THE ESSENTIALLY CHIEF SERIES OF A COMPACTLY GENERATED LOCALLY COMPACT GROUP

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ABSTRACT. We first obtain finiteness properties for the collection of closed normal subgroups of a compactly generated locally compact group. Via these properties, every compactly generated locally compact group admits an essentially chief series – i.e. a finite normal series in which each factor is either compact, discrete, or a topological chief factor. A Jordan–Hölder theorem additionally holds for the 'large' factors in an essentially chief series.

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1. INTRODUCTION

Within the theory of locally compact groups, an important place is occupied by the compactly generated groups. From a topological group theory perspective, every locally compact group is the directed union of its compactly generated subgroups, so problems of a 'local' nature can be reduced to the compactly generated case. From a geometric perspective, such groups admit a well-defined geometric structure and are the natural generalization of finitely generated groups. Finally and most concretely, many examples of locally compact groups of independent interest are compactly generated. Any locally compact group that acts continuously, properly, and cocompactly on a proper metric space is compactly generated; for example $\mathrm{Aut}(\Gamma)$ is such a group for *any* Cayley graph Γ of a finitely generated group.

There is an emerging structure theory of compactly generated locally compact groups which reveals that they have special properties, often in a form

that has no or trivial counterpart in the theory of finitely generated discrete groups. This theory could be said to begin with the paper *Decomposing locally compact groups into simple pieces* of P.-E. Caprace and N. Monod, [1], in which general results on the normal subgroup structure of compactly generated locally compact groups are derived.

The key insight of Caprace and Monod is to study compactly generated locally compact groups as *large-scale topological objects*. That is to say, they observe that non-trivial interactions between local structure and large-scale structure place significant restrictions on compactly generated locally compact groups. Of course, these restrictions will always be up to compact groups and discrete groups; e.g. these results are insensitive to, say, taking a direct product with a discrete group. We stress that this theory nonetheless can yield non-trivial results for discrete groups; consider [9] or [10].

The work at hand is a further contribution to the (large-scale topological) structure theory of compactly generated locally compact groups. We first establish finiteness conditions for the lattice of closed normal subgroups. These conditions are then used to prove the existence of a finite chief series, up to compact groups and discrete groups, in any compactly generated locally compact group.

Remark 1.1. Compactly generated locally compact groups are second countable modulo a compact normal subgroup; see [3, Theorem 8.7]. We thus restrict to the second countable case whenever convenient.

1.1. **Statement of results.** A **normal factor** of a topological group G is a quotient K/L such that K and L are distinct closed normal subgroups of G with $L < K$. We say that K/L is a (topological) **chief factor** of G if there are no closed normal subgroups of G strictly between L and K.

Definition 1.2. An **essentially chief series** for a locally compact group G is a finite series

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of closed normal subgroups such that each normal factor G_{i+1}/G_i is either compact, discrete, or a topological chief factor of G.

Every compactly generated locally compact group G admits an essentially chief series; indeed, any finite normal series can be refined to an essentially chief series.

Theorem 1.3 (See Theorem 4.4)**.** *Suppose that G is a compactly generated locally compact group. If $(G_1, G_2, \ldots, G_m)$ is an increasing sequence of closed normal subgroups of G, then there exists an essentially chief series*

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_n = G$$

for G such that $\{G_1, \ldots, G_m\}$ is a subset of $\{K_0, \ldots, K_n\}$.

Corollary 1.4. *Every compactly generated locally compact group admits an essentially chief series.*

A Jordan–Hölder theorem additionally holds for essentially chief series. For our Jordan–Hölder theorem, the *association* classes of chief factors are uniquely determined:

Definition 1.5 (See [7]). For a topological group G, normal factors K_1/L_1 and K_2/L_2 are **associated** if $\overline{K_1 L_2} = \overline{K_2 L_1}$ and $K_i \cap \overline{L_1 L_2} = L_i$ for $i = 1, 2$.

If chief factors K_1/L_1 and K_2/L_2 of G are associated, then there is a third normal factor K/L of G such that each K_i/L_i admits a G-equivariant continuous monomorphism into K/L with a dense normal image; see [7, Lemma 6.6]. Associated chief factors are not in general isomorphic as topological groups. The association relation is also not an equivalence relation in general, but it becomes one when restricted to non-abelian chief factors; see [7, Proposition 6.8].

We must also ignore certain small chief factors.

Definition 1.6. For an l.c.s.c. group G, a chief factor K/L is called **negligible** if it is either abelian or associated to a compact or discrete factor.

Theorem 1.7 (see Theorem 4.8). *Suppose that G is a locally compact second countable group and that G has two essentially chief series $(A_i)_{i=0}^m$ and $(B_j)_{j=0}^n$. Define*

$$I := \{i \in \{1, \ldots, m\} \mid A_i/A_{i-1} \text{ is a non-negligible chief factor of } G\}; \text{and}$$

$$J := \{j \in \{1, \ldots, n\} \mid B_j/B_{j-1} \text{ is a non-negligible chief factor of } G\}.$$

Then there is a bijection $f : I \to J$ where $f(i)$ is the unique element $j \in J$ such that A_i/A_{i-1} is associated to B_j/B_{j-1}.

In [8], we show negligible non-abelian chief factors are limited in topological complexity. Specifically, they are either connected and compact or totally disconnected with dense quasi-center. We also show the chief factors themselves have a tractable internal normal subgroup structure.

Remark 1.8. The essentially chief series seems to be a useful tool with which to study compactly generated locally compact groups. Via the existence of the series, questions can often be reduced to chief factors, which are topologically characteristically simple. The essentially chief series can also be used to establish non-existence results. For example, the uniqueness given by Theorem 4.8 ensures any finitely generated just infinite branch group has no infinite commensurated subgroups of infinite index; see [10].

Our results follow from a finiteness property of the lattice of closed normal subgroups, which generalizes results of Caprace–Monod in [1]. The essential tool is given by the **Cayley–Abels graph**; this graph is a connected locally finite graph on which the group in question acts vertex-transitively with compact open point stabilizers. The *finite degree* of such graphs along with the usual dimension from Lie theory provide the finiteness from which all other finiteness properties in this paper are deduced.

A family of closed normal subgroups $\mathcal{F}$ is **filtering** if for all $N, M \in \mathcal{F}$, there is $K \in \mathcal{F}$ with $K \leq N \cap M$. A family $\mathcal{D}$ is **directed** if for all $N, M \in \mathcal{D}$, there is $K \in \mathcal{D}$ with $N \cup M \subseteq K$.

Theorem 1.9 (See Theorem 3.3). *Let G be a compactly generated locally compact group.*

(1) *If $\mathcal{F}$ is a filtering family of closed normal subgroups of G, then there exists $N \in \mathcal{F}$ and a closed normal subgroup K of G such that $\bigcap \mathcal{F} \leq K \leq N$, $K/\bigcap \mathcal{F}$ is compact, and N/K is discrete.*

(2) *If $\mathcal{D}$ is a directed family of closed normal subgroups of G, then there exists $N \in \mathcal{D}$ and a closed normal subgroup K of G such that $N \leq K \leq \overline{\langle \mathcal{D} \rangle}$, K/N is compact, and $\overline{\langle \mathcal{D} \rangle}/K$ is discrete.*

Theorem 1.9 implies additionally the existence of interesting quotients. For a property of groups P, a topological group G is called **just-non-P** if G does not have P, but every proper non-trivial Hausdorff quotient G/N has property P.

Theorem 1.10 (See Theorem 3.5). *Let P be a property of compactly generated locally compact groups. If all groups with P are compactly presented and P is closed under compact extensions, then for any compactly generated locally compact group G, exactly one of the following holds:*

(1) *Every non-trivial quotient of G (including G itself) has P; or*

(2) *G admits a quotient that is just-non-P.*

Any quasi-isometry invariant property is stable under compact extensions, so these properties occasionally fall under the previous theorem.

