Irrationality of certain numbers that contain values of the di- and trilogarithm

Kh. Hessami Pilehrood* (Shahrekord),
T. Hessami Pilehrood† (Shahrekord) and W. Zudilin‡ (Moscow)

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Abstract

We prove that, for $z \in \{1/2, 2/3, 3/4, 4/5\}$, at least one of the two numbers

$$\sum_{l=1}^{\infty} \frac{z^l}{l^2} + \log(1-z)\log z \qquad \text{and} \qquad \sum_{l=1}^{\infty} \frac{z^l}{l^3} + \frac{1}{2}\log(1-z)\log^2 z$$

is irrational.

The irrationality result that we prove here, involves the values of the functions

$$f_j(z) = \mathrm{Li}_j(z) - \mathrm{Li}_1(z) \cdot \frac{\log^{j-1} z}{(j-1)!}, \qquad j = 2, 3, \dots .$$

Here, as usual,

$$\mathrm{Li}_j(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^j}, \quad |z| < 1, \qquad j = 1, 2, 3, \dots,$$

denote the polylogarithmic functions; $\mathrm{Li}_1(z) = -\log(1-z)$.

Theorem 1. *For $z \in \{1/2, 2/3, 3/4, 4/5\}$, at least one of the two numbers $f_2(z)$ and $f_3(z)$ is irrational.*

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Using the classical formulae (see, e.g., [5], eqs. (1.11) and (6.10)) we may also express the functions $f_2(z)$ and $f_3(z)$ as follows:

$$f_2(z) = \frac{\pi^2}{6} - \mathrm{Li}_2(1-z),$$
$$f_3(z) = \zeta(3) + \frac{1}{6}\log^3 z + \frac{\pi^2}{6}\log z - \mathrm{Li}_3(1-z) - \mathrm{Li}_3(1-z^{-1}).$$

Moreover, the values of these functions at the point $z = 1/2$ are computed by means of $\log 2$, $\zeta(2) = \pi^2/6$ and $\zeta(3)$ (see [5], eqs. (1.16) and (6.12)):

$$f_2(1/2) = \frac{1}{2}\zeta(2) + \frac{1}{2}\log^2 2, \qquad f_3(1/2) = \frac{7}{8}\zeta(3) - \frac{1}{2}\zeta(2)\log 2 - \frac{1}{3}\log^3 2.$$

Using these formulae we obtain the following corollary of Theorem 1.

Theorem 2. *At least one of the two numbers*

$$\pi^2 + 6\log^2 2 \qquad \textit{and} \qquad \zeta(3) - \frac{2}{21}(\pi^2 + 4\log^2 2)\log 2$$

is irrational.

Relative results, 'at least one of the numbers $3\zeta(3) - c\zeta(2)$, $\zeta(2) - 2c\log 2$ ($c \in \mathbb{Q}$) is irrational' and 'at least one of the numbers $\mathrm{Li}_2(1/q)$, $\mathrm{Li}_3(1/q)$ ($q \in \mathbb{Z} \setminus \{0, 2\}$) is irrational', are proved in [3] and [4], respectively.

Our proof relies on a general hypergeometric construction of two linear forms in the polylogarithms and positive powers of the logarithm, respectively. This idea was recently used in [2] for proving that at least one of the three numbers $f_2(1/2)$, $f_3(1/2)$ and $f_4(1/2)$ is irrational. We are able to improve this earlier result and present the related ones due to the powerful group-structure arithmetic method introduced in [6] and [7] in order to prove new estimates for irrationality measures of $\zeta(2)$ and $\zeta(3)$ (see also [8] and [9]).

1 Hypergeometric series and integral

Let $a_0, a_1, \dots, a_s$ and $b_0, b_1, \dots, b_s$ be collections of positive integers satisfying

$$\begin{gathered} b_0 = 1 < a_0, a_1, \dots, a_s < b_1, \dots, b_s, \qquad \sum_{j=0}^{s} a_j < \sum_{j=0}^{s} b_j, \\ \min_{0 \leqslant j \leqslant s}\{a_j\} + \max_{0 \leqslant j \leqslant s}\{a_j\} \leqslant b_l \qquad \text{for all } l = 1, \dots, s, \end{gathered} \tag{1}$$

We will also consider the ordered versions $\{a_0^*, a_1^*, \dots, a_s^*\} = \{a_0, a_1, \dots, a_s\}$ and $\{b_0^*, b_1^*, \dots, b_s^*\} = \{b_0, b_1, \dots, b_s\}$ of the collections:

$$a_0^* \leqslant a_1^* \leqslant \dots \leqslant a_s^*, \qquad b_0^* = 1, \quad b_1^* \geqslant b_2^* \geqslant \dots \geqslant b_s^*.$$

