi \circ q$ is continuous and hence also ϕ. Thus $\phi^{-1}(W)$ is an identity neighbourhood in $Q := G/N$. Since Q is a torsion group and W is torsion-free, we must have $\phi^{-1}(W) \subseteq \ker\phi$ and thus $\ker\phi$ is open. Hence

$$Q = \bigcup_{k \in \mathbb{N}_0} \bar{\alpha}^{-k}(\ker\phi),$$

where $\bar{\alpha}$ denotes the contractive automorphism of Q induced by α. Since, as a simple calculation shows, $\bar{\alpha}^{-1}(\ker\phi) \subseteq \ker\phi$, it follows that $Q = \ker\phi$. Hence $xyx^{-1} = \phi(xN)(y) = y$ for each $x \in G$ and thus $N \subseteq Z(G)$. □

Lemma 10.3. *Let (G, α) be a totally disconnected contraction group, $A \subseteq G$ be a central, α-stable closed subgroup and $q : G \to Q$ be a quotient morphism with kernel A. Assume that $A \cong \mathbb{Q}_p^d$ and $Q \cong F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift σ, for a finite group F. Then* $\mathrm{tor}(G)$ *is a subgroup of G, and $G = A \times \mathrm{tor}(G)$ internally as a topological group.*

Proof. Without loss of generality $Q = F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$. We set $F_k := F^{\{-k,\dots,k\}}$ for $k \in \mathbb{N}$, $G_k := q^{-1}(F_k)$, and consider A as an F_k-module with the trivial action. For each $n \in \mathbb{N}$, the n-th cohomology group $H^n(F_k, A)$ with coefficients in A (as in [18], §6.9) is a $\mathbb{Q}_p$-vector space in a natural way and hence a torsion-free group. On the other hand, F_k being finite, $H^n(F_k, A)$ is a torsion group by [18], Theorem 6.14. Hence $H^n(F_k, A) = \{0\}$ for each $n \in \mathbb{N}$ and thus $H^2(F_k, A) = \{0\}$ in particular, entailing that the extension $A \to G_k \to F_k$ splits and thus $G_k = A \times S_k$ internally for some subgroup $S_k \leqq G_k$ (cf. [18], Theorem 6.15). Since A is torsion-free and $S_k \cong F_k$ a torsion group, we deduce that $S_k = \mathrm{tor}(G_k)$. In particular, S_k is uniquely determined and $S_k = \mathrm{tor}(G_k) \subseteq \mathrm{tor}(G_{k+1}) = S_{k+1}$, whence $S_\infty := \bigcup_{k \in \mathbb{N}} S_k$ and its closure $S := \overline{S_\infty}$ are subgroups of G. Each $S_k \cong F_k$ being a torsion group of exponent dividing $|F|$, also S_∞ and its closure S are torsion groups of exponent dividing $|F|$ and thus $S \cap A = \mathbf{1}$, because A is torsion-free.

From $q\big(\alpha(G_k)\big) = \sigma\big(q(G_k)\big) = \sigma(F_k) \subseteq F_{k+1}$ we deduce that $\alpha(G_k) \subseteq G_{k+1}$ and thus $\alpha(S_k) = \alpha\big(\mathrm{tor}(G_k)\big) \subseteq \mathrm{tor}(G_{k+1}) = S_{k+1}$, entailing that $\alpha(S_\infty) \subseteq S_\infty$. Likewise, $\alpha^{-1}(S_\infty) \subseteq S_\infty$, whence $\alpha(S_\infty) = S_\infty$ and $\alpha(S) = S$. By Corollary 3.6, $q(S)$ is closed in Q. Since $q(S) \supseteq q(S_\infty) = \bigcup_{n \in \mathbb{N}} F_n = F^{(\mathbb{Z})}$, where $F^{(\mathbb{Z})}$ is dense in Q, we see that $q(S) = Q$. Hence $G = A \times S$ internally as a group and hence also as a topological group (cf. Corollary 3.6). Since A is torsion-free and $S \cong Q$ a torsion group, $S = \mathrm{tor}(G)$ follows. □

Proof of Lemma 10.1, *completed.* Since G_n/G_{n-1} is a torsion group, and is a factor of an $\langle\alpha\rangle$-composition series, it is isomorphic to a restricted product of copies of a finite group. Lemmas 10.2 and 10.3 show that G_n has a closed subgroup isomorphic to this restricted product. This subgroup is characteristic in G_n and can be chosen as G_1. We have reached a contradiction. $\square$

11. Proof that D has the desired properties

In this section, we prove Lemma 8.9, thus completing the details of Step 4 from Section 8.

