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Construction of new larger (a, d)-edge antimagic vertex graphs by using adjacency matrices

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Abstract

Let $G = G(V, E)$ be a finite simple undirected graph with vertex set V and edge set E, where $|E|$ and $|V|$ are the number of edges and vertices on G. An (a, d)-edge antimagic vertex ((a, d)-EAV) labeling is a one-to-one mapping f from $V(G)$ onto $\{1, 2, \ldots, |V|\}$ with the property that for every edge $xy \in E$, the edge-weight set is equal to $\{f(x) + f(y) : x, y \in V\} = \{a, a+d, a+2d, \ldots, a+(|E|-1)d\}$, for some integers $a > 0$, $d \geq 0$. An (a, d)-edge antimagic total ((a, d)-EAT) labeling is a one-to-one mapping f from $V \cup E$ onto $\{1, 2, \ldots, |V| + |E|\}$ with the property that for every edge $xy \in E$, the edge-weight set is equal to $\{f(x)+f(y)+$

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$f(xy) : x, y \in V, xy \in E\} = \{a,\ a+d,\ a+2d,\ \ldots,\ a+(|E|-1)d\}$, where $a > 0$, $d \geq 0$ are two fixed integers. Such a labeling is called a super (a, d)-edge antimagic total ((a, d)-SEAT) labeling if $f(V) = \{1, 2, \ldots, |V|\}$. A graph that has an (a, d)-EAV ((a, d)-EAT or (a, d)-SEAT) labeling is called an (a, d)-EAV ((a, d)-EAT or (a, d)-SEAT) graph. For an (a, d)-EAV (or (a, d)-SEAT) graph G, an adjacency matrix of G is a $|V| \times |V|$ matrix $A_G = [a_{ij}]$ such that the entry a_{ij} is 1 if there is an edge from vertex with index i to vertex with index j, and entry a_{ij} is 0 otherwise. This paper shows the construction of new larger (a, d)-EAV graph from an existing (a, d)-EAV graph using the adjacency matrix, for $d = 1, 2$. The results will be extended for (a, d)-SEAT graphs with $d = 0, 1, 2, 3$.

1 Introduction

In this paper, we consider finite simple undirected graphs. The set of vertices and edges of a graph G is denoted by V and E, respectively. Let $|V| = n$ and $|E| = m$.

Simanjuntak, Miller and Bertault [9] defined an (a, d)-*edge-antimagic vertex* ((a, d)-EAV) labeling for a graph $G(V, E)$ as an injective mapping f from V onto the set $\{1, 2, \ldots, n\}$ with the property that the edge-weights $\{w(xy) : w(xy) = f(x) + f(y), xy \in E\}$, form an arithmetic sequence with the first term a and difference d, where $a > 0$ and $d \geq 0$ are two fixed integers.

Acharya and Hegde [1] (see also [6]) introduced the concept of a *strongly* (a, d)-*indexable labeling* which is equivalent to (a, d)-EAV labeling. The relationship between the sequential graphs and the graphs having an (a, d)-EAV labeling is shown in [3].

An (a, d)-edge antimagic total ((a, d)-EAT) labeling is a bijection f from $V \cup E$ onto $\{1, 2, \ldots, n + m\}$ with the property that the sums of the label on the edges and the labels of their end points form an arithmetic sequence starting from a and having a common difference d. This labeling is a natural extension of the notion of *edge-magic labeling* which was originally introduced by Kotzig and Rosa in [7], where edge-magic labeling is called *magic valuation*. Relationships between (a, d)-EAT labeling and other labelings, namely, (a, d)-EAV labeling are presented in [2].

An (a, d)-EAT labeling is called *super* (a, d)-*edge antimagic total* ((a, d)-SEAT) labeling if $f(V) = \{1, 2, \ldots, n\}$. This labeling is a natural extension of the notion of a *super edge-magic labeling* defined by Enomoto *et al.* in [5]. A graph that has an (a, d)-EAV ((a, d)-EAT or (a, d)-SEAT) labeling is called an (a, d)-EAV ((a, d)-EAT or (a, d)-SEAT) graph.

An adjacency matrix of G is a symmetric matrix $A_G = [a_{ij}]$ of order n such that the entry a_{ij} is 1 if there is an edge from the vertex with index i to the vertex with index j, and the entry a_{ij} is 0 otherwise.

There are many results on graph labeling, including on edge antimagic vertex labeling. Sugeng and Miller in [10] have explained the relationship between (a, d)-EAV labeling and adjacency matrix and shown how to manipulate this matrix to construct new (a, d)-EAV graphs, for $d = 1$. In this paper, we give a construction of new larger (a, d)-EAV graph from an existing (a, d)-EAV graph by using adjacency matrix, for $d = 1$ and $d = 2$. The results will be extended for (a, d)-SEAT graphs.