To this data we assign the rational function

$$\widetilde{R}(t)=\widetilde{R}(\boldsymbol{a},\boldsymbol{b};t)=\prod_{j=0}^{s}\frac{\Gamma(t+a_j)}{\Gamma(t+b_j)}=\frac{(t+1)(t+2)\cdots(t+a_0-1)}{\prod_{j=1}^{s}(t+a_j)\cdots(t+b_j-1)} \tag{2}$$

and its 'arithmetically normalized' version

$$R(t)=R(\boldsymbol{a},\boldsymbol{b};t)=\frac{\prod_{j=1}^{s}\Gamma(b_j-a_j)}{\Gamma(a_0)}\widetilde{R}(\boldsymbol{a},\boldsymbol{b};t)=\sum_{j=1}^{s}\sum_{k=a_j^*}^{b_j^*-1}\frac{A_{jk}}{(t+k)^j}. \tag{3}$$

Our main objects then are as follows:

$$\begin{aligned}
I=I(\boldsymbol{a},\boldsymbol{b};z)&=z^{a_0^*}\sum_{t=0}^{\infty}R(t)z^t=\sum_{t=1-a_0^*}^{\infty}R(t)z^{t+a_0^*}=\sum_{j=1}^{s}\sum_{k=a_j^*}^{b_j^*-1}A_{jk}z^{-k+a_0^*}\sum_{t=1-a_0^*}^{\infty}\frac{z^{t+k}}{(t+k)^j}\\
&=\sum_{j=1}^{s}\sum_{k=a_j^*}^{b_j^*-1}A_{jk}z^{-k+a_0^*}\sum_{l=1+k-a_0^*}^{\infty}\frac{z^l}{l^j}=\sum_{j=1}^{s}\sum_{k=a_j^*}^{b_j^*-1}A_{jk}z^{-k+a_0^*}\biggl(\mathrm{Li}_j(z)-\sum_{l=1}^{k-a_0^*}\frac{z^l}{l^j}\biggr)\\
&=\sum_{j=1}^{s}\mathrm{Li}_j(z)\cdot\sum_{k=a_j^*}^{b_j^*-1}A_{jk}z^{-k+a_0^*}-\sum_{j=1}^{s}\sum_{k=a_j^*}^{b_j^*-1}A_{jk}\sum_{l=1}^{k-a_0^*}\frac{z^{-(k-a_0^*-l)}}{l^j}\\
&=\sum_{j=1}^{s}P_j(z^{-1})\,\mathrm{Li}_j(z)-P_0(z^{-1}),
\end{aligned} \tag{4}$$

and (the closed contour $\mathcal{L}$ below surrounds all poles $t=-k$ for $a_1^*\leqslant k<b_1^*$ of the rational function $R(t)$)

$$\begin{aligned}
J=J(\boldsymbol{a},\boldsymbol{b};z)&=\frac{z^{a_0^*}}{2\pi i}\oint_{\mathcal{L}}R(t)z^t\,\mathrm{d}t=z^{a_0^*}\sum_{k}\mathrm{Res}_{t=-k}(R(t)z^t)\\
&=\sum_{k}z^{-k+a_0^*}\,\mathrm{Res}_{t=-k}(R(t)z^{t+k})=\sum_{j=1}^{s}\sum_{k}A_{jk}z^{-k+a_0^*}\cdot\mathrm{Res}_{t=-k}\,\frac{z^{t+k}}{(t+k)^j}\\
&=\sum_{j=1}^{s}\sum_{k}A_{jk}z^{-k+a_0^*}\cdot\mathrm{Res}_{t=0}\,\frac{z^t}{t^j}=\sum_{j=1}^{s}\sum_{k}A_{jk}z^{-k+a_0^*}\cdot\frac{\log^{j-1}z}{(j-1)!}\\
&=\sum_{j=1}^{s}P_j(z^{-1})\frac{\log^{j-1}z}{(j-1)!}.
\end{aligned} \tag{5}$$

All this means that we arrange to construct 'simultaneous' approximations to the set of polylogarithms $\mathrm{Li}_1(z),\dots,\mathrm{Li}_s(z)$ and the set of logarithm powers $\log z,\dots,\frac{1}{(s-1)!}\log^{s-1}z$. This is essentially the idea from [2], Theorem 3.