Corollary 1.11. *Let G be a compactly generated locally compact group that does not have polynomial growth. Then G admits a non-trivial quotient that is just-not-(of polynomial growth).*

2. Preliminaries

2.1. **Notations and generalities.** All groups are taken to be Hausdorff topological groups and are written multiplicatively. Topological group isomorphism is denoted by $\simeq$. We use "t.d.", "l.c.", and "s.c." for "totally disconnected", "locally compact", and "second countable."

The collection of closed normal subgroups of a topological group G is denoted by $\mathcal{N}(G)$. The connected component of the identity is denoted by G°. For any subset $K \subseteq G$, $C_G(K)$ is the collection of elements of G that centralize every element of K. We denote the collection of elements of G that normalize K by $N_G(K)$. The topological closure of K in G is denoted by $\overline{K}$. If G acts on a set X, $G_{(x)}$ denotes the stabilizer of $x \in X$ in G.

A topological group is **Polish** if it is separable and admits a complete, compatible metric. A locally compact group is Polish if and only if it is second countable; cf. [4, (5.3)].

For a poset $\mathcal{P}$, a **filtering family** $\mathcal{F} \subseteq \mathcal{P}$ in $\mathcal{P}$ is a subset of $\mathcal{P}$ such that for all $N, M \in \mathcal{F}$, there exists $L \in \mathcal{F}$ with $L \leq M$ and $L \leq N$. Dual to this notion, $\mathcal{D} \subseteq \mathcal{P}$ is a **directed family** if for all $M, N \in \mathcal{D}$, there is $L \in \mathcal{D}$ with $M \leq L$ and $N \leq L$.

2.2. **Chief factors and chief blocks.** We here recall the basic theory established in [7]. In the present work, this theory is lightly used to establish the Jordan–Hölder theorem.

Definition 2.1. A **normal factor** of a topological group G is a quotient K/L such that K and L are distinct closed normal subgroups of G with $L < K$. We say that K/L is a (topological) **chief factor** of G if there are no closed normal subgroups of G strictly between L and K.

There is a natural notion of 'equivalence' between chief factors:

Definition 2.2. For a topological group G, normal factors K_1/L_1 and K_2/L_2 are **associated** if $\overline{K_1 L_2} = \overline{K_2 L_1}$ and $K_i \cap \overline{L_1 L_2} = L_i$ for $i = 1, 2$.

Association is not an equivalence relation in general, but it becomes one when restricted to the set of non-abelian chief factors of a topological group G; see [7, Proposition 6.8]. For a non-abelian chief factor K/L, the equivalence class of non-abelian chief factors equivalent to K/L is denoted by $[K/L]$. The class $[K/L]$ is called a **chief block** of G, and the set of chief blocks of G is denoted by $\mathfrak{B}_G$.

Given a Polish group G and normal subgroups $N \leq M$ of G, we say that M/N **covers** $[K/L]$ if there exist closed normal subgroups $N \leq B < A \leq M$ of G for which A/B is a non-abelian chief factor associated to K/L. Otherwise we say that M/N **avoids** $[K/L]$. We say that M covers or avoids $[K/L]$ if $M/\{1\}$ does.

Our Jordan–Hölder theorem is a consequence of the following general refinement theorem.

Theorem 2.3 ([7, Theorem 1.14])**.** *Let G be a Polish group, K/L be a non-abelian chief factor of G, and*

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

be a series of closed normal subgroups in G. Then there is exactly one $i \in \{0, \ldots, n-1\}$ such that G_{i+1}/G_i covers $[K/L]$.

2.3. **Background on locally compact groups.** A closed subgroup K of a locally compact group G is **cocompact** if the coset space G/K is compact when equipped with the quotient topology.

A locally compact group G is **locally elliptic** if every finite subset of G is contained in a compact subgroup. The **locally elliptic radical**, denoted

by $\mathrm{Rad}_{\mathcal{LE}}(G)$, is the union of all closed normal locally elliptic subgroups of G.

Theorem 2.4 (Platonov, [6]). *For G a locally compact group, $\mathrm{Rad}_{\mathcal{LE}}(G)$ is the unique largest locally elliptic closed normal subgroup of G, and*

$$\mathrm{Rad}_{\mathcal{LE}}(G/\mathrm{Rad}_{\mathcal{LE}}(G)) = \{1\}.$$

A (real) **Lie group** is a topological group that is a finite-dimensional analytic manifold over $\mathbb{R}$ such that the group operations are analytic maps. A Lie group G can have any number of connected components, but G° is always an *open* subgroup of G. The group G/G° of components is thus discrete.

Theorem 2.5 (Gleason–Yamabe; see [5, Theorem 4.6]). *Let G be a locally compact group. If G/G° is compact, then $\mathrm{Rad}_{\mathcal{LE}}(G)$ is compact, and the quotient $G/\mathrm{Rad}_{\mathcal{LE}}(G)$ is a Lie group with finitely many connected components.*

Theorem 2.5 suggests a notion of dimension applicable to all locally compact groups.

Definition 2.6. For a locally compact group G, the **non-compact real dimension**, denoted by $\dim_{\mathbb{R}}^\infty(G)$, is the dimension of $G^\circ/\mathrm{Rad}_{\mathcal{LE}}(G^\circ)$ as a real manifold.

The non-compact real dimension is always finite. It is *superadditive*, not subadditive, with respect to extensions. Additionally, $\dim_{\mathbb{R}}^\infty(G) = 0$ if and only if G is compact-by-(totally disconnected).

The following technical consequence of Theorem 2.5 will be useful later:

Lemma 2.7. *Suppose G is a locally compact group with closed normal subgroups $H < L$. If L is connected, then there is a closed normal subgroup $K \trianglelefteq G$ such that $H^\circ \leq K \leq H$ with K/H° compact and H/K discrete. In particular, K/K° is compact.*

Proof. Let $R := \mathrm{Rad}_{\mathcal{LE}}(L)$. The group R is a compact normal subgroup of G and is such that L/R is a Lie group via Theorem 2.5. The group HR/R is a closed subgroup of the Lie group L/R, hence HR/R is a Lie group. Additionally, the connected component of HR/R equals $H^\circ R/R$. The group $K := H^\circ R \cap H$ satisfies the lemma. $\square$

The basic structural properties of Lie groups stated in the proposition below are classical and will be used without further comment.

Proposition 2.8.
(1) *The only connected abelian Lie groups are groups of the form $\mathbb{R}^a \times \mathbb{T}^b$ for $a, b \geq 0$ where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the circle group.*
(2) *The factors in the closed derived series of a connected solvable Lie group are themselves connected abelian Lie groups.*

(3) *A Lie group L has a largest connected solvable normal subgroup, called the **solvable radical** of L.*

(4) *A connected Lie group with trivial solvable radical is **semisimple**. A semisimple Lie group L has discrete center, and $L/\mathrm{Z}(L)$ is a finite direct product of abstractly simple groups.*

(5) *Every closed subgroup of a Lie group is a Lie group.*

(6) *The dimension of a Lie group L regarded as a real manifold, denoted by $\dim_{\mathbb{R}}(L)$, is additive with respect to extensions: if $\dim_{\mathbb{R}}(L) = n$ and K is a closed normal subgroup of dimension k, then $\dim_{\mathbb{R}}(L/K) = n - k$.*

We shall need one further observation about abelian Lie groups.