2 Some Properties

2.1 Adjacency Matrix

Let $G = G(V, E)$ be a graph with an (a, d)-EAV labeling f. Label the vertices in G such that $f(v_i) = i$, for $i = 1, 2, \dots, n$. An $n \times n$ matrix $A_G = [a_{ij}]$, $i, j = 1, 2, \dots, n$, is called an *adjacency matrix* of G if

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be an (a, d)-EAV graph with adjacency matrix A_G. Since G is an undirected graph, A_G is a symmetric matrix. Beside that, A_G has another characteristic that shows that A_G is a matrix of an (a, d)-EAV graph. A skew diagonal S_r, $r = 3, 4, \dots, 2n - 1$, of A_G is $\{a_{ij} : i + j = r;\ i, j = 1, 2, \dots, n\}$ (see Figure 1).

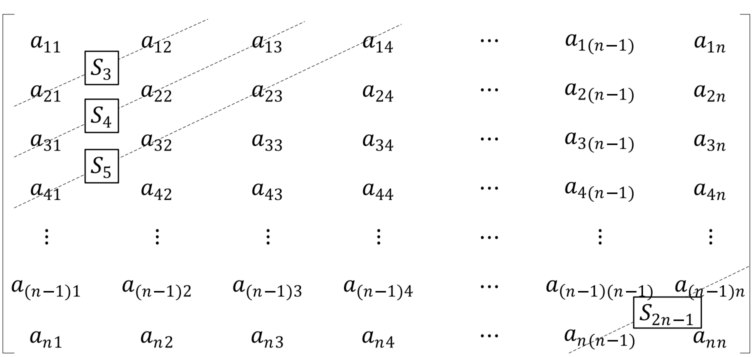


Figure 1: Skew diagonal S_r in a matrix A_G.

A skew diagonal S_r contains all entries of A_G that are related to edges with weight r. With respect to the symmetry of A_G, every skew diagonal of A_G has either zero or exactly two "1" elements. A skew diagonal that only contains the zero elements is called zero skew diagonal, while a skew diagonal that contains exactly two "1" elements is called non-zero skew diagonal.

Sugeng and Miller [10] explained that the set of edge-weights $\{f(x)+f(y) : x, y \in V\}$ in skew diagonal lines generate a sequence of integers of difference d. If $d = 1$ then the nonzero skew diagonal lines form a band of consecutive integers. If $d = 2$ then the non-zero skew diagonal lines form a band of difference 2 with a zero skew diagonal line in between. We have similar skew diagonal line bands for $d = 3, 4, \ldots$ and denote such a skew diagonal band as d-band.

2.2 Maximal (a, d)-EAV Graph

A maximal (a, d)-EAV graph of order n is a graph that has an (a, d)-EAV labeling and has the maximum possible number of edges. If G is a maximal $(a, 1)$-EAV graph then $a = 3$. From the adjacency matrix of a maximal $(3, 1)$-EAV graph, we can see that the first "1" elements will be in the position of (1,2) and (2,1).

Observation 1. [10] *The number of edges of a maximal* (a, d)*-EAV graph of order* n *is* $\lceil \frac{n-1}{d} \rceil + \lceil \frac{n-2}{d} \rceil$.

Consequently, a maximal (a, d)-EAV graph of order n cannot be connected for $d > 2$ since the maximum number of edges is less than the maximum number of edges for $d = 2$, i.e., $\lceil \frac{n-1}{2} \rceil + \lceil \frac{n-2}{2} \rceil = n - 1$.

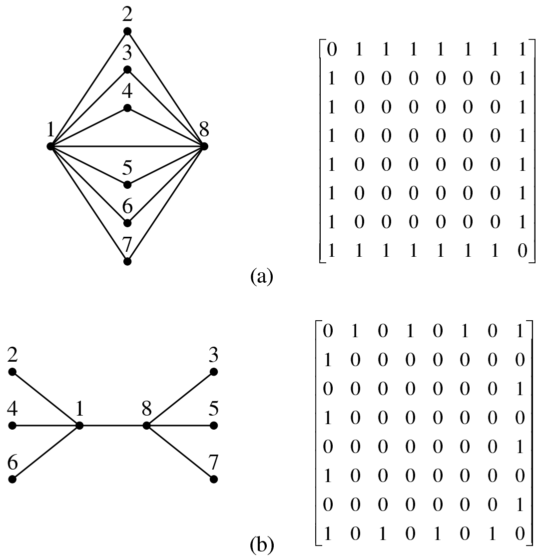
We can construct adjacency matrices of maximal $(3, d)$-EAV graphs for $d = 1, 2$ by putting "1" elements at the ends of each non zero skew diagonal. *A triangular book* $B_{n-2}(C_3)$ is the complete tripartite graph $K_{1,1,n-2}$. It is a graph consisting of $n - 2$ triangles all sharing a common edge. A double star obtained from two vertex disjoint copies of the star $S_{\frac{n}{2}}$ by connecting their centers we call the *twin star graph*, $Twin(n)$. Both $B_{n-2}(C_3)$ and $Twin(n)$ are maximal $(3, 1)$-EAV graphs and maximal $(3, 2)$-EAV graphs of order n, respectively. Figure 2(a) depicts the triangular book graph $B_6(C_3)$ of order 8 with $(3, 1)$-EAV labeling and its adjacency matrix. Figure 2(b) shows the twin star graph $Twin(8)$ with $(3, 1)$-EAV labeling and its adjacency matrix.