2 Arithmetic ingredients

Lemma 1. *Let*

$$m_1 = \max\Big\{a_0 - 1, \max_{1\leqslant j\leqslant s}\{b_j\} - \min_{1\leqslant j\leqslant s}\{a_j\} - 1\Big\}. \tag{6}$$

Then, for any $j = 1, 2, \dots, s$, we have the inclusions

$$D_{m_1}^{s-j} \cdot A_{jk} \in \mathbb{Z},$$

where A_{jk} are the coefficients in the partial-fraction decomposition (3)*.*

Proof. The rational function in (3) may be written as $R(t) = \prod_{j=0}^{s} R_j(s)$, where

$$R_0(t) = \frac{(t+1)(t+2)\cdots(t+a_0-1)}{(a_0-1)!}$$

and

$$R_j(t) = \frac{(b_j - a_j - 1)!}{(t+a_j)\cdots(t+b_j-1)} \quad \text{for } j = 1, \dots, s.$$

The polynomial $R_0(t)$ (of degree $a_0 - 1$) is integer-valued, with the property

$$D_{a_0-1}^{l} \cdot \frac{1}{l!}\frac{\mathrm{d}^l R_0(t)}{\mathrm{d}t^l}\bigg|_{t=-k} \in \mathbb{Z}, \qquad k \in \mathbb{Z}, \quad l = 0, 1, 2, \dots \tag{7}$$

(see [9], Lemma 15). All poles of the rational functions $R_1(t), \dots, R_s(t)$ are of the form $t = -k$ for some integer k in the range $a' \leqslant k < b'$, where $a' = \min_{1\leqslant j\leqslant s}\{a_j\}$ and $b' = \max_{1\leqslant j\leqslant s}\{b_j\}$. Therefore, for any $j = 1, \dots, s$, we have

$$D_{b'-a'-1}^{l} \cdot \frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}t^l}\big(R_j(t)(t+k)\big)\bigg|_{t=-k} \in \mathbb{Z}, \qquad k = a', a'+1, \dots, b'-1, \quad l = 0, 1, 2, \dots \tag{8}$$

(see [9], Lemma 16). Finally, by the Leibniz rule

$$\begin{aligned} A_{s-l,k} &= \frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}t^l}\big(R(t)(t+k)^s\big)\bigg|_{t=-k} \\ &= \sum_{\substack{l_0, l_1, \dots, l_s \geqslant 0 \\ l_0 + l_1 + \dots + l_s = l}} \frac{1}{l_0!}\frac{\mathrm{d}^{l_0} R_0(t)}{\mathrm{d}t^{l_0}}\bigg|_{t=-k} \cdot \frac{1}{l_1!}\frac{\mathrm{d}^{l_1}}{\mathrm{d}t^{l_1}}\big(R_1(t)(t+k)\big)\bigg|_{t=-k} \cdots \\ &\qquad \times \frac{1}{l_s!}\frac{\mathrm{d}^{l_s}}{\mathrm{d}t^{l_s}}\big(R_s(t)(t+k)\big)\bigg|_{t=-k} \end{aligned}$$

for any $k = a', a'+1, \dots, b'-1$ and $l = 0, 1, \dots, s-1$. In view of inclusions (7) and (8), the last formula yields the required result. □

Lemma 2. *Let* $m = \max\{m_1, b_2^* - a_0^* - 1\}$ *with* m_1 *defined in* (6) *and*

$$m_0 = b_1^* - a_0^* - 1 = \max_{1\leqslant j\leqslant s}\{b_j\} - \min_{0\leqslant j\leqslant s}\{a_j\} - 1.$$

Then the polynomials

$$P_j(x) = \sum_{k=a_j^*}^{b_j^*-1} A_{jk} x^{k-a_0^*}, \quad j = 1, \dots, s, \qquad P_0(x) = \sum_{j=1}^{s} \sum_{k=a_j^*}^{b_j^*-1} A_{jk} \sum_{l=1}^{k-a_0^*} \frac{x^{k-a_0^*-l}}{l^j},$$

defined in (4) *and* (5) *as the coefficients of linear forms satisfy*

$$\deg P_j(x) \leqslant m_0, \quad D_{m_0} D_m^{s-1} \cdot P_j(x) \in \mathbb{Z}[x], \qquad j = 0, 1, \dots, s.$$

Proof. The claim follows immediately for the polynomials $P_1(x), \dots, P_s(x)$ from Lemma 1 since $m \geqslant m_1$. In the case of $P_0(x)$ we require the inclusions of Lemma 1, also the obvious ones

$$D_{b_1^*-a_0^*-1} \cdots D_{b_j^*-a_0^*-1} \cdot \sum_{l=1}^{k-a_0^*} \frac{x^{k-a_0^*-l}}{l^j} \in \mathbb{Z}[x], \qquad k = a_j^*, \dots, b_j^* - 1, \quad j = 1, \dots, s,$$