Lemma 2.9. *Let A be a connected abelian Lie group. If $A = \overline{\langle \mathcal{D} \rangle}$ where $\mathcal{D}$ is a directed family of closed subgroups of A, then some $D \in \mathcal{D}$ is cocompact in A.*

Proof. Write A as $A = \mathbb{R}^a \times \mathbb{T}^b$ for some non-negative integers a and b. Since $\mathbb{T}^b$ is compact, we can pass to the quotient $A/\mathbb{T}^b$ and assume that $A = \mathbb{R}^a$. The conclusion now follows by considering the $\mathbb{R}$-linear span of D for each $D \in \mathcal{D}$. $\square$

2.4. Cayley–Abels graphs.

Cayley–Abels graphs play an essential role in the present work. Our discussion of Cayley–Abels graphs is somewhat more technical than usual. This additional complication is necessary to ensure the degree behaves well under quotients.

A **graph** $\Gamma = (V, E, o, r)$ consists of a vertex set $V = V\Gamma$, a directed edge set $E = E\Gamma$, a map $o : E \to V$ assigning to each edge an **initial vertex**, and a bijection $r : E \to E$, denoted by $e \mapsto \bar{e}$ and called **edge reversal**, such that $r^2 = \mathrm{id}$.

The **terminal vertex** of an edge is $t(e) := o(\bar{e})$. A **loop** is an edge e such that $o(e) = t(e)$. For e a loop, we allow both $\bar{e} = e$ and $\bar{e} \neq e$ as possibilities. The **degree** of a vertex $v \in V$ is $\deg(v) := |o^{-1}(v)|$, and the graph is **locally finite** if every vertex has finite degree. The **degree** of the graph is defined to be

$$\deg(\Gamma) := \sup_{v \in V\Gamma} \deg(v).$$

The graph is **simple** if the map $E \to V \times V$ defined by $e \mapsto (o(e), t(e))$ is injective and no edge is a loop.

An **automorphism** of a graph is a pair of permutations $\alpha_V : V \to V$ and $\alpha_E : E \to E$ that respect initial vertices and edge reversal: $\alpha_V(o(e)) = o(\alpha_E(e))$ and $\overline{\alpha_E(e)} = \alpha_E(\bar{e})$. For simple graphs, automorphisms are just permutations of V that respect the edge relation in $V \times V$.

For G a group acting on a graph Γ and $v \in V\Gamma$, the orbit of v under G is denoted by Gv. We denote the orbit of an edge $e \in E\Gamma$ under G by Ge. The action of G gives a **quotient graph** Γ/G as follows: the vertex set V_G is the set of G-orbits on V and the edge set E_G is the set of G-orbits on E. The origin map $\tilde{o} : E_G \to E_G$ is defined by $\tilde{o}(Ge) := Go(e)$;

this is well-defined since graph automorphisms send initial vertices to initial vertices. The reversal $\tilde{r} : E_G \to E_G$ is given by $Ge \mapsto G\bar{e}$; this map is also well-defined. We will abuse notation and write o and r for $\tilde{o}$ and $\tilde{r}$.

We stress an important feature of quotient graphs: If N is a normal subgroup of G, then Γ/N is naturally equipped with an action of G with kernel containing N. The action of G on Γ/N therefore factors through G/N.

Lemma 2.10. *Let G be a group acting on a graph Γ with N a normal subgroup of G.*

(1) If $\deg(\Gamma)$ is finite, then $\deg(\Gamma/N) \leq \deg(\Gamma)$, with equality if and only if there exists a vertex $v \in V$ of maximal degree such that the elements of $o^{-1}(v)$ all lie in distinct N-orbits.

(2) For $v \in V$, the vertex stabilizer in G of Nv is $NG_{(v)}$.

Proof. (1) Take $v \in V\Gamma$ and let Ne be an edge of Γ/N such that $o(Ne) = Nv$. There then exists $v' \in Nv$ and $e' \in Ne$ such that $o(e') = v'$, and hence $o(ge') = v$ where $g \in N$ is such that $gv' = v$. In other words, all edges of Γ/N starting at Nv are represented by edges of Γ starting at v. Hence $\deg(Nv) \leq \deg(v)$, and $\deg(Nv) = \deg(v)$ if and only if every edge in $o^{-1}(v)$ is mapped to a distinct edge of Γ/N. Since $v \in V\Gamma$ was arbitrary, the conclusions for the degree of Γ/N are clear.

(2) Let H be the vertex stabilizer of Nv in G; that is, H is the setwise stabilizer of Nv. That N is normal ensures Nv is a block of imprimitivity for the action of G on $V\Gamma$. We thus deduce that $G_{(v)} \leq H$, so $G_{(v)} = H_{(v)}$. Since N is transitive on Nv and $N \leq H$, it follows that $NG_{(v)} = H$. $\square$

Definition 2.11. For G a t.d.l.c. group, a **Cayley–Abels** graph for G is a connected graph of finite degree on which G acts vertex-transitively such that the vertex stabilizers are open and compact. A Cayley–Abels graph for a locally compact group G is a Cayley–Abels graph for the t.d.l.c. group G/G°. That is to say, a Cayley–Abels graph for a locally compact group G is a locally finite connected graph on which G acts vertex-transitively and such that the vertex stabilizers are open and connected-by-compact.

The following proposition is a standard result; see for example [2, Proposition 2.E.9].

Proposition 2.12. *Let G be a locally compact group. The group G has a Cayley–Abels graph if and only if G is compactly generated. Moreover, if G is compactly generated, then for every compact open subgroup U/G° of G/G°, there exists a Cayley–Abels graph Γ for G such that U is a vertex stabilizer.*

The vertices of a Cayley–Abels graph have the same degree, since G acts vertex-transitively by graph automorphisms. This leads to an invariant for compactly generated groups.

Definition 2.13. If G is a compactly generated locally compact group, the **degree** $\deg(G)$ of G is the smallest degree of a Cayley–Abels graph for G.

We see that $\deg(G) = 0$ if and only if G is connected-by-compact, hence we infer the following:

Observation 2.14. *For G a compactly generated locally compact group $\deg(G) + \dim_{\mathbb{R}}^{\infty}(G) = 0$ if and only if G is compact.*

Essential to the work at hand are the groups which act on (Cayley–Abels) graphs like discrete groups.

Definition 2.15. Given a group G acting on a graph Γ, we say that G acts **freely modulo kernel** on Γ if the vertex stabilizer $G_{(v)}$ acts trivially on both the vertices and the edges of Γ for all $v \in V$.

Proposition 2.16. *Let G be a compactly generated locally compact group, N be a closed normal subgroup of G, and Γ be a connected graph of finite degree on which G acts vertex-transitively.*

(1) If Γ is a Cayley–Abels graph for G, then Γ/N is a Cayley–Abels graph for G/N.
(2) We have $\deg(\Gamma/N) \leq \deg(\Gamma)$, with equality if and only if N acts freely modulo kernel on Γ.

Proof. (1) The graph Γ/N is connected, and G acts vertex-transitively on Γ/N. Lemma 2.10(1) ensures that $\deg(\Gamma/N)$ is also finite. The fact that the vertex stabilizers are connected-by-compact and open in G/N follows from Lemma 2.10(2).

(2) Fix $v \in V$. Lemma 2.10 ensures $\deg(\Gamma/N) \leq \deg(\Gamma)$ with equality if and only if the elements of $o^{-1}(v)$ all lie in distinct N-orbits. We thus deduce the first claim of (2), and for the second, it suffices to show the elements of $o^{-1}(v)$ all lie in distinct N-orbits if and only if N acts freely modulo kernel on Γ.

Suppose N acts freely modulo kernel on Γ. For an edge e of Γ, if $o(e)$ is fixed by $g \in N$, then $ge = e$. The elements of $o^{-1}(v)$ thus all lie in distinct N-orbits.