3 Constructing New Larger (a, d)-EAV Graph Using Adjacency Matrix

We can construct new (a^*, d)-EAV graphs from an existing (a, d)-EAV graph by using adjacency matrices manipulation. Here we only consider how adjacency matrix manipulation can be used to construct a new larger maximal $(3, d)$-EAV graph. Given an (a, d)-EAV graph G, there are several ways to obtain a larger (a, d)-EAV graph, such as adding some vertices and edges, combining two (or more) given (a, d)-EAV graphs, and combining two (or more) given (a, d)-EAV graphs and adding some vertices and edges.

3.1 Constructing New Larger Maximal (3,1)-EAV Graph Using Adjacency Matrix

We will construct a new larger maximal (3,1)-EAV graph by adding some vertices and some edges to an existing maximal (3,1)-EAV graph using adjacency matrix. It



$$\begin{bmatrix} 0&1&1&1&1&1&1&1\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1\\ 1&1&1&1&1&1&1&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&1&0&1&0&1&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 1&0&1&0&1&0&1&0 \end{bmatrix}$$

Figure 2: Graphs $B_6(C_3)$ and $Twin(8)$ with $(3,1)$-EAV labelings and corresponding adjacency matrices.

can be done by adding some columns and rows in the adjacency matrix and make it deal with the properties of maximal (3,1)-EAV graph. Let us note that the *transpose* A' of a matrix A is the matrix obtained from A by writing its rows as columns.

Theorem 1. *Let G be a maximal $(3,1)$-EAV graph of order n, $n \geq 2$, with adjacency matrix A_G. Let $t = [t_{i1}]$ be $n \times 1$ matrix with*

$$t_{i1} = \begin{cases} 1, & \text{for} \quad i = 1,2 \\ 0, & \text{for} \quad i = 3,4,\ldots,n. \end{cases}$$

Then the matrix

$$M = \begin{bmatrix} 0 & t' \\ t & A_G \end{bmatrix}$$

is the adjacency matrix of maximal $(3,1)$-EAV graph of order $n+1$.

Proof. Matrix M contains A_G as its diagonal block matrix starting in position $(2,2)$. Therefore, each vertex v_i in G with label i is now labeled with $i+1$ and it makes the S_r, $r = 5,6,\ldots,2n+1$, of M non-zero skew diagonals. The matrices t' and t in M fill the S_3 and S_4 and make them non-zero skew diagonals. Now M is an $(n+1)\times(n+1)$ symmetric matrix with nonzero skew diagonal lines induce a band of consecutive integers started with S_3 until S_{2n+1}. ■

Since the matrix M in Theorem 1 is an adjacency matrix of a $(3,1)$-EAV graph, it can be considered as A_G. Thus repeating the construction from Theorem 1 leads to the following corollary.

Corollary 1. *Let G be a maximal $(3,1)$-EAV graph of order n, $n \geq 2$, with adjacency matrix A_G. Let $t_k = [t_{i1}]$, $k = 1, 2, \dots,$ be a $(n+k-1) \times 1$ matrix with*

$$t_{i1} = \begin{cases} 1, & \text{for} \quad i = 1, 2 \\ 0, & \text{for} \quad i = 3, 4, \dots, n+k-1 \end{cases}$$

and let M_1 be a $(n+1) \times (n+1)$ matrix with

$$M_1 = \begin{bmatrix} 0 & t_1' \\ t_1 & A_G \end{bmatrix}.$$

Then the matrix

$$M_k = \begin{bmatrix} 0 & t_k' \\ t_k & M_{k-1} \end{bmatrix}, \quad k = 2, 3, \dots$$

is the adjacency matrix of a maximal $(3,1)$-EAV graph of order $n+k$.

Theorem 1 and Corollary 1 show a construction of new larger maximal $(3,1)$-EAV graphs by adding several columns and rows on the left and top side of the adjacency matrix of an existing maximal $(3,1)$-EAV graph. We also can add several columns and rows on the right and bottom sides of an adjacency matrix.

Theorem 2. *Let G be a maximal $(3,1)$-EAV graph of order n, $n \geq 2$, with adjacency matrix A_G. Let $t = [t_{i1}]$ and $t^* = [t^*_{i1}]$ be $n \times 1$ matrices with*

$$t_{i1} = \begin{cases} 1, & \text{for} \quad i = 1, 2 \\ 0, & \text{for} \quad i = 3, 4, \dots, n \end{cases}$$

$$t^*_{i1} = \begin{cases} 0, & \text{for} \quad i = 1, 2, \dots, n-2 \\ 1, & \text{for} \quad i = n-1, n. \end{cases}$$

Then the matrix

$$M = \begin{bmatrix} 0 & t' & 0 \\ t & A_G & t^* \\ 0 & (t^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal $(3,1)$ EAV graph of order $n+2$.

Proof. Matrix M contains A_G as its diagonal block matrix starting in position $(2,2)$. Therefore, each vertex v_i in G with label i is labeled with $i+1$ and it makes the S_r, $r = 5, 6, \dots, 2n+1$, of M non-zero skew diagonals. The matrices t' and t in M fill the S_3 and S_4 and make them non-zero skew diagonals and the matrices $(t^*)'$ and t^* in M fill the S_{2n+2} and S_{2n+3} and make them non-zero skew diagonals. Now M is an $(n+2) \times (n+2)$ symmetric matrix with nonzero skew diagonal lines induce a band of consecutive integers starting with S_3 until S_{2n+3}. ■

Theorem 2 can be done repeatedly which leads to the following corollary.