To prove Lemma 8.9, we use induction on the length of the canonical α-stable series of G; the induction starts because the case $G = \mathbf{1}$ is trivial. For general G, let $S := S_1^{\alpha}(G)$ be the first term in the canonical series for G. Then G/S has a shorter canonical series and so, by the inductive hypothesis, $G/S = \tilde{T} \times \tilde{D}$, where $\tilde{T}$ is the torsion subgroup of G/S and $\tilde{D} := \overline{\langle x^{\tilde{t}_\alpha} : x \in G/S \rangle}$ is a divisible subgroup, where $\tilde{t}_\alpha$ is the module of the automorphism of $\tilde{T}$ induced by α^{-1}. Let $q : G \to G/S$ be the quotient map. Then $T \subseteqq q^{-1}(\tilde{T})$.

By Proposition 4.1 (a), there is a compact, open subgroup $N \leqq S$ such that $N \lhd G$, $\alpha(N) \subseteqq N$ and $S = \bigcup_{j \in \mathbb{Z}} \alpha^{-j}(N)$. Since α is an automorphism, $\alpha^j(N) \lhd G$ for every $j \in \mathbb{N}$. Consider, for each $j \in \mathbb{N}$, the finite group $N/\alpha^j(N)$ and define a homomorphism

$$\phi_j : G \to \operatorname{Aut}\bigl(N/\alpha^j(N)\bigr) \quad \text{by } \phi_j(x) : y\alpha^j(N) \mapsto xyx^{-1}\alpha^j(N) \quad \text{for } y \in N.$$

Then $\ker(\phi_j)$ is a finite index normal subgroup of G for each j and so there is a positive integer, d_j, such that $x^{d_j} \in \ker(\phi_j)$ for every $x \in G$. Since α is an automorphism, we have that $x^{d_j} \in \alpha^{\ell}\bigl(\ker(\phi_j)\bigr)$ for every $x \in G$ and $\ell \in \mathbb{Z}$. Define $M_j := \bigcap_{\ell \in \mathbb{Z}} \alpha^{\ell}\bigl(\ker(\phi_j)\bigr)$. Then M_j is a closed α-stable normal subgroup of G for each j.

It is clear that $\ker(\phi_{j+1}) \subseteqq \ker(\phi_j)$ for each j. Hence $(M_j)_{j \in \mathbb{N}}$ is a decreasing sequence of closed α-stable normal subgroups of G. As $\Delta_{M_{j+1}}(\alpha^{-1}|_{M_{j+1}})$ is a positive integer strictly less than $\Delta_{M_j}(\alpha^{-1}|_{M_j})$ if M_{j+1} is a proper normal subgroup of M_j (by Proposition 1.1 (e) and Lemma 3.4), this sequence eventually stabilizes. Thus there is a J such that $M_j = M_J$ for all $j \geqq J$.

Lemma 11.1. *M_J is equal to the centralizer of S.*

Proof. It is clear from the definition of M_j that the centralizer of S is a subgroup of M_j for every j.

For the converse, let $s \in S$. By the definition of N, there is an $\ell \in \mathbb{Z}$ such that $s \in \alpha^{\ell}(N)$. If $x \in M_J$, then for every $j \geqq J$ we have $xsx^{-1}\alpha^{\ell+j}(N) = s\alpha^{\ell+j}(N)$. Since $\bigcap_{j \geqq J} \alpha^{\ell+j}(N) = \mathbf{1}$ (because α is compactly contractive), it follows that $xsx^{-1} = s$. $\square$

The subgroup M_J is not trivial if G has a torsion-free composition factor.

Lemma 11.2. *The map $\phi : D \cap M_J \to G/T$, $\phi(x) := xT$ is surjective.*

Proof. Let d be the least common multiple of t_α (from Lemma 8.7) and d_J. Exploiting that $x^d \in D \cap M_J$ for every $x \in G$, we can repeat the argument used to prove Lemma 8.8. $\square$

The kernel of the homomorphism in Lemma 11.2 is equal to $D \cap M_J \cap T$ and so there is an exact sequence

$$\mathbf{1} \to D \cap M_J \cap T \to D \cap M_J \to G/T \to \mathbf{1}. \tag{12}$$

Recalling that D is defined to be $D = \overline{\langle x^{t_\alpha} \mid x \in G \rangle}$ and similarly for $\tilde{D}$, the following lemma implies that $q(D) \subseteqq \tilde{D}$. Hence $q(D \cap T) \subseteqq \tilde{D} \cap \tilde{T} = \mathbf{1}$ and thus $D \cap T \subseteqq S$, whence $D \cap M_J \cap T$ is contained in the centre of $D \cap M_J$, by Lemma 11.1. Thus, (12) is a central extension.