and the fact that $m \geqslant b_j^* - a_0^* - 1$ for $j = 2, \dots, s$. □

Denote by

$$\Pi = \Pi(\boldsymbol{a}, \boldsymbol{b}) = \frac{\prod_{j=1}^{s} \Gamma(b_j - a_j)}{\Gamma(a_0)} = \frac{\prod_{j=1}^{s} (b_j - a_j - 1)!}{(a_0 - 1)!} \tag{9}$$

the arithmetic normalization in (3). By (2)–(5), the quantities

$$\frac{I(\boldsymbol{a}, \boldsymbol{b}; z)}{\Pi(\boldsymbol{a}, \boldsymbol{b})} \qquad \text{and} \qquad \frac{J(\boldsymbol{a}, \boldsymbol{b}; z)}{\Pi(\boldsymbol{a}, \boldsymbol{b})} \tag{10}$$

do not depend on rearranging the parameters $a_0, a_1, \dots, a_s$. Let us denote by $\mathfrak{S}$ the permutation group (of order $(s+1)!$) of the parameters $a_0, a_1, \dots, a_s$. For any $\sigma \in \mathfrak{S}$, introduce the corresponding normalization factor $\Pi^{(\sigma)} = \Pi(\sigma\boldsymbol{a}, \boldsymbol{b})$ and the quantities

$$I^{(\sigma)} = I(\sigma\boldsymbol{a}, \boldsymbol{b}; z) = \sum_{j=1}^{s} P_j^{(\sigma)}(z^{-1}) \operatorname{Li}_j(z) - P_0^{(\sigma)}(z^{-1}),$$
$$J^{(\sigma)} = J(\sigma\boldsymbol{a}, \boldsymbol{b}; z) = \sum_{j=1}^{s} P_j^{(\sigma)}(z^{-1}) \frac{\log^{j-1} z}{(j-1)!}.$$

Using Lemma 2, the inequality $m_0 \geqslant m$, and invariance of (10) and $m_0 = b_1^* - a_0^* - 1$ under the $\mathfrak{S}$-action we arrive at the following claim.

Lemma 3. *The following inclusions*

$$D_{m_0}^s \cdot \frac{\Pi^{(\sigma)}}{\Pi} \cdot P_j(x) = D_{m_0}^s \cdot P_j^{(\sigma)}(x) \in \mathbb{Z}[x], \quad j = 0, 1, \ldots, s,$$

are valid for any $\sigma \in \mathfrak{S}$*.*

Corollary (cf. [9], Lemma 10)**.** *If*

$$\Phi = \Phi(\boldsymbol{a}, \boldsymbol{b}) = \prod_{\sqrt{m_0} < p \leqslant m} p^{e_p}, \qquad \textit{where} \quad e_p = \max_{\sigma \in \mathfrak{S}} \operatorname{ord}_p \frac{\Pi}{\Pi^{(\sigma)}},$$

then

$$D_{m_0} D_m^{s-1} \cdot \Phi^{-1} \cdot P_j(x) \in \mathbb{Z}[x], \quad j = 0, 1, \ldots, s. \tag{11}$$

Since $\operatorname{ord}_p N! = \lfloor N/p \rfloor$ for primes $p > \sqrt{N}$, we have

$$\operatorname{ord}_p \Pi(\boldsymbol{a}, \boldsymbol{b}) = \sum_{j=1}^{s} \left\lfloor \frac{b_j - a_j - 1}{p} \right\rfloor - \left\lfloor \frac{a_0 - 1}{p} \right\rfloor, \qquad p > \sqrt{m_0},$$

hence

$$e_p = \max_{\sigma \in \mathfrak{S}} \varpi_p^{(\sigma)},$$

where

$$\varpi_p^{(\sigma)} = \sum_{j=1}^{s} \left(\left\lfloor \frac{b_j - a_j - 1}{p} \right\rfloor - \left\lfloor \frac{b_j - \sigma a_j - 1}{p} \right\rfloor \right) - \left(\left\lfloor \frac{a_0 - 1}{p} \right\rfloor - \left\lfloor \frac{\sigma a_0 - 1}{p} \right\rfloor \right).$$