Conversely, suppose the elements of $o^{-1}(v)$ all lie in distinct N-orbits. Any element of N that fixes v must also fix $o^{-1}(v)$ pointwise. Each $g \in N_{(v)}$ then fixes $t(e)$, so g fixes all the neighbors of v. We thus conclude $N_{(v)} \leq N_{(w)}$ where $w \in V\Gamma$ is adjacent to v.

As G acts vertex-transitively on Γ and N is normal in Γ, the choice of v is not important, so in fact, $N_{(v')} \leq N_{(w')}$ for (v', w') any pair of adjacent vertices of Γ. That Γ is connected now implies all vertex stabilizers of N acting on Γ are equal. Moreover, they fix every edge of Γ, since every edge lies in $o^{-1}(w)$ for some $w \in V\Gamma$. The group N therefore acts freely modulo kernel on Γ. $\square$

3. Finiteness properties of the lattice of closed normal subgroups

We here establish a finiteness property of the lattice of closed normal subgroups. As an immediate consequence, we obtain the existence of certain quotients with an interesting minimality condition.

3.1. Directed and filtering families of normal subgroups.

Claim (1) of the next lemma is more or less a restatement of [1, Proposition 2.5]. We give a proof for completeness.

Lemma 3.1. *Let G be a compactly generated t.d.l.c. group and Γ be a Cayley–Abels graph for G.*

(1) Let $\mathcal{F}$ be a filtering family of closed normal subgroups of G and set $M := \bigcap \mathcal{F}$. Then there exists $N \in \mathcal{F}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$.
(2) Let $\mathcal{D}$ be a directed family of closed normal subgroups of G and set $M := \overline{\langle \mathcal{D} \rangle}$. Then there exists $N \in \mathcal{D}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$.

Proof. Fix $v \in V\Gamma$. For $N \in \mathcal{N}(G)$, the value $\deg(\Gamma/N)$ is determined by the number of orbits of the action of $N_{(v)}$ on the set X of edges issuing from v. Defining $\alpha(N)$ to be the subgroup of $\mathrm{Sym}(X)$ induced by $N_{(v)}$ on X, we see that $\deg(\Gamma/N) = \deg(\Gamma/M)$ if $\alpha(N) = \alpha(M)$. The assignment $N \mapsto \alpha(N)$ is also order-preserving. For a filtering or directed family $\mathcal{N} \subseteq \mathcal{N}(G)$, the family $\alpha(\mathcal{N}) := \{\alpha(N) \mid N \in \mathcal{N}\}$ is then a filtering or directed family of subgroups of $\mathrm{Sym}(X)$. That $\mathrm{Sym}(X)$ is a finite group ensures $\alpha(\mathcal{N})$ is a finite family, so it admits a minimum or maximum, according to whether $\mathcal{N}$ is filtering or directed.

Claim (2) is now immediate. The directed family $\alpha(\mathcal{D})$ admits a maximal element $\alpha(N)$. It then follows that $\alpha(N) = \alpha(M)$, so $\deg(\Gamma/N) = \deg(\Gamma/M)$.

For (1), an additional compactness argument is required. If G acts freely modulo kernel on Γ, then $N \in \mathcal{F}$ and M also act as such. The desired result then follows since $\deg(\Gamma/N) = \deg(\Gamma) = \deg(\Gamma/M)$. Let us assume that G does not act freely modulo kernel, so $G_{(v)}$ acts non-trivially on Γ for any $v \in V\Gamma$.

Take $\alpha(N) \in \alpha(\mathcal{F})$ to be the minimum. Given $r \in \alpha(N)$, let Y be the set of elements of $G_{(v)}$ that *do not* induce the permutation r on X. If $r \neq 1$, then plainly $Y \neq G_{(v)}$. If $r = 1$, then $Y \neq G_{(v)}$ since $G_{(v)}$ acts non-trivially on Γ. The set Y is a proper open subset of $G_{(v)}$, and thus $G_{(v)} \setminus Y$ is a non-empty compact set.

Letting $\mathcal{K}$ be a finite subset of $\mathcal{F}$, the group $K := \bigcap_{F \in \mathcal{K}} F$ contains some element of $\mathcal{F}$, so $\alpha(K) \geq \alpha(N)$. In particular, $K_{(v)} \not\subseteq Y$. The intersection

$$\bigcap_{F \in \mathcal{K}} (F_{(v)} \cap (G_{(v)} \setminus Y))$$

is therefore non-empty. Compactness now implies that

$$M_{(v)} \cap (G_{(v)} \setminus Y) = \bigcap_{F \in \mathcal{F}} (F_{(v)} \cap (G_{(v)} \setminus Y)) \neq \emptyset;$$

that is, some element of $M_{(v)}$ induces the permutation r on X. Since $r \in \alpha(N)$ is arbitrary, we conclude that $\alpha(M) = \alpha(N)$, and so $\deg(\Gamma/N) = \deg(\Gamma/M)$. $\qquad\square$

The conclusion of claim (1) in Lemma 3.1 implies that the factor N/M is "discrete" from the point of view of the Cayley–Abels graph.

Lemma 3.2. *Let G be a compactly generated t.d.l.c. group with N a closed normal subgroup of G. If there is a Cayley–Abels graph Γ for G such that $\deg(\Gamma/N) = \deg(\Gamma)$, then there exists a compact normal subgroup L of G acting trivially on Γ such that L is an open subgroup of N.*

Proof. In view of Proposition 2.16, N acts freely modulo kernel on Γ. For U the pointwise stabilizer of the star $o^{-1}(v)$ for some vertex v, the subgroup U is a compact open subgroup of G, and its core K is the kernel of the action of G on Γ. Since N acts freely modulo kernel, we deduce that $N \cap U \leq K$. The group $L := K \cap N$ now satisfies the lemma. $\qquad\square$

Combining our results on Cayley–Abels graphs with the Gleason–Yamabe Theorem, we obtain a result that applies to compactly generated locally compact groups without dependence on a choice of Cayley–Abels graph.

Theorem 3.3. *Let G be a compactly generated locally compact group.*
(1) If $\mathcal{F}$ is a filtering family of closed normal subgroups of G, then there exists $N \in \mathcal{F}$ and a closed normal subgroup K of G such that $\bigcap \mathcal{F} \leq K \leq N$, $K/\bigcap \mathcal{F}$ is compact, and N/K is discrete.
(2) If $\mathcal{D}$ is a directed family of closed normal subgroups of G, then there exists $N \in \mathcal{D}$ and a closed normal subgroup K of G such that $N \leq K \leq \overline{\langle \mathcal{D} \rangle}$, K/N is compact, and $\overline{\langle \mathcal{D} \rangle}/K$ is discrete.

Proof. (1) The group $G/\bigcap \mathcal{F}$ is a compactly generated locally compact group, so we assume that $\bigcap \mathcal{F} = \{1\}$. Fix a Cayley–Abels graph Γ for G and let E be the kernel of the action of G on Γ. Since $G^\circ \leq E$, we have $G^\circ = E^\circ$, and E/G° is compact, since E/G° is the core of a compact open subgroup of G/G°. Theorem 2.5 thus ensures $R := \mathrm{Rad}_{\mathcal{LE}}(E)$ is compact and the quotient E/R is a Lie group with finitely many connected components. Observe additionally that $R \trianglelefteq G$.

For $N \in \mathcal{N}(G)$, set $a_N := \deg(\Gamma/\overline{NG^\circ})$ and $b_N := \dim_{\mathbb{R}}((N \cap E)R/R)$. Both a_N and b_N are natural numbers depending on N in a monotone fashion: given $N, N' \in \mathcal{N}(G)$ such that $N \leq N'$, then $a_N \geq a_{N'}$ and $b_N \leq b_{N'}$. By Lemma 3.1, there exists $N \in \mathcal{F}$ such that $a_N = \deg(\Gamma/\overline{NG^\circ}) = \deg(\Gamma)$. Since $\mathcal{F}$ is a filtering family, we can choose N such that additionally b_N is minimized across $N \in \mathcal{F}$. Fix such an N.