Lemma 11.3. *$\tilde{t}_\alpha$ divides t_α.*

Proof. By the proof of Lemma 8.7, t_α is the product of the modules of the automorphisms induced by α^{-1} on the composition factors of T. By Lemma 8.5, the latter coincide with those composition factors of G which are torsion groups. Likewise, $\tilde{t}_\alpha$ is the product of the modules of the automorphisms induced by α^{-1} on those composition factors of G/S which are torsion groups. As the latter are among the composition factors of G which are torsion groups, we deduce that $\tilde{t}_\alpha$ divides t_α. □

Two algebraic results will help us to discuss the central extension (12).

Lemma 11.4. *Let H be a nilpotent group which admits a central series $\mathbf{1} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ such that H/H_j is torsion-free for each $j \in \{0, \ldots, n\}$. Then roots in H are unique: If $x^m = y^m$ for certain $x, y \in H$ and $m \in \mathbb{N}$, then $x = y$.*

Proof. The proof is by induction on n. If $n = 1$, then H is abelian and torsion-free. Thus $x^m = y^m$ entails that $(y^{-1}x)^m = 1$, whence $y^{-1}x = 1$ and $x = y$. If $n > 1$, then the inductive hypothesis applies to H/H_1, whence $xH_1 = yH_1$. Since $H_1 \subseteqq Z(H)$, we have $x = yz$ for some element z in the centre of H. Thus $y^m = x^m = y^m z^m$ and hence $z^m = 1$. Since H is torsion-free, we infer that $z = 1$ and thus $x = y$. □

Lemma 11.5. *Let $\mathbf{1} \to C \to H \to H/C \to \mathbf{1}$ be a central extension, where C is a torsion group of finite exponent and $Q := H/C$ a divisible nilpotent group admitting a central series $\mathbf{1} = Q_0 \lhd Q_1 \lhd \cdots \lhd Q_m = Q$ such that Q_j is divisible and Q/Q_j is torsion-free for each $j \in \{0, \ldots, m\}$. Then there exists a divisible subgroup, L, of H such that $H = C \times L$. Furthermore, $C = \operatorname{tor}(H)$ and $L = \operatorname{div}(H)$ are fully invariant subgroups of H.*

Proof. It suffices to prove the first assertion (because the final assertion is an immediate consequence). We first note that $C = \operatorname{tor}(H)$ because H/C is torsion-free. Hence C is a fully invariant subgroup. Let $d \in \mathbb{N}$ be such that $c^d = 1$ for every $c \in C$. We first show that each coset, xC, in H/C contains a unique element that is divisible by kd for every $k \in \mathbb{N}$. Since H/C is divisible, there is $yC \in H/C$ such that $(yC)^{kd} = xC$. Hence $y^{kd} \in xC$. Because H/C satisfies the hypotheses of the preceding lemma, the coset yC is unique. If yc is another element of yC, then $(yc)^{kd} = y^{kd}$ because c belongs to the centre of H and $c^d = 1$. Hence y^{kd} is the unique element of xC that is divisible by kd. If k' is another element of $\mathbb{N}$, then there is $z \in H$ such that $z^{kk'd} = y^{kd}$. Hence y^{kd} is also the unique element of xC divisible by $k'd$.

Let $L := \{y^d : y \in H\}$. Then $LC/C = H/C$, and the above argument shows that each element of L is divisible by kd for each $k \in \mathbb{N}$. Moreover, $L \cap C = \mathbf{1}$, as we know that C contains a unique element u divisible by kd for each $k \in \mathbb{N}$ (and the neutral element is such an element u). We assert: *L is a group and is complementary to C*. The proof is by induction on m.