Now, take the parameters $\boldsymbol{a}, \boldsymbol{b}$ as follows:

$$a_j = \alpha_j n + 1, \quad j = 0, 1, \ldots, n, \qquad \text{and} \qquad b_j = \beta_j n + 2, \quad j = 1, \ldots, n, \tag{12}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are positive integers satisfying

$$\begin{gathered} 0 < \alpha_0, \alpha_1, \ldots, \alpha_s < \beta_1, \ldots, \beta_s, \qquad \sum_{j=0}^{s} \alpha_j < \sum_{j=1}^{s} \beta_j, \\ \min_{0 \leqslant j \leqslant s} \{\alpha_j\} + \max_{0 \leqslant j \leqslant s} \{\alpha_j\} \leqslant \beta_l \qquad \text{for all } l = 1, \ldots, s, \end{gathered} \tag{13}$$

to ensure conditions (1). As before, we introduce the ordered collections

$$\begin{gathered} \{\alpha_0^*, \alpha_1^*, \ldots, \alpha_s^*\} = \{\alpha_0, \alpha_1, \ldots, \alpha_s\}, \quad \{\beta_1^*, \ldots, \beta_s^*\} = \{\beta_1, \ldots, \beta_s\}, \\ \alpha_0^* \leqslant \alpha_1^* \leqslant \ldots \leqslant \alpha_s^*, \quad \beta_1^* \geqslant \ldots \geqslant \beta_s^*. \end{gathered}$$

Then $m_0 = \nu_0 n$, $m = \nu n$, where

$$\begin{gathered} \nu_0 = \beta_1^* - \alpha_0^* = \max_{1 \leqslant j \leqslant s} \{\beta_j\} - \min_{0 \leqslant j \leqslant s} \{\alpha_j\}, \\ \nu = \max \Bigl\{ \alpha_0, \max_{1 \leqslant j \leqslant s} \{\beta_j\} - \min_{1 \leqslant j \leqslant s} \{\alpha_j\}, \max_{2 \leqslant j \leqslant s} \{\beta_j\} - \min_{0 \leqslant j \leqslant s} \{\alpha_j\} \Bigr\}, \end{gathered}$$

and

$$\varpi_p^{(\sigma)} = \varphi^{(\sigma)}\left(\frac{n}{p}\right),$$

where

$$\varphi^{(\sigma)}(x) = \sum_{j=1}^{s} \bigl(\lfloor(\beta_j - \alpha_j)x\rfloor - \lfloor(\beta_j - \sigma\alpha_j)x\rfloor\bigr) - \bigl(\lfloor\alpha_0 x\rfloor - \lfloor\sigma\alpha_0 x\rfloor\bigr), \qquad \sigma \in \mathfrak{G}. \tag{14}$$

With a help of identity $\lfloor y\rfloor = y - \{y\}$ (where $\{\cdot\}$ denotes the fractional part of a number) we see that functions (14) are 1-periodic. Moreover, since $\lfloor kx\rfloor$ (for a positive integer k) changes its value only at the points $x = l/k$ for $l \in \mathbb{Z}$, we conclude that any of the functions in (14) takes constant values in demi-intervals $[u, v) \subset [0, 1)$, where u, v is any pair of neighbour fractions with denominators $\leqslant \nu_0$. These properties are inherited by the function

$$\varphi(x) = \max_{\sigma\in\mathfrak{G}} \varphi^{(\sigma)}(x),$$

whence its computation is a pure machinery.

Denoting $\Phi_n = \Phi(\boldsymbol{a}, \boldsymbol{b})$ and using the prime number theorem and the standard arithmetic argument (see, e.g., [9], Lemma 11) we obtain

Lemma 4. *The following limit relations are valid:*

$$\begin{gathered}
\lim_{n\to\infty} \frac{\log \Phi_n}{n} = \int_0^1 \varphi(x)\,\mathrm{d}\psi(x) - \int_0^{1/\nu} \varphi(x)\frac{\mathrm{d}x}{x^2}, \\
\lim_{n\to\infty} \frac{\log D_{\nu_0 n}}{n} = \nu_0, \qquad \lim_{n\to\infty} \frac{\log D_{\nu n}}{n} = \nu,
\end{gathered} \tag{15}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function.

In accordance with (11), formulae (15) completely determine the arithmetic behaviour of the constructed approximations (4) and (5).

3 Asymptotics

Consider the integral

$$J = J(\boldsymbol{a}, \boldsymbol{b}; z) = \frac{z^{a_0^*}}{2\pi i} \oint_{\mathcal{L}} R(t) z^t \,\mathrm{d}t, \tag{16}$$

where $\mathcal{L}$ is a closed clockwise contour surrounding all poles $t = -k$ for $a_1^* \leqslant k < b_1^*$ of the rational function $R(t) = R(\boldsymbol{a}, \boldsymbol{b}; t)$.

Lemma 5. *Let $0 < z < 1$ and $s \geqslant 1$. Then the following integral representation is valid:*

$$J = -\frac{z^{a_0^*}}{2\pi i}\int_{M-i\infty}^{M+i\infty} R(t)z^t\,\mathrm{d}t, \tag{17}$$

where M is an arbitrary real constant in the interval $(-\infty, -b_1^*)$.