Corollary 2. *Let G be a maximal $(3,1)$-EAV graph of order n, $n \geq 2$, with adjacency matrix A_G. Let $t_k = [t_{i1}]$ and $t_k^* = [t_{i1}^*]$, $k = 1, 2, \ldots,$ be $(n+2k-2) \times 1$ matrices with*

$$t_{i1} = \begin{cases} 1, & \text{for} \quad i = 1, 2 \\ 0, & \text{for} \quad i = 3, 4, \ldots, n+2k-2 \end{cases}$$

$$t_{i1}^* = \begin{cases} 0, & \text{for} \quad i = 1, 2, \ldots, n+2k-4 \\ 1, & \text{for} \quad i = n+2k-3, n+2k-2 \end{cases}$$

and let M_1 be a $(n+2) \times (n+2)$ matrix with

$$M_1 = \begin{bmatrix} 0 & t_1' & 0 \\ t_1 & A_G & t_1^* \\ 0 & (t_1^*)' & 0 \end{bmatrix}.$$

Then the matrix

$$M_k = \begin{bmatrix} 0 & t_k' & 0 \\ t_k & M_{k-1} & t_k^* \\ 0 & (t_k^*)' & 0 \end{bmatrix}, \quad k = 2, 3, \ldots$$

is the adjacency matrix of a maximal $(3,1)$-EAV graph of order $n+2k$.

Let us start with the triangular book graph $B_2(C_3)$. According to Corollary 2, the matrix M_k, k even, produces a new maximal $(3,1)$-EAV graph of order $4+2k$. This graph is a *triangular ladder* $\mathbb{L}_{2+k}$ which can be obtained from Cartesian product of two paths P_{2+k} and P_2 with $V(P_{2+k} \times P_2) = \{u_i, v_i : 1 \leq i \leq 2+k\}$ and $E(P_{2+k} \times P_2) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq 1+k\} \cup \{u_i v_i : 1 \leq i \leq 2+k\}$ by completing the edges $u_{2i-1}v_{2i}$, for $1 \leq i \leq \frac{k}{2}+1$, and $v_{2i}u_{2i+1}$, for $1 \leq i \leq \frac{k}{2}$, (see Figure 3).

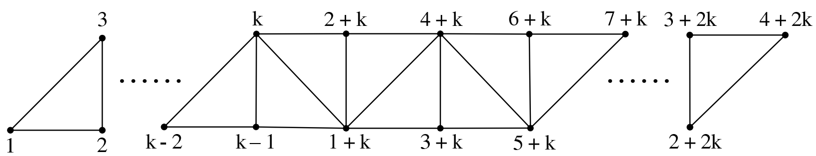


Figure 3: Constructing new larger $(3,1)$-EAV graphs by using Theorem 2.

Corollary 3. *Every triangular ladder $\mathbb{L}_{2+k}$, $k \geq 2$ even, is a maximal $(3,1)$-EAV graph.*

Graph $G(H, \mathbb{L}_{2+k})$ is called a *triangular ladder towered graph* if it is obtained from a graph H and the disjoint union of two copies of the triangular ladder $\mathbb{L}_{2+k}$ in such a way that only two different edges in G are mutual with the edges $u_{2+k}v_{2+k}$ in each copy of $\mathbb{L}_{2+k}$.

Let us start with $B_6(C_3)$, see Figure 4(a). A triangular ladder towered graph $G(B_6(C_3), \mathbb{L}_4)$, see Figure 4(b), is a maximal $(3,1)$-EAV graph. The form of the triangular ladder towered graph $G(B_{n-2}(C_3), \mathbb{L}_{2+k})$, $n \geq 4$ and $k \geq 2$ even, is shown in Figure 4(c). For any maximal $(3,1)$-EAV graph H, the general form of the triangular ladder towered graph $G(H, \mathbb{L}_{2+k})$ is shown in Figure 5.

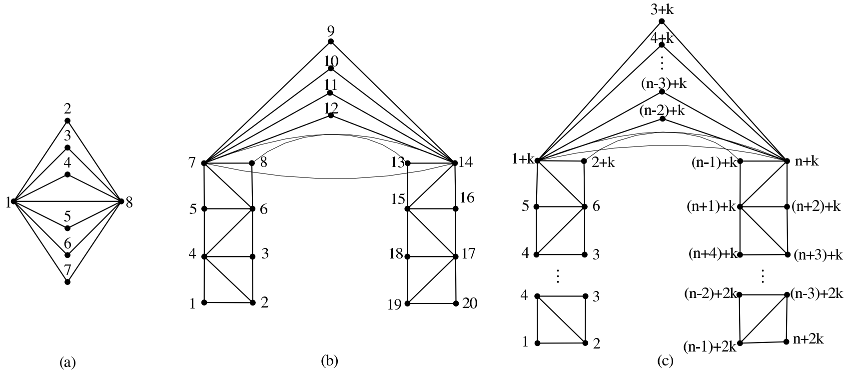


Figure 4: Triangular ladder towered graphs.