We argue claim (1) holds for N and $K := N \cap E \cap R$. Since R is compact, it suffices to show N/K is discrete. Consider first $N \cap E$. By Proposition 2.16, $\overline{NG^\circ}$ acts freely modulo kernel on Γ, hence N acts freely modulo kernel on Γ. The group $N \cap E$ is thus a vertex stabilizer of the action of N on Γ. In particular, $N \cap E$ is open in N. It now suffices to show that K is open in $N \cap E$.

Consider $((N \cap E)R/R)^\circ$ and suppose for contradiction that there is an infinite compact identity neighborhood T/R of $((N \cap E)R/R)^\circ$; we assume that $TR = T$. Find $U \subsetneq T$ open containing 1 with $U = UR$. For $F \in \mathcal{F}$ with $F \leq N$, the minimality of b_N implies $(F \cap E)R/R$ is a subgroup of $(N \cap E)R/R$ with the same dimension b_N. Consequently, $(F \cap E)R/R$ contains $((N \cap E)R/R)^\circ$, and

$$T = FR \cap T = (F \cap T)R.$$

We infer that $(F \cap T) \not\subseteq U$, so F intersects the non-empty compact set $T \setminus U$. As $\mathcal{F}$ is a filtering family, it follows by compactness that $\bigcap \mathcal{F} \cap T$ intersects $T \setminus U$. However, this is absurd since $\bigcap \mathcal{F} \cap T = \{1\}$.

All compact identity neighborhoods of $((N \cap E)R/R)^\circ$ are thus finite. It follows that a singleton set is open, so $((N \cap E)R/R)^\circ$ is discrete. The only discrete connected group is the trivial group, so $((N \cap E)R/R)^\circ$ is in fact trivial. Since $(N \cap E)R/R$ is a Lie group, we conclude the group $(N \cap E)R/R \simeq (N \cap E)/K$ is discrete, verifying (1) holds for N and K.

(2) Set $M := \overline{\langle \mathcal{D} \rangle}$ and let Γ be a Cayley–Abels graph for G. By Lemma 3.1, there exists $N \in \mathcal{D}$ such that $\deg(\Gamma/\overline{NG^\circ}) = \deg(\Gamma/\overline{MG^\circ})$. Moreover, $\Gamma/\overline{NG^\circ} = \Gamma/N$ is a Cayley–Abels graph for G/N by Proposition 2.16. Passing to the quotient G/N and replacing Γ with Γ/N, we may assume that $\deg(\Gamma) = \deg(\Gamma/\overline{MG^\circ})$ and that M acts freely modulo kernel on Γ.

Let E be the kernel of the action of G on Γ and let $L := M \cap E$. Our assumptions ensure that L is open in M. We will now find an $F \in \mathcal{D}$ and a closed $J \trianglelefteq G$ such that J is an open subgroup of L and that JF/F is compact; this will prove (2) with $K := JF$ and $N := F$.

As in (1), the group $R := \mathrm{Rad}_{\mathcal{LE}}(E)$ is compact, and the quotient E/R is a Lie group with finitely many connected components. Let $N \in \mathcal{D}$ witness the minimum of the set

$$\{\dim_{\mathbb{R}}(E/(N \cap E)R) \mid N \in \mathcal{D}\}$$

and consider $S := (N \cap E)R$. For all $D \in \mathcal{D}$ such that $D \geq N$, the quotient $(D \cap E)R/S$ is discrete, by our choice of S. Every element of $(D \cap E)R/S$ thereby has an open centralizer in G, and hence $(D \cap E)R/S$ is centralized by $(G/S)^\circ$. As $L = M \cap E$ is open in M,

$$L = \overline{\bigcup_{D \in \mathcal{D}} (D \cap M \cap E)} = \overline{\bigcup_{D \in \mathcal{D}} (D \cap E)}.$$

The group LR/S is thus centralized by $(G/S)^\circ$, and a fortiori, $(LR/S)^\circ$ is abelian.

The group E/S is a Lie group, so LR/S is also a Lie group. We infer that $(LR/S)^\circ$ is an open subgroup of LR/S. The component $(LR/S)^\circ$ thus contains a dense subgroup which is formed of a directed union of discrete subgroups of the form $(D\cap E)R/S\cap(LR/S)^\circ$. Since $(LR/S)^\circ$ is a connected abelian Lie group, Lemma 2.9 ensures there is some $F\in\mathcal{D}$ such that $(F\cap E)R/S\cap(LR/S)^\circ$ is cocompact in $(LR/S)^\circ$. Fix $F\in\mathcal{D}$ with this property.

There is a compact set $A\subseteq LR/S$ such that $(LR/S)^\circ\subseteq A(F\cap E)R/S$. Letting $\pi:LR/S\to LR/(F\cap E)R$ be the usual projection, $\pi((LR/S)^\circ)\leq\pi(A)$ which is compact. Let $J\trianglelefteq L$ be the preimage in L of the connected component of $L/(F\cap E)(R\cap L)\simeq LR/(F\cap E)R$. We see that $J/(F\cap E)(R\cap L)$ is compact, and since R is compact, in fact $J/(F\cap E)$ is compact.

The subgroup J is invariant under continuous automorphisms of L that fix $(F\cap E)(R\cap L)$, so $J\trianglelefteq G$. Additionally, J is an open subgroup of L. Forming JF, the quotient $JF/F\simeq J/F\cap J$ is a quotient of the compact group $J/(F\cap E)$. We have thus found the desired subgroups, and (2) follows. $\quad\square$

3.2. **Just-non-P quotients.** As an immediate application of Theorem 3.3, we show certain quotients exist.

Definition 3.4. For P a property of groups, we say that a non-trivial locally compact group G is **just-non-P** if every proper non-trivial quotient of G has P, but G itself does not have P.

The properties P we consider must enjoy the following permanence property: A property P of locally compact groups is **closed under compact extensions** if for a compactly generated locally compact group G with a compact normal subgroup N, the group G has P if and only if G/N has P. Quasi-isometry invariant properties are closed under compact extensions; see [2, Proposition 1.D.4].

A compactly generated locally compact group G is called **compactly presented** if there is a compact generating set S such that G has a presentation $\langle S|R\rangle$ where the relators R have bounded length. Being compactly presented is a quasi-isometry invariant property of compactly generated locally compact groups; see [2, Corollary 8.A.4].

Theorem 3.5. *Let P be a property of locally compact groups. If all groups with P are compactly presented and P is closed under compact extensions, then for any compactly generated locally compact group G, exactly one of the following holds:*

(1) Every non-trivial quotient of G (including G itself) has P; or
(2) G admits a quotient that is just-non-P.

Proof. Let $\mathcal{F}$ be the collection of proper closed normal subgroups N of G such that G/N fails to have property P. If $\mathcal{F}=\emptyset$, then G satisfies (1), and we are done. Assuming $\mathcal{F}\neq\emptyset$, we claim increasing chains in $\mathcal{F}$ have upper

bounds. Let $(N_\alpha)_{\alpha \in I}$ be an $\subseteq$-increasing chain and put $L := \overline{\bigcup_{\alpha \in I} N_\alpha}$. Suppose for contradiction G/L has property P; in particular, G/L is compactly presented.