Proof. Take the contour $\mathcal{L}$ in (16) as the rectangle with vertices $M \pm iT$, $T \pm iT$, where M is an arbitrary fixed number, $M \leqslant -b_1^*$, and T is sufficiently large. In accordance with (2), (12) we have that $R(t) = o(T^{-s-1})$ on the segments $[M - iT, T - iT]$, $[T - iT, T + iT]$ and $[T + iT, M + iT]$. Therefore, for horizontal sides of the rectangle we obtain

$$\left|\int_{M\pm iT}^{T\pm iT} R(t)z^t\,\mathrm{d}t\right| = \left|\int_{M}^{T} R(x\pm iT)z^{x\pm iT}\,\mathrm{d}x\right| \leqslant \frac{c_1}{T^2}|z|^M(T-M), \tag{18}$$

while on the vertical segment $[T - iT, T + iT]$ we have the trivial bound $|z|^t = |z^{T+iy}| = |z|^T < 1$, hence

$$\left|\int_{T-iT}^{T+iT} R(t)z^t\,\mathrm{d}t\right| \leqslant \frac{c_2}{T}. \tag{19}$$

Finally, letting T tend to infinity and taking into account estimates (18) and (19) we arrive at the required statement. □

Lemma 6. *The integral $J = J(\boldsymbol{a}, \boldsymbol{b}; z)$ admits the following representation:*

$$J = c_3\int_{\mu-i\infty}^{\mu+i\infty} g(\tau)e^{nf(\tau)}\big(1 + O(n^{-1})\big)\,\mathrm{d}t, \tag{20}$$

where the functions

$$\begin{aligned} f(\tau) = \sum_{j=0}^{s}\big((\tau-\beta_j)\log(\tau-\beta_j) - (\tau-\alpha_j)\log(\tau-\alpha_j)\big) \\ - \alpha_0\log\alpha_0 + \sum_{j=1}^{s}(\beta_j-\alpha_j)\log(\beta_j-\alpha_j) + (\alpha_0^*-\tau)\log z \end{aligned}$$

and

$$g(\tau) = \exp\left\{\frac{1}{2}\sum_{j=0}^{s}\big(\log(\tau-\alpha_j) - 3\log(\tau-\beta_j)\big)\right\}$$

are defined in the cut τ-plane $\mathbb{C}\setminus(-\infty, \beta_1^*]$, μ is an arbitrary number in the interval $(\beta_1^*, +\infty)$, and the constants $c_3 \neq 0$ and in $O(1/n)$ depend only on the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

Proof. Changing the variable t by $-t$ and applying the formula

$$\Gamma(a-t)\Gamma(t-a+1) = \frac{\pi}{\sin(\pi(a-t))},$$

we can write (17) in the form

$$J(\boldsymbol{a},\boldsymbol{b};z) = \pm\frac{z^{a_0^*}}{2\pi i}\frac{\prod_{j=1}^{s}\Gamma(b_j-a_j)}{\Gamma(a_0)}\int_{-M-i\infty}^{-M+i\infty}\prod_{j=0}^{s}\frac{\Gamma(t-b_j+1)}{\Gamma(t-a_j+1)}z^{-t}\,\mathrm{d}t$$

for some choice of the sign in '$\pm$'. Set $-M=\mu n$, $t=n\tau$, where $\tau=\mu+iy$, $\mu\in(\beta_1^*,\infty)$, $y\in(-\infty,+\infty)$, and $n\in\mathbb{N}$, use (12) and the following asymptotic formula for the Γ-function:

$$\log\Gamma(u) = \Bigl(u-\frac{1}{2}\Bigr)\log u - u + \log\sqrt{2\pi} + r(u), \qquad |r(u)|\leqslant K|\operatorname{Re}u|^{-1}, \qquad (21)$$

where K is an absolute constant. Then we obtain

$$\begin{aligned}\prod_{j=0}^{s}\frac{\Gamma(t-b_j+1)}{\Gamma(t-a_j+1)} &= \prod_{j=0}^{s}\frac{\Gamma(n(\tau-\beta_j)-1)}{\Gamma(n(\tau-\alpha_j))}\\ &= \exp\Biggl\{\sum_{j=0}^{s}\Bigl((\alpha_j-\beta_j)n\log n + n(\tau-\beta_j)\log(\tau-\beta_j) - n(\tau-\alpha_j)\log(\tau-\alpha_j)\\ &\quad -\frac{3}{2}\log(\tau-\beta_j)+\frac{1}{2}\log(\tau-\alpha_j)-\frac{1}{2}\log n + O(n^{-1})\Bigr)\Biggr\}.\end{aligned}$$

Finally, estimating the normalizing factor $\Pi(\boldsymbol{a},\boldsymbol{b})$ in (9) by (21) we obtain expression (20) as required. □