Assume $m = 1$ first; then H/C is abelian. Given $x_1, x_2 \in L$, we have $x_i = y_i^{2d}$ for some $y_i \in H$. Since $z := y_1^{-1}y_2^{-1}y_1y_2$ belongs to C, we have $z^d = 1$ and thus $x_1x_2 = (y_1y_2)^{2d}z^{\frac{1}{2}2d(2d-1)} = (y_1y_2)^{2d}z^{d(2d-1)} = (y_1y_2)^{2d} \in L$. It is clear that L is closed under inverses and so L is a group. It is also clear from its definition that L is a characteristic subgroup. Since $L \cap C = \mathbf{1}$ and $LC/C = H/C$, we have $LC = H$ and hence $H = C \times L$.

Now let $m > 1$ and assume that the assertion holds if m is replaced by $m - 1$. Let H' be the preimage of Q_1 under the quotient map $H \to H/C$. Then $H'/C = Q_1$ is a divisible, torsion-free abelian group, whence the extension $\mathbf{1} \to C \to H' \to H'/C \to \mathbf{1}$ satisfies the hypotheses of the lemma. By the abelian case already discussed, $H' = C \times L'$ with L' a divisible and characteristic subgroup of H'. The group L' is normal in H and the extension $\mathbf{1} \to CL'/L' \to H/L' \to (H/L')/(CL'/L') \to \mathbf{1}$ satisfies the hypotheses of the lemma, because $CL'/L' \cong C/(C \cap L')$ is a torsion group of finite exponent and $(H/L')/(CL'/L') \cong H/(CL') = H/H' \cong Q_m/Q_1$ is a nilpotent group isomorphic to Q_m/Q_1 which admits the central series $\mathbf{1} = Q_1/Q_1 \lhd \cdots \lhd Q_m/Q_1$ of length $m - 1$ where Q_j/Q_1 is divisible and $(Q_m/Q_1)/(Q_j/Q_1) \cong Q_m/Q_j$ torsion-free for all $j \in \{1, \dots, m\}$. Hence $H/L' = (CL'/L') \times L''$ for a divisible subgroup L'' of H/L', by the case $m - 1$. We claim: *The preimage of L'' under the quotient map $q : H \to H/L'$ is equal to L*. If this is true, then L is a characteristic subgroup of H. Since $L \cap C = \mathbf{1}$ and $LC/C = H/C$, we have $LC = H$ and hence $H = C \times L$. In particular, $L \cong H/C$ is divisible.

It only remains to verify the claim. Since CL'/L' has exponent dividing d, we have $q(y^d) \in L''$ for each $y \in H$ and thus $q(L) \subseteq L''$. If $x \in L''$, then there exists $w \in L''$ such that $w^d = x$. Taking $y \in H$ such that $w = q(y)$, we have $y^d \in L$ and $q(y^d) = x$, showing that $q(L) = L''$. Hence $L = q^{-1}(L'')$ will follow if we can show that $LL' \subseteq L$. To this end, let $x_1 \in L$, $x_2 \in L'$. Then $x_1 = y_1^{2d}$ for some $y_1 \in H$, and $x_2 = y_2^{2d}$ for some $y_2 \in L'$. Since $L'C/C \subseteq Q_1$ is contained in the centre of H/C, we have $z := y_1^{-1}y_2^{-1}y_1y_2 \in C$ and thus $L \ni (y_1y_2)^{2d} = y_1^{2d}y_2^{2d}z^{d(2d-1)} = y_1^{2d}y_2^{2d} = x_1x_2$. This completes the proof. □

To obtain information concerning the central extension (12), we now consider the canonical α-stable series $\mathbf{1} = Q_0 \lhd \cdots \lhd Q_m = G/T$ of G/T. Since all $\langle\alpha\rangle$-composition factors of G/T are torsion-free by Lemma 10.1, also all $\langle\alpha\rangle$-composition factors of Q_m/Q_j are torsion-free, as well as those of Q_j, for $j \in \{0, \dots, m\}$. Hence Q_j and Q_m/Q_j are torsion-free, divisible, nilpotent groups by Proposition 8.3 and Lemma 8.4. Thus Lemma 11.5 implies that (12) splits as an extension of abstract groups. Write

$$D \cap M_J = (D \cap M_J \cap T) \times L,$$

say, where $L \cong G/T$. Then $L = \operatorname{div}(D \cap M_J)$ is a characteristic subgroup of $D \cap M_J$. Since $D \cap M_J$ is a normal and α-stable subgroup of G, it follows that L is a normal and α-stable subgroup of G. We also have that $G = LT$, by Lemma 11.2, and that $L \cap T = \mathbf{1}$, since L is torsion-free. Therefore, $G = T \times L$ as an abstract group. We need more:

Lemma 11.6. *L is closed in G and $G = T \times L$ as a topological group.*

Proof. Pick a compact, open subgroup $U \leqq G$ and let $\pi : G \to G/T$ be the quotient map. Since G/T is a product of p-adic Lie groups for certain primes p, we find primes $p_1, \ldots, p_m$ and continuous homomorphisms $\xi_i : \mathbb{Z}_{p_i} \to G/T$ such that, for each $k \in \mathbb{N}_0$,

$$V_k := \xi_1(p_1^k \mathbb{Z}_{p_1})\xi_2(p_2^k \mathbb{Z}_{p_2}) \cdots \xi_m(p_m^k \mathbb{Z}_{p_m})$$

is a compact, open subgroup of $\pi(U)$. To see this, recall that each p-adic Lie group admits coordinates of the second kind, and apply the ultrametric inverse function theorem. Since $\pi|_U : U \to \pi(U)$ is a quotient homomorphism between pro-finite groups, each ξ_i lifts to a continuous homomorphism $\theta_i : \mathbb{Z}_{p_i} \to U$ such that $\pi \circ \theta_i = \xi_i$ (this is clear from standard facts of pro-finite Sylow theory, notably [37], Proposition 2.2.3 (b)). There exists $k \in \mathbb{N}$ such that $t_\alpha^{-1} p_i^k \in \mathbb{Z}_{p_i}$ for all $i \in \{1, \ldots, m\}$, entailing that all elements of $\theta_i(p_i^k \mathbb{Z}_{p_i})$ are divisible by t_α in G. Hence $\theta_i(p_i^k \mathbb{Z}_{p_i}) \subseteqq L$ and thus

$$W := \theta_1(p_1^k \mathbb{Z}_{p_1})\theta_2(p_2^k \mathbb{Z}_{p_2}) \cdots \theta_m(p_m^k \mathbb{Z}_{p_m}) \subseteqq L.$$

Note that $\pi(W) = V_k$. If $x, y \in W$, then $\pi(x)\pi(y) \in V_k$ and thus $\pi(x)\pi(y) = \pi(z)$ for some $z \in W$. Since also $\pi(xy) = \pi(z)$, where both xy and z are in L, we deduce from the injectivity of $\pi|_L$ that $xy = z$. Hence W is a subgroup of L. Since W is compact, the bijective continuous homomorphism $\pi|_W^{V_k}$ is an isomorphism of topological groups. Therefore the homomorphism $(\pi|_L)^{-1} : G/T \to L$ is continuous on the identity neighbourhood V_k and hence continuous. Thus $G = T \times L$ as a topological group, and thus L is closed. $\square$

Proof of Lemma 8.9, *completed.* Since $G = T \times L$, Lemma 8.7 implies that $x^{t_\alpha} \in L$ for each $x \in G$. Thus $D \overset{\text{def}}{=} \overline{\langle x^{t_\alpha} : x \in G \rangle} \subseteqq \overline{L} = L$, using Lemma 11.6. Conversely, $L \subseteqq D$, because L is divisible. Hence $L = D$ and thus $G = T \times D$ internally as a topological group. Since T has finite exponent and D is divisible, it follows that $D = \operatorname{div}(G)$. $\square$

References

[1] *U. Baumgartner* and *G. A. Willis*, Contraction groups and scales of automorphisms of totally disconnected locally compact groups, Israel J. Math. **142** (2004), 221–248.

[2] *N. Bourbaki*, Lie Groups and Lie Algebras, Chapters 1–3, Springer-Verlag, 1989.

[3] *S. G. Dani* and *R. Shah*, Contraction subgroups and semistable measures on p-adic Lie groups, Math. Proc. Camb. Phil. Soc. **110** (1991), 299–306.

[4] *J. D. Dixon*, *M. P. F. du Sautoy*, *A. Mann* and *D. Segal*, Analytic pro-p Groups, Cambridge Univ. Press, 1991.

[5] *R. Engelking*, General Topology, Heldermann Verlag, Berlin 1989.

[6] *S. Evans-Riley* and *G. A. Willis*, Automorphisms of abelian locally compact topological groups, manuscript, August 2000.

[7] *J. M. G. Fell* and *R. S. Doran*, Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles, Vol. I, Academic Press, 1988.