Appealing to Theorem 3.3, we may find $\gamma \in I$ and a closed $K \trianglelefteq G$ such that $N_\gamma \leq K \leq L$ with L/K discrete and K/N_γ compact. The group G/L is a quotient of G/K with kernel L/K. Moreover, since G/L is compactly presented, [2, Proposition 8.A.10] implies the discrete group L/K is finitely normally generated as a subgroup of G/K.

Let $X \subseteq L/K$ be a finite normal generating set. Since $\bigcup_{\alpha \in I} N_\alpha$ is dense in L and K is open in L, we may find N_β for some $\beta > \gamma$ such that $N_\beta K/K$ contains X. The normality of N_β implies that indeed $N_\beta K = L$. The group G/L is thus a quotient of G/N_β with kernel KN_β/N_β. The kernel KN_β/N_β is compact, so G/N_β is a compact extension of G/L. Hence, G/N_β has property P, an absurdity.

We conclude that increasing chains in $\mathcal{F}$ have upper bounds. Applying Zorn's lemma, we may find $N \in \mathcal{F}$ maximal. The maximality of N ensures every proper non-trivial quotient of G/N has property P, verifying (2). $\square$

Let G be a compactly generated locally compact group with compact symmetric generating set X. The group G has **polynomial growth** if there are constants $C, k > 0$ such that $\mu(X^n) \leq Cn^k$ for all $n \geq 1$, where μ is a Haar measure. Having polynomial growth is a quasi-isometry invariant property, and compactly generated locally compact groups with polynomial growth are *necessarily* compactly presented, [2, Proposition 8.A.25].

The following corollary is now immediate from Theorem 3.5.

Corollary 3.6. *Let G be a compactly generated locally compact group.*

(1) *If some quotient of G is not compactly presented, then G admits a quotient that is just-non-(compactly presented).*
(2) *If G does not have polynomial growth, then G admits a quotient that is just-not-(of polynomial growth).*

We remark that groups in which all proper quotients are compactly presented satisfy a stronger version of Theorem 3.3(2).

Proposition 3.7. *Let G be a compactly generated locally compact group such that every proper quotient of G is compactly presented. If $\mathcal{D}$ is a directed family of closed normal subgroups of G, then there exists $N \in \mathcal{D}$ such that $\overline{\langle \mathcal{D} \rangle}/N$ is compact.*

Proof. Without loss of generality $\overline{\langle \mathcal{D} \rangle} \neq \{1\}$. By Theorem 3.3(2), there exists $M \in \mathcal{D}$ and a closed normal subgroup K of G such that $M \leq K \leq \overline{\langle \mathcal{D} \rangle}$, $\overline{\langle \mathcal{D} \rangle}/K$ is discrete, and K/M is compact. The group $G/\overline{\langle \mathcal{D} \rangle}$ is compactly presented, so $\overline{\langle \mathcal{D} \rangle}/K$ is normally generated in G by a finite set. It now follows there exists $M \leq N \in \mathcal{D}$ such that $\overline{\langle \mathcal{D} \rangle} = NK$. In particular, $\overline{\langle \mathcal{D} \rangle}/N$ is compact, as required. $\square$

4. ESSENTIALLY CHIEF SERIES

Definition 4.1. An **essentially chief series** for a compactly generated locally compact group G is a finite series

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of closed normal subgroups such that each normal factor G_{i+1}/G_i is either compact, discrete, or a topological chief factor of G.

We now show that any compactly generated locally compact group admits an essentially chief series; more precisely, any finite normal series can be refined to an essentially chief series.

4.1. **Existence of essentially chief series.** We begin with two technical results which prove the existence of essentially chief refinements of normal series, with bounds on the number of factors required. These lemmas deal with the connected case and totally disconnected case, respectively.

Lemma 4.2. *Suppose that G is a compactly generated locally compact group, $H \leq L$ are closed normal subgroups of G, and $d := \dim_{\mathbb{R}}^{\infty}(L) - \dim_{\mathbb{R}}^{\infty}(H)$. If L/H is connected-by-compact, then there is a series*

$$H = G_0 \leq G_1 \leq \cdots \leq G_k = L$$

of closed normal subgroups of G with $k \leq 2d + 1$ such that each factor G_{i+1}/G_i is either compact, discrete, or a chief factor of G. Additionally, at most d factors are neither compact nor discrete.

Proof. Let us first assume that L/H is connected and $\mathrm{Rad}_{\mathcal{LE}}(L/H) = \{1\}$; we will here prove the result with $k \leq 2d$. In this situation, L/H is a Lie group and $d = \dim_{\mathbb{R}}(L/H)$. There is then $i \leq d$ and a series $H = M_0 < M_1 \cdots < M_i = L$ of closed G-invariant subgroups such that each of the factors $M_j/M_{j-1} =: V_j$ is connected, has positive dimension, and has no proper closed G-invariant subgroup of positive dimension.

For each $1 \leq j \leq i$, the factor V_j is either compact, a non-compact semisimple Lie group such that G permutes transitively the simple factors, or $\mathbb{R}^n$ such that G acts irreducibly. If V_j is compact, we do nothing. In the second case, we take N_j such that $N_j/M_{j-1} = Z(V_j)$. The factor N_j/M_{j-1} is then discrete, and M_j/N_j is a chief factor of G. For the last case, every proper closed G-invariant subgroup of V_j is either trivial or a lattice. If G does not preserve a lattice, then V_j is already a chief factor. If G preserves a lattice N_j/M_{j-1}, then N_j/M_{j-1} is discrete, and M_j/N_j is compact.

By including the N_j terms as needed, we obtain a G-invariant series from H to L with at most $k \leq 2d$ factors such that each factor is compact, discrete, or a chief factor. Additionally, at most d of the factors are neither compact nor discrete, since each V_j contributes at most one such factor. Let us make a further observation for later use: If there are exactly $2d$ factors in the series, then each V_j is one-dimensional, hence abelian, and divided into a discrete factor N_j/M_{j-1} and a compact factor M_j/N_j.

For the general case, let L' be the preimage of $(L/H)^{\circ}$ and let H' be the preimage of $\mathrm{Rad}_{\mathcal{LE}}(L'/H)$. By our work above, there is a G-invariant series from H' to L' with at most $2d$ factors of the appropriate form. Since both H'/H and L/L' are compact, we immediately obtain the required G-invariant series from H to L with at most $l+2$ factors where $l \leq 2d$ such that at most d factors are non-compact. If $l = 2d$, the uppermost factor in the series from H' to L' can be combined with L/L', so we find a series with $2d+1$ factors. We thus deduce that there is an essentially chief series from H to L with at most $2d+1$ factors such that at most d factors are non-compact. $\square$

Lemma 4.3. *Suppose that G is a compactly generated locally compact group, $H \leq L$ are closed normal subgroups of G, and Γ is a Cayley–Abels graph for G such that $\deg(G) = \deg(\Gamma)$. If $G^{\circ} \leq H$, then there exists a series*

$$H =: C_0 \leq K_0 \leq D_0 \leq \cdots \leq C_n \leq K_n \leq D_n = L$$

of closed normal subgroups of G with $n \leq \deg(\Gamma/H) - \deg(\Gamma/L)$ such that

(1) for $0 \leq i \leq n$, K_i/C_i is compact, and D_i/K_i is discrete; and
(2) for $1 \leq i \leq n$, C_i/D_{i-1} is a chief factor of G.

Proof. Set $k := \deg(\Gamma/H)$ and $m := \deg(\Gamma/L)$. By induction on $i \leq k - m$, we prove there exists a series of normal subgroups of G

$$H =: C_0 \leq K_0 \leq D_0 \leq \cdots \leq C_i \leq K_i \leq D_i \leq L$$

such that claims (1) and (2) hold of all factors up to i and that there is $i \leq j \leq k - m$ for which D_i is maximal among normal subgroups of G such that $\deg(\Gamma/D_i) = k - j$ and $D_i \leq L$.