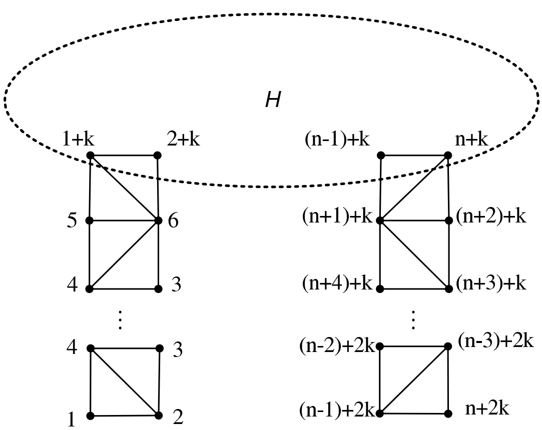


Figure 5: General form of triangular ladder towered graph $G(H, \mathbb{L}_{2+k})$.

Corollary 4. *Let H be any maximal $(3,1)$-EAV graph. Then the triangular ladder towered graph $G(H, \mathbb{L}_{2+k})$, $k \geq 2$ even, is also a maximal $(3,1)$-EAV graph.*

3.2 Constructing New Larger Maximal (3,2)-EAV Graph Using Adjacency Matrix

Similarly to the construction of a new larger $(3,1)$-EAV graph, a new larger $(3,2)$-EAV graph will be constructed by adding some vertices and some edges to an $(a,2)$-

EAV graph by using adjacency matrix. It can be done by adding some columns and rows in the adjacency matrix and make it deal with the properties of $(3, 2)$-EAV graph. Some of the results presented in this subsection are already discussed in [8].

Theorem 3. *Let G be a maximal $(3, 2)$-EAV graph of order n, $n \geq 1$, with adjacency matrix A_G. Let $s = [s_{i1}]$ and $s^* = [s^*_{i1}]$ be $n \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for} \quad i = 1 \\ 0, & \text{for} \quad i = 2, 3, \ldots, n \end{cases}$$

$$s^*_{i1} = \begin{cases} 0, & \text{for} \quad i = 1, 2, \ldots, n-1 \\ 1, & \text{for} \quad i = n. \end{cases}$$

Then the matrix

$$M = \begin{bmatrix} 0 & s' & 0 \\ s & A_G & s^* \\ 0 & (s^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal $(3, 2)$-EAV graph of order $n + 2$.

Proof. Matrix M contains A_G as its diagonal block matrix starting in position $(2, 2)$. Therefore, each vertex v_i in G with label i is labeled with $i + 1$ and it makes the S_r, $r = 5, 7, \ldots, 2n + 1$, of M non-zero skew diagonals. The matrices s' and s in M fill the S_3 and make it non-zero skew diagonal and the matrices $(s^*)'$ and s^* in M fill the S_{2n+3} and make it non-zero skew diagonal. Now M is an $(n + 2) \times (n + 2)$ symmetric matrix with nonzero skew diagonal lines induce a band of the arithmetic sequence of difference 2 starting with S_3 until S_{2n+3}. ■

Since the matrix M in Theorem 3 is an adjacency matrix of a $(3, 2)$-EAV graph, it can be considered as A_G. Thus repeating the construction from Theorem 3 leads to the following corollary.

Corollary 5. *Let G be a maximal $(3, 2)$-EAV graph of order n, $n \geq 1$, with adjacency matrix A_G. Let $s_k = [s_{i1}]$ and $s^*_k = [s^*_{i1}]$, $k = 1, 2, \ldots,$ be $(n + 2k - 2) \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for} \quad i = 1 \\ 0, & \text{for} \quad i = 2, 3, \ldots, n + 2k - 2 \end{cases}$$

$$s^*_{i1} = \begin{cases} 0, & \text{for} \quad i = 1, 2, \ldots, n + 2k - 3 \\ 1, & \text{for} \quad i = n + 2k - 2 \end{cases}$$

and let M_1 be a $(n + 2) \times (n + 2)$ matrix with

$$M_1 = \begin{bmatrix} 0 & s'_1 & 0 \\ s_1 & A_G & s^*_1 \\ 0 & (s^*_1)' & 0 \end{bmatrix}.$$

Then the matrix

$$M_k = \begin{bmatrix} 0 & s'_k & 0 \\ s_k & M_{k-1} & s^*_k \\ 0 & (s^*_k)' & 0 \end{bmatrix}, \quad k = 2, 3, \ldots$$

is the adjacency matrix of a maximal $(3,2)$*-EAV graph of order* $n + 2k$*.*

Graph $G(H, P_k)$ is called *path towered graph* if it is obtained from a graph H of order n and the disjoint union of two copies of the path P_k in such a way that an end vertex of each path P_k is adjoined to a vertex of the graph H. Thus $G(H, P_k)$ is graph of order $n + 2k - 2$.