Lemma 7. *Let $0<z<1$, $s=3$, the parameters $\boldsymbol{\alpha},\boldsymbol{\beta}$ be defined in* (13)*, and*

$$h(\tau) = z\prod_{j=0}^{s}(\tau+\alpha_j) - \prod_{j=0}^{s}(\tau+\beta_j). \qquad (22)$$

Suppose furthermore that

$$h'(0)<0, \quad h''(0)<0, \quad h'(-\beta_1^*)<0, \quad \text{and} \quad h''(-\beta_1^*)>0.$$

Then the polynomial equation $h(\tau)=0$ has a unique positive root τ_1 belonging to the interval $(0, z\alpha_0^*/(1-z))$ and a unique root τ_0 in the interval $(-\infty,-\beta_1^*)$.

Proof. A straightforward verification shows that in the case $s=3$ the second derivative $h''(\tau)$ is a quadratic polynomial with negative coefficients:

$$h''(\tau) = 12(z-1)\tau^2 + 6\tau\sum_{j=0}^{3}(z\alpha_j-\beta_j) + 2\sum_{0\leqslant j<k\leqslant 3}(z\alpha_j\alpha_k-\beta_j\beta_k). \qquad (23)$$

In particular, $h''(\tau) < 0$ for all $\tau \geqslant 0$. This means that the function $h'(\tau)$ is strictly decreasing on $[0, +\infty)$ and, since $h'(0) < 0$, we have $h'(\tau) < 0$ on this interval. Whence $h(\tau)$ is strictly decreasing on $[0, +\infty)$ and, noting that $h(0) > 0$ and $h(+\infty) = -\infty$, we deduce that the equation $h(\tau) = 0$ has a unique positive root τ_1. If $\tau \geqslant z\alpha_0^*/(1-z)$, then $z(\tau + \alpha_0^*) \leqslant \tau$ implies $h(\tau) < 0$ by (13). Therefore, $\tau_1 \in (0, z\alpha_0^*/(1-z))$.

Consider now the interval $(-\infty, -\beta_1^*)$. First, note that the equation $h''(\tau) = 0$ has two negative roots $\tau_* < \tau_{**} < 0$ by (23). Since $h''(-\beta_1^*) > 0$, it follows that $-\beta_1^* \in (\tau_*, \tau_{**})$, therefore $h''(\tau)$ takes negative values on $(-\infty, \tau_*)$ and positive values on $(\tau_*, -\beta_1^*)$. This means that the function $h'(\tau)$ is strictly increasing on $(\tau_*, -\beta_1^*)$. Taking into account that $h'(-\infty) = +\infty$, $h'(-\beta_1^*) < 0$, $h(-\infty) = -\infty$, and $h(-\beta_1^*) > 0$, we conclude that the equation $h(\tau) = 0$ has a unique root τ_0 in $(-\infty, -\beta_1^*)$, and the lemma is proved. □

Lemma 8. *Assume the conditions of Lemma 7 and, moreover, that $f''(-\tau_0) \neq 0$. Then the following asymptotic formulae are valid:*

$$\lim_{n\to\infty} \frac{\log|J(\boldsymbol{a}, \boldsymbol{b}; z)|}{n} = f_0(\tau_0), \qquad \lim_{n\to\infty} \frac{\log|I(\boldsymbol{a}, \boldsymbol{b}, z)|}{n} = \operatorname{Re} f_0(\tau_1),$$

where

$$\begin{aligned} f_0(\tau) &= f(-\tau) + \tau \cdot f'(-\tau) \\ &= \sum_{j=0}^{s} \big(\alpha_j \log(\tau + \alpha_j) - \beta_j \log(\tau + \beta_j)\big) + \sum_{j=1}^{s} (\beta_j - \alpha_j) \log(\beta_j - \alpha_j) \\ &\qquad - \alpha_0 \log \alpha_0 + \alpha_0^* \log z. \end{aligned}$$

Proof. Note that $f'(-\tau) = 0$ implies $f(-\tau) = f_0(\tau)$.