For $i = 0$, it follows from Lemma 3.1 and Zorn's lemma that there exists D_0 maximal such that $\deg(\Gamma/D_0) = k$ and $H \leq D_0 \leq L$. The graph Γ/H is a Cayley–Abels graph for G/H with degree k, and

$$\deg((\Gamma/H)/(D_0/H)) = \deg(\Gamma/D_0) = k.$$

Applying Lemma 3.2, there is $K_0 \trianglelefteq G$ such that $H \leq K_0 \leq D_0$ with K_0/H compact, open, and normal in D_0/H. We conclude that $C_0 = H$, K_0, and D_0 satisfy the inductive claim when $i = 0$ with $j = 0$.

Suppose we have built our sequence up to i. By the inductive hypothesis, there is $i \leq j \leq k - m$ such that D_i is maximal with $\deg(\Gamma/D_i) = k - j$ and $D_i \leq L$. If $j = k - m$, then the maximality of D_i implies $D_i = L$, and we stop. Else, let $j' > j$ be least such that there is $M \trianglelefteq G$ with $\deg(\Gamma/M) = k - j'$ and $D_i \leq M \leq L$. We take $C_{i+1} \trianglelefteq G$ to be minimal such that $\deg(\Gamma/C_{i+1}) = k - j'$ and $D_i < C_{i+1} \leq L$; Lemma 3.1 ensures such a subgroup exists.

Consider a closed $N \trianglelefteq G$ with $D_i \leq N < C_{i+1}$. Putting $\deg(\Gamma/N) = l$, Proposition 2.16 implies $k - j' \leq l \leq k - j$, and the minimality of C_{i+1} further implies $k - j' < l$. On the other hand, we chose $j' > j$ least such that there is $M \trianglelefteq G$ with $\deg(\Gamma/M) = k - j'$ and $D_i \leq M \leq L$. Therefore,

$l = k - j$. In view of the maximality of D_i, we deduce that $D_i = N$ and that C_{i+1}/D_i is a chief factor.

Applying again Lemma 3.1, there is $D_{i+1} \trianglelefteq G$ maximal such that

$$\deg(\Gamma/D_{i+1}) = k - j'$$

and $C_{i+1} \leq D_{i+1} \leq L$. Lemma 3.2 supplies $K_{i+1} \trianglelefteq G$ such that $C_{i+1} \leq K_{i+1} \leq D_{i+1}$ with K_{i+1}/C_{i+1} compact and open in D_{i+1}/C_{i+1}. This completes the induction.

Our procedure halts at some $n \leq k - m$. At this stage, $D_n = L$, verifying the theorem. $\square$

We now use Lemmas 4.2 and 4.3 to refine a normal series factor by factor to produce an essentially chief series.

Theorem 4.4. *Let G be a compactly generated locally compact group and let $(G_i)_{i=1}^{m-1}$ be a finite ascending sequence of closed normal subgroups of G. Then there exists an essentially chief series for G*

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_l = G,$$

such that

(1) $\{G_1, \ldots, G_{m-1}\}$ is a subset of $\{K_0, \ldots, K_l\}$; and
(2) if $G^\circ \in \{G_1, \ldots, G_{m-1}\}$, then $l \leq 2m + 2\dim_{\mathbb{R}}^\infty(G) + 3\deg(G)$, and at most $\dim_{\mathbb{R}}^\infty(G) + \deg(G)$ of the factors K_{i+1}/K_i are neither compact nor discrete.

Proof. Let us extend the series by $G_0 := \{1\}$ and $G_m := G$ obtaining the series

$$\{1\} =: G_0 \leq G_1 \leq \cdots \leq G_{m-1} \leq G_m := G.$$

For each $j \in \{0, \ldots, m-1\}$, we apply Lemma 4.2 to $H := G_j$ and L such that $L/G_j = (G_{j+1}/G_j)^\circ$ to refine the series. We then refine again by applying Lemma 4.3 to $L := G_{j+1}$ and H such that $H/G_j = (G_{j+1}/G_j)^\circ$. This yields the desired refined series claimed in (1).

For (2), suppose $G^\circ \in \{G_0, \ldots, G_{m-1}\}$; say that $G_k = G^\circ$ for some $0 \leq k \leq m-1$. For each $i < k$, Lemma 2.7 yields a closed normal H_{i+1} of G with $G_i \leq H_{i+1} \leq G_{i+1}$ such that G_{i+1}/H_{i+1} is discrete and H_{i+1}/G_i is connected-by-compact. We may thus apply Lemma 4.2 to each of the pairs $G_i \leq H_{i+1}$ with $i < k$ to produce a refined series. (If $k = 0$, there is nothing to do at this stage.)

In the refined series, there are at most $2(\dim_{\mathbb{R}}^\infty(G_{i+1}) - \dim_{\mathbb{R}}^\infty(G_i)) + 2$ terms T such that $G_i \leq T < G_{i+1}$. Additionally, at most $\dim_{\mathbb{R}}^\infty(G_{i+1}) - \dim_{\mathbb{R}}^\infty(G_i)$ factors are neither compact nor discrete. The number of terms in our refined series strictly below G_k is thus at most

$$\sum_{i=0}^{k-1} \left(2(\dim_{\mathbb{R}}^\infty(G_{i+1}) - \dim_{\mathbb{R}}^\infty(G_i)) + 2\right) = 2k + 2\dim_{\mathbb{R}}^\infty(G),$$

and the total number of factors that are neither compact nor discrete is at most

$$\sum_{i=0}^{k-1}(\dim_{\mathbb{R}}^{\infty}(G_{i+1}) - \dim_{\mathbb{R}}^{\infty}(G_i)) = \dim_{\mathbb{R}}^{\infty}(G).$$

We now consider the series $G_k \leq \ldots G_{m-1} \leq G_m = G$. If $G = G_k$, we are done; we thus suppose that $G_k < G$. Let Γ be a Cayley–Abels graph for G with $\deg(\Gamma) = \deg(G)$ and put $k_j := \deg(\Gamma/G_j)$. For each $j \in \{k, \ldots, m-1\}$, we apply Lemma 4.3 to each pair $G_j \leq G_{j+1}$ to obtain a refined series. This results in a series in which the number of terms T such that $G_j \leq T < G_{j+1}$ is at most $3(k_j - k_{j+1}) + 2$, and at most $k_j - k_{j+1}$ of the factors are neither compact nor discrete. The total number of terms in the refined series not including G_m is thus at most

$$
\begin{aligned}
\sum_{j=k}^{m-1}(3(k_j - k_{j+1}) + 2) \;\; &= \;\; 2(m - k) + 3(\deg(\Gamma) - \deg(\Gamma/G)) \\
&\leq \;\; 2(m - k) + 3\deg(G),
\end{aligned}
$$

and the total number of non-compact, non-discrete factors is at most

$$\sum_{j=1}^{m-1}(k_j - k_{j+1}) \leq \deg(G).$$

Putting together our two refined series, we obtain an essentially chief series

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_l = G$$

such that

$$
\begin{aligned}
l \;\; &\leq \;\; 2k + 2\dim_{\mathbb{R}}^{\infty}(G) + 2(m - k) + 3\deg(G) \\
&\leq \;\; 2m + 2\dim_{\mathbb{R}}^{\infty}(G) + 3\deg(G).
\end{aligned}
$$

Furthermore, the number of factors that are neither compact nor discrete is at most $\dim_{\mathbb{R}}^{\infty}(G) + \deg(G)$. $\square$

Remark 4.5. For any $n \geq 0$, the direct product of n compactly generated non-discrete simple locally compact groups gives a compactly generated l.c.s.c. group such that any essentially chief series has n chief factors.