Let us start with the twin star graph $Twin(8)$, see Figure 6(a). Then, forming the matrix M_1 by using Corollary 5 produces the new $(3,2)$-EAV graph $G(Twin(8), P_2)$ of order 10, see Figure 6(b). Forming the matrix M_k produces the new maximal $(3,2)$-EAV graph $G(Twin(8), P_{k+1})$ of order $8 + 2k$, see Figure 6(c).

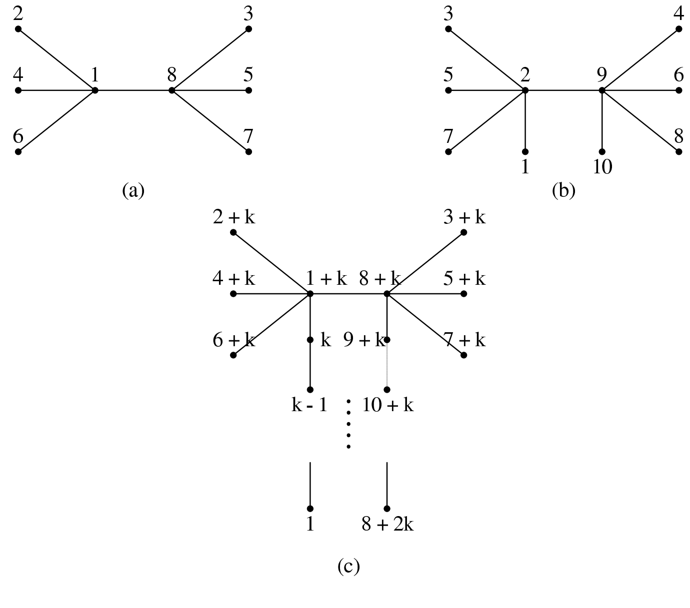


Figure 6: Constructing new larger $(3,2)$-EAV graph by using Theorem 3.

As an immediate consequence of Theorem 3 and Corollary 5 we can state the following corollary.

Corollary 6. *Let* H *be any maximal* $(3,2)$*-EAV graph. Then the path towered graph* $G(H, P_k)$*,* $k \geq 2$*, is also a maximal* $(3,2)$*-EAV graph.*

3.3 Other Constructions

Sugeng and Miller [10] have proved the following theorem.

Theorem 4. [10] *Let G_i, $i = 1, 2, \dots, p$, be an $(a, 1)$-EAV graph of order n_i. Then there are $(a, 1)$-EAV graphs of order w, where $\sum_{i=1}^{p} n_i - 2(p-1) \leq w \leq \sum_{i=1}^{p} n_i$, and each contains G_i as induced subgraph.*

The proof of Theorem 4 uses a construction of a new adjacency matrix where its main diagonal contains adjacency matrices of graphs G_i, $i = 1, 2, \dots, p$, to obtain a new adjacency matrix of a maximal $(a, 1)$-EAV graph.

Let $B_{n_i-2}(C_3)$ be the triangular book of order n_i with adjacency matrix A_i, $i = 1, 2, \dots, p$. Then combining the graphs using manipulation of adjacency matrix as the main diagonal block matrices produces a new class of maximal $(3, 1)$-EAV graphs with order $\sum_{i=1}^{p} n_i - 2(p-1)$, see Figure 7, or with order $\sum_{i=1}^{p} n_i - (p-1)$, see Figure 8. In the first case we obtain a *ladder of triangular books* $LB(n_1 - 2, n_2 - 2, \dots, n_p - 2)$ and in the second case we obtain a *chain of triangular books* $CB(n_1 - 2, n_2 - 2, \dots, n_p - 2)$.

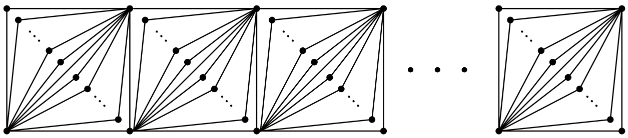


Figure 7: Ladder of triangular books.

Using the same construction as in Theorem 4 when the main diagonal of an adjacency matrix contains adjacency matrices of $(b, 2)$-EAV graphs we are able to obtain a $(b, 2)$-EAV graph. Thus we have

Theorem 5. *Let G_i, $i = 1, 2, \dots, p$, be $(b, 2)$-EAV graphs of order n_i, respectively. Then there are $(b, 2)$-EAV graphs of order w, where $\sum_{i=1}^{p} n_i - 2(p-1) \leq w \leq \sum_{i=1}^{p} n_i$, and each contains G_i as induced subgraph.*

Another way to construct a new larger graph that has the same labeling as a given graph was introduced by Cavalier [4]. Using a similar idea, we have the following theorem.