[8] *H. Glöckner*, Contraction groups for tidy automorphisms of totally disconnected groups, Glasgow Math. J. **47** (2005), 329–333.

[9] *H. Glöckner*, Locally compact groups built up from p-adic Lie groups, for p in a given set of primes, J. Group Th. **9** (2006), 427–454.

[10] *H. Glöckner*, Lectures on Lie groups over local fields, preprint, arXiv:0804.2234v3.

[11] *H. Glöckner*, Invariant manifolds for analytic dynamical systems over ultrametric fields, preprint, arXiv:0808.2709v2.

[12] *S. Grosser* and *M. Moskowitz*, Compactness conditions in topological groups, J. reine angew. Math. **246** (1971), 1–40.

[13] *W. Hazod*, Remarks on (semi-) stable probabilities, Heyer, H., ed., Probability Measures on Groups VII, Lect. Notes Math. **1064**, Springer-Verlag (1984), 182–203.

[14] *W. Hazod* and *E. Siebert*, Automorphisms on a Lie group contracting modulo a compact subgroup and applications to semistable convolution semigroups, J. Theor. Probab. **1** (1988), 211–225.

[15] *W. Hazod* and *S. Siebert*, Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups, Kluwer, Dordrecht 2001.

[16] *E. Hewitt* and *K. A. Ross*, Abstract Harmonic Analysis I, Springer-Verlag, 1963.

[17] *N. Jacobson*, Basic Algebra I, Freeman and Co., New York 1985.

[18] *N. Jacobson*, Basic Algebra II, Freeman and Co., New York 1989.

[19] *M. Lazard*, Groupes analytiques p-adiques, Publ. Math. IHES **26** (1965), 5–219.

[20] *W. Jaworski*, On contraction groups of automorphisms of totally disconnected locally compact groups, Israel J. Math. **172** (2009), 1–8.

[21] *G. A. Margulis*, Discrete Subgroups of Semisimple Lie Groups, Springer-Verlag, 1991.

[22] *F. J. Mautner*, Geodesic flows on symmetric Riemannian spaces, Ann. Math. **65** (1957), 416–431.

[23] *D. Montgomery* and *L. Zippin*, Topological Transformation Groups, Interscience, New York 1955.

[24] *C. C. Moore*, The Mautner phenomenon for general unitary representations, Pac. J. Math. **86** (1980), 155–169.

[25] *P. R. Müller-Römer*, Kontrahierende Erweiterungen und kontrahierbare Gruppen, J. reine angew. Math. **283/284** (1976), 238–264.

[26] *T. W. Palmer*, Banach Algebras and the General Theory of $*$-Algebras, Vol. 2, Cambridge University Press, 2001.

[27] *L. Ribes* and *P. Zalesskii*, Profinite Groups, Springer-Verlag, 2000.

[28] *D. J. S. Robinson*, A Course in the Theory of Groups, Springer-Verlag, 1982.

[29] *J.-P. Serre*, Lie Algebras and Lie Groups, Springer-Verlag, 1992.

[30] *J.-P. Serre*, Trees, Springer-Verlag, 2003, corrected second printing of the First English Edition of 1980.

[31] *R. Shah*, Semistable measures and limit theorems on real and p-adic groups, Monatsh. Math. **115** (1993), 191–213.

[32] *E. Siebert*, Contractive automorphisms on locally compact groups, Math. Z. **191** (1986), 73–90.

[33] *E. Siebert*, Semistable convolution semigroups and the topology of contraction groups, H. Heyer, ed., Probability Theory on Groups IX, Lect. Notes Math. **1379**, Springer-Verlag (1989), 325–343.

[34] *K. Telöken*, Limit theorems for probability measures on totally disconnected groups, Semigr. Forum **58** (1999), 69–84.

[35] *J. S. P. Wang*, The Mautner phenomenon for p-adic Lie groups, Math. Z. **185** (1984), 403–412.

[36] *G. A. Willis*, The structure of totally disconnected, locally compact groups, Math. Ann. **300** (1994), 341–363.

[37] *J. S. Wilson*, Profinite Groups, Oxford University Press, 1998.

Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany
e-mail: glockner@math.upb.de

Department of Mathematics, University of Newcastle, Callaghan, NSW 2308, Australia
e-mail: George.Willis@newcastle.edu.au

Eingegangen 27. Mai 2008, in revidierter Fassung 12. Februar 2009