One can also construct compactly generated l.c.s.c. groups such that any essentially chief series has at least n compact factors or n discrete factors. This can be arranged by taking wreath products. For example, let K be a non-abelian finite simple group and V be an infinite finitely generated simple group. The group V acts on K^V, so we may form $G := K^V \rtimes V$, which is locally compact. Fixing U a proper open subgroup of K^V, G acts faithfully and transitively on the cosets $X := G/U$. We obtain a second locally compact group $G' := \bigoplus_X V \rtimes G$. One checks that any chief series for G has one compact factor and two discrete factors. By taking more wreath products, one produces any desired number of discrete or compact factors.

4.2. **Uniqueness of essentially chief series.** We finally obtain a Jordan–Hölder theorem for essentially chief series, using the general properties of associated chief factors obtained in [7]. For this uniqueness result, we need to exclude chief factors associated to compact and discrete factors.

Definition 4.6. For G a Polish group and K/L a chief factor of G, we say that K/L is **negligible** if K/L is either abelian or associated to a compact or discrete chief factor. A chief block $\mathfrak{a} \in \mathfrak{B}_G$ is **negligible** if $\mathfrak{a}$ has a compact or discrete representative. The collection of non-negligible chief blocks of G is denoted by $\mathfrak{B}_G^*$. See Subsection 2.2 for the definitions of the association relation and chief blocks.

Using the general methods outlined in [1, Appendix II], one can produce negligible chief factors which are neither abelian, compact, nor discrete.

Remark 4.7. In [8], we show that all negligible chief factors K/L in an l.c.s.c. group are either quasi-discrete, meaning that the elements of K/L with open centralizer form a dense subgroup of K/L, or compact. The work [1] furthermore shows quasi-discrete groups have restrictive topological structure.

In contrast to the results about existence of chief series, we do not need to assume that G is compactly generated for our Jordan–Hölder theorem.

Theorem 4.8. *Suppose that G is an l.c.s.c. group and that G has two essentially chief series $(A_i)_{i=0}^m$ and $(B_j)_{j=0}^n$. Define*

$I := \{i \in \{1,\ldots,m\} \mid A_i/A_{i-1}$ *is a non-negligible chief factor of G*\}; *and*

$J := \{j \in \{1,\ldots,n\} \mid B_j/B_{j-1}$ *is a non-negligible chief factor of G*\}.

Then there is a bijection $f : I \to J$, where $f(i)$ is the unique element $j \in J$ such that A_i/A_{i-1} is associated to B_j/B_{j-1}.

Proof. Theorem 2.3 provides a function $f : I \to \{1,\ldots,n\}$ where $f(i)$ is the unique element of $\{1,\ldots,n\}$ such that A_i/A_{i-1} is associated to a subquotient of $B_{f(i)}/B_{f(i)-1}$.

If $B_{f(i)}/B_{f(i)-1}$ is compact or discrete, then A_i/A_{i-1} is associated to a compact or discrete factor of G, which contradicts our choice of I. The factor $B_{f(i)}/B_{f(i)-1}$ is thus chief, and A_i/A_{i-1} is associated to $B_{f(i)}/B_{f(i)-1}$. Theorem 2.3 implies $B_{f(i)}/B_{f(i)-1}$ is also non-abelian. Since association is an equivalence relation for non-abelian chief factors, we conclude that $B_{f(i)}/B_{f(i)-1}$ is non-negligible, and therefore, $f(i) \in J$.

We thus have a well-defined function $f : I \to J$. The same argument with the roles of the series reversed produces a function $f' : J \to I$ such that B_j/B_{j-1} is associated to $A_{f'(j)}/A_{f'(j)-1}$. Since each factor of the first series is associated to at most one factor of the second by Theorem 2.3, we conclude that f' is the inverse of f, hence f is a bijection. $\square$

Corollary 4.9. *If G is a compactly generated l.c.s.c. group, then each $\mathfrak{a} \in \mathfrak{B}_G^*$ is represented exactly once in every essentially chief series for G, and $|\mathfrak{B}_G^*| \leq \dim_{\mathbb{R}}^{\infty}(G) + \deg(G)$.*

Proof. Let $(G_i)_{i=0}^n$ be an essentially chief series for G. For $\mathfrak{a} \in \mathfrak{B}_G^*$, fix a representative $A/B \in \mathfrak{a}$ and use Theorem 4.4 to refine the series $\{1\} \leq A < B \leq G$ to a chief series $(H_i)_{i=0}^k$. Theorem 4.8 now implies there is a unique $0 \leq i < n$ such that A/B is associated to G_{i+1}/G_i. Hence, $G_{i+1}/G_i \in \mathfrak{a}$. On the other hand, association is an equivalence relation, so the uniqueness of i implies that G_{i+1}/G_i is the only representative of $\mathfrak{a}$ appearing in $(G_i)_{i=0}^n$. The chief block $\mathfrak{a} \in \mathfrak{B}_G^*$ is thus represented exactly once in any essentially chief series for G.

For the second claim, we use Theorem 4.4 to produce $(K_i)_{i=0}^m$ an essentially chief series for G that refines the series $\{1\} \leq G^{\circ} \leq G$. By the previous paragraph, each $\mathfrak{a} \in \mathfrak{B}_G^*$ admits exactly one representative with the form K_{i+1}/K_i for some $0 \leq i < m$. Moreover, such a representative must be neither compact nor discrete. Theorem 4.4 ensures that we can choose $(K_i)_{i=0}^m$ such that the number of non-compact, non-discrete factors is at most $\dim_{\mathbb{R}}^{\infty}(G) + \deg(G)$, hence $|\mathfrak{B}_G^*| \leq \dim_{\mathbb{R}}^{\infty}(G) + \deg(G)$. $\square$

To conclude this section, we note that each non-negligible block of a compactly generated group admits a unique smallest closed normal subgroup covering it; this somewhat technical observation is important in [8]. See Subsection 2.2 for the definition of a normal subgroup covering a chief block.

Proposition 4.10. *Let G be a compactly generated l.c.s.c. group with $\mathfrak{a} \in \mathfrak{B}_G^*$. Then there is a closed normal subgroup $G_{\mathfrak{a}}$ of G such that for every closed normal subgroup K of G, K covers $\mathfrak{a}$ if and only if $K \geq G_{\mathfrak{a}}$.*

Proof. Let $\mathcal{K}$ be the set of closed normal subgroups of G that cover $\mathfrak{a}$ and set $G_{\mathfrak{a}} := \bigcap \mathcal{K}$. By [7, Lemma 7.10], the set $\mathcal{K}$ is a filtering family, and thus, Theorem 3.3 ensures there exists $L \in \mathcal{K}$ and $G_{\mathfrak{a}} \leq M \leq L$ such that M is G-invariant and open in L and $M/G_{\mathfrak{a}}$ is compact.

We now consider the series

$$\{1\} \leq G_{\mathfrak{a}} \leq M \leq L \leq G.$$

Theorem 2.3 implies that one of L/M, $M/G_{\mathfrak{a}}$, or $G_{\mathfrak{a}}/\{1\}$ covers $\mathfrak{a}$. As $\mathfrak{a}$ is non-negligible, neither the discrete factor L/M nor the compact factor $M/G_{\mathfrak{a}}$ covers $\mathfrak{a}$. We deduce that $G_{\mathfrak{a}}$ covers $\mathfrak{a}$, hence every closed normal subgroup of G that contains $G_{\mathfrak{a}}$ covers $\mathfrak{a}$. Since every closed normal subgroup of G that covers $\mathfrak{a}$ contains $G_{\mathfrak{a}}$ by construction, the proposition is verified. $\square$

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