Theorem 6. *Let G be a maximal $(3, 2)$-EAV graph of order n with adjacency matrix A_G. Let $s = [s_{i1}]$ and $s^* = [s^*_{i1}]$ be $n \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for} \quad i = 1 \\ 0, & \text{for} \quad i = 2, 3, \dots, n \end{cases}$$

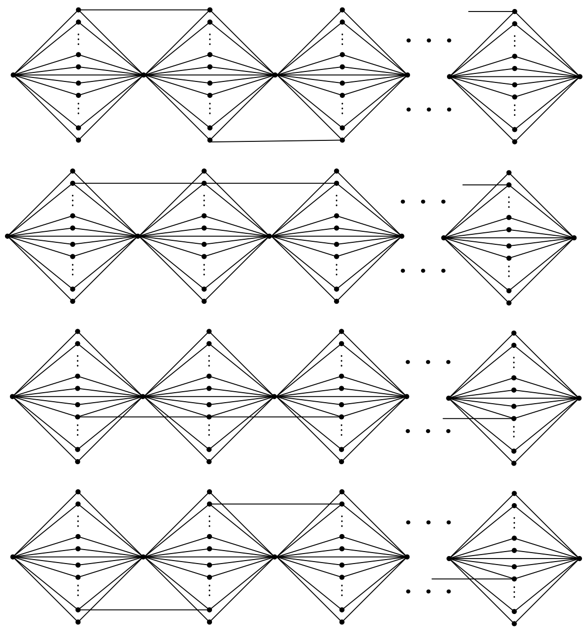


Figure 8: Chain of triangular books.

$$s^*_{i1} = \begin{cases} 0, & \text{for} \quad i = 1, 2, \ldots, n-1 \\ 1, & \text{for} \quad i = n, \end{cases}$$

and let $\mathbf{0}$ *be the* $n \times 1$ *matrix of all zeros and* $\mathbb{O}$ *be the* $n \times n$ *matrix of all zeros. Then a* $(2pn + 2) \times (2pn + 2)$ *matrix* M *constructed from* $2p$ *copies of* A_G*'s*

$$M = \begin{bmatrix} 0 & s' & \mathbf{0}' & s' & \cdots & \mathbf{0}' & 1 \\ s & A_G & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & A_G & \mathbb{O} & \cdots & \mathbb{O} & s^* \\ s & \mathbb{O} & \mathbb{O} & A_G & \cdots & \mathbb{O} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & A_G & s^* \\ 1 & \mathbf{0}' & (s^*)' & \mathbf{0}' & \cdots & (s^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal $(3, 2)$*-EAV graph of order* $2pn + 2$.

Proof. According to Theorem 5 if we add the element 1 as the last element in the first row and as the first element in the last row, and also the matrices s and s^* then the resulting matrix M with A_G as its main diagonal block matrices forms a new adjacency matrix for $(3, 2)$-EAV graph. ■

Theorem 6 can be done repeatedly and it leads to the following corollary.

Corollary 7. *Let G be a maximal $(3,2)$-EAV graph of order n with adjacency matrix $A_G = M_0$ and let q_k, $k = 0,1,2,\ldots$ be the order of the matrix M_k. Let $s_k = [s_{i1}]$ and $s_k^* = [s_{i1}^*]$ be $q_k \times 1$ matrices with*

$$s_{i1} = \begin{cases} 1, & \text{for} \quad i = 1 \\ 0, & \text{for} \quad i = 2,3,\ldots,q_k \end{cases}$$

$$s_{i1}^* = \begin{cases} 0, & \text{for} \quad i = 1,2,\ldots,q_k - 1 \\ 1, & \text{for} \quad i = q_k, \end{cases}$$

and $\mathbf{0}$ be the $q_k \times 1$ matrix of all zeros and $\mathbb{O}$ be the $q_k \times q_k$ matrix of all zeros and let M_1 be a matrix of order $q_1 = 2pn + 2$ constructed from $2p$ copies of M_0,

$$M_1 = \begin{bmatrix} 0 & s' & \mathbf{0}' & s' & \cdots & \mathbf{0}' & 1 \\ s & M_0 & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & M_0 & \mathbb{O} & \cdots & \mathbb{O} & s^* \\ s & \mathbb{O} & \mathbb{O} & M_0 & \cdots & \mathbb{O} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & M_0 & s^* \\ 1 & \mathbf{0}' & (s^*)' & \mathbf{0}' & \cdots & (s^*)' & 0 \end{bmatrix}.$$

Then the matrix M_k constructed from $2p$ copies of M_{k-1}

$$M_k = \begin{bmatrix} 0 & s' & \mathbf{0}' & s' & \cdots & \mathbf{0}' & 1 \\ s & M_{k-1} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & M_{k-1} & \mathbb{O} & \cdots & \mathbb{O} & s^* \\ s & \mathbb{O} & \mathbb{O} & M_{k-1} & \cdots & \mathbb{O} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & M_{k-1} & s^* \\ 1 & \mathbf{0}' & (s^*)' & \mathbf{0}' & \cdots & (s^*)' & 0 \end{bmatrix}$$

is the adjacency matrix of a maximal $(3,2)$-EAV graph of order $q_k = 2pq_{k-1} + 2$.

A graph containing only one vertex is a trivial $(3,2)$-EAV graph. We can combine a finite even number of copies of that trivial graph and construct a new $(3,2)$-EAV graph with adjacency matrix M_1 by using Corollary 7. For example, we combine 6 graphs of one vertex and produce a new $(3,2)$-EAV graph of order 8, see Figure 9(a). Then we construct the adjacency matrix M_2 by combining 6 M_1 and produce the new larger $(3,2)$-EAV graph of order 50, see Figure 9(b). Constructing M_3 by combining 6 M_2 will produce a new larger $(3,2)$-EAV graph, see Figure 10.

4 Results for (a,d)-Super Edge Antimagic Total Graphs

In [2] there is proved the following theorem.

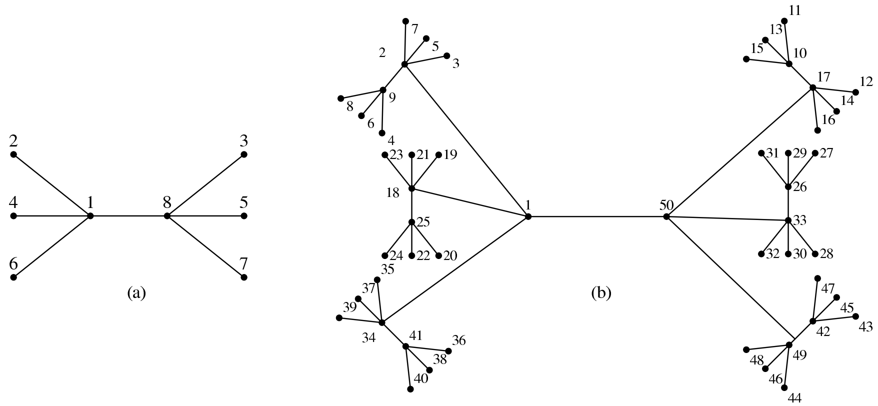


Figure 9: Constructing new larger $(3, 2)$-EAV graph by using Corollary 7.

Theorem 7. [2] *If G of order n and size m has an (a, d)-EAV labeling, then*
(i) G has an $(a + n + 1, d + 1)$-SEAT labeling, and
(ii) G has an $(a + n + m, d - 1)$-SEAT labeling.

According to Theorem 7 from the previous results it follows that

Corollary 8. *Triangular book graph $B_{n-2}(C_3)$ and triangular ladder $\mathbb{L}_n$, both of order n and size $2n - 3$, admit an $(n + 4, 2)$-SEAT labeling and a $(3n, 0)$-SEAT labeling.*

Corollary 9. *The triangular ladder towered graph $G(B_{n-2}(C_3), \mathbb{L}_{2+k}))$ of order $n+2k$ and size $2n + 8k + 5$, $k \geq 2$ even, admits an $(n + 2k + 4, 2)$-SEAT labeling and a $(3n + 10k + 8, 0)$-SEAT labeling.*

Corollary 10. *The ladder of triangular books $LB(n_1 - 2, n_2 - 2, \ldots, n_p - 2)$ of order $\sum_{i=1}^{p} n_i - 2(p - 1)$ admits a $(\sum_{i=1}^{p} n_i - 2(p - 1) + 4, 2)$-SEAT labeling and a $(3\sum_{i=1}^{p} n_i - 2(p - 1), 0)$-SEMT labeling.*

Corollary 11. *The chain of triangular books $CB(n_1 - 2, n_2 - 2, \ldots, n_p - 2)$ of order $\sum_{i=1}^{p} n_i - (p - 1)$ admits a $(\sum_{i=1}^{p} n_i - (p - 1) + 4, 2)$-SEAT labeling and a $(3\sum_{i=1}^{p} n_i - (p - 1), 0)$-SEMT labeling.*

Corollary 12. *The twin star graph $Twin(n)$ of order n and size $n - 1$ admits a $(2n + 2, 1)$-SEAT labeling and an $(n + 4, 3)$-SEAT labeling.*

Corollary 13. *The path towered graph $G(H, P_k)$, $k \geq 2$, of order $n + 2k - 2$ and size $n + 2k - 3$ admits a $(2n + 4k - 2, 1)$-SEAT labeling and an $(n + 2k + 2, 3)$-SEAT labeling.*

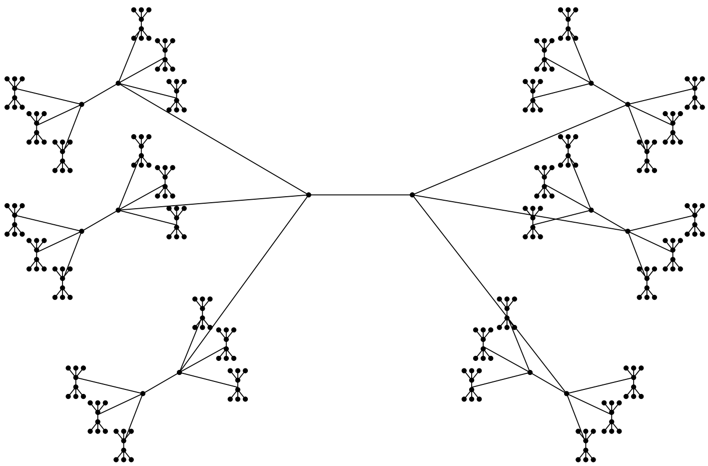


Figure 10: Graph given by adjacency matrix M_3.

5 Conclusion

In this paper we showed how to construct a new larger (a, d)-EAV graph from a given graph with an (a, d)-EAV labeling, $d = 1, 2$, by using adjacency matrices. We also extended the results to the SEAT labelings.

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