The proof of the first equality is based on application of the saddle-point method to the integral representation of Lemma 6. Take $\mu = -\tau_0$ in (20), where τ_0 is defined in Lemma 7. Since $\tau_0 \in (-\infty, -\beta_1^*)$ is a zero of the polynomial (22) we see that

$$z \prod_{j=0}^{s} (-\tau_0 - \alpha_j) = \prod_{j=0}^{s} (-\tau_0 - \beta_j) > 0,$$

hence $f'(\mu) = f'(-\tau_0) = 0$. Let us show that the function $\operatorname{Re} f(\mu + iv)$ of the real variable $v \in (-\infty, +\infty)$ attains its maximum at the unique point $v = 0$. Indeed, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}v} \operatorname{Re} f(\mu + iv) &= -\operatorname{Im} \frac{\mathrm{d}}{\mathrm{d}\tau} f(\mu + iv) \\
&= \sum_{j=0}^{1} \bigl(\arg(\mu + iv - \alpha_j^*) - \arg(\mu + iv - \beta_j)\bigr) \\
&\quad + \sum_{j=2}^{s} \bigl(\arg(\mu + iv - \alpha_j^*) - \arg(\mu + iv - \beta_j)\bigr) \\
&= \Sigma_1 + \Sigma_2, \qquad (24)
\end{aligned}
$$

where $\beta_0 = 0$. Note that the function defined in (24) is odd with respect to the variable v. We assume that $v > 0$ and consider the parametrization $\tau = \mu + iv = \mu(1 + i \tan\varphi)$ for $\varphi = \arg\tau \in (0, \pi/2)$. Using the inequalities $0 < \alpha_0^* \leqslant \alpha_1^* < \beta_1 \leqslant \beta_1^* < \mu$ we see that

$$0 < \arg\tau < \arg(\tau - \alpha_0^*) \leqslant \arg(\tau - \alpha_1^*) < \arg(\tau - \beta_1) < \frac{\pi}{2},$$

hence

$$0 < \arg(\tau - \alpha_0^*) + \arg(\tau - \alpha_1^*) < \pi, \qquad 0 < \arg\tau + \arg(\tau - \beta_1) < \pi.$$

Furthermore, we have

$$\cot\arg(\tau - \alpha_0^*) = \frac{\mu - \alpha_0^*}{\mu \cdot \tan\varphi}, \qquad \cot\arg(\tau - \alpha_1^*) = \frac{\mu - \alpha_1^*}{\mu \cdot \tan\varphi};$$

denote

$$
\begin{aligned}
C_1 &= \cot\bigl(\arg(\tau - \alpha_0^*) + \arg(\tau - \alpha_1^*)\bigr) = \frac{\mu^2(1 - \tan^2\varphi) - \mu(\alpha_0^* + \alpha_1^*) + \alpha_0^*\alpha_1^*}{\mu \cdot \tan\varphi \cdot (2\mu - \alpha_0^* - \alpha_1^*)}, \\
C_2 &= \cot\bigl(\arg\tau + \arg(\tau - \beta_1)\bigr) = \frac{\mu(1 - \tan^2\varphi) - \beta_1}{(2\mu - \beta_1)\tan\varphi}.
\end{aligned}
$$

Then

$$C_1 - C_2 = \frac{\mu^2(\beta_1 - \alpha_0^* - \alpha_1^*)(1 + \tan^2\varphi) + \alpha_0^*\alpha_1^*(2\mu - \beta_1)}{\mu(2\mu - \alpha_0^* - \alpha_1^*)(2\mu - \beta_1) \cdot \tan\varphi} > 0$$

by (13). Thus, $C_1 > C_2$ or, equivalently,

$$\arg(\tau - \alpha_0^*) + \arg(\tau - \alpha_1^*) < \arg\tau + \arg(\tau - \beta_1),$$

i.e. $\Sigma_1 < 0$. If $v > 0$, then in accordance with our choice of the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ given in (13),

$$0 < \arg(\mu - \alpha_j^* + iv) < \arg(\mu - \beta_j + iv) < \frac{\pi}{2}, \qquad j = 2, \ldots, s,$$

yielding $\Sigma_2 < 0$. Finally, $\frac{\mathrm{d}}{\mathrm{d}v}\operatorname{Re} f(\mu+iv) = \Sigma_1 + \Sigma_2 < 0$ for $v > 0$ and $\frac{\mathrm{d}}{\mathrm{d}v}\operatorname{Re} f(\mu+iv) > 0$ for $v < 0$ (by oddness of the derivative). Therefore, μ is the unique maximum of the function $\operatorname{Re} f(\tau)$ on the vertical line $(\mu - i\infty, \mu + i\infty)$. This implies that the integral J is determined by the contribution of the saddle point μ, namely,

$$J(\boldsymbol{a},\boldsymbol{b};z) = c_3 g(\mu)|f''(\mu)|^{-1/2} e^{nf(\mu)}\bigl(1 + O(n^{-1})\bigr),$$

whence the first equality of Lemma 8 follows.

In order to prove the second asymptotic relation, we write the sum I in the form

$$\begin{aligned} I(\boldsymbol{a},\boldsymbol{b};z) &= z^{a_0^*}\sum_{t=0}^{\infty} R(t) z^t \\ &= \frac{\prod_{j=1}^{s}\Gamma((\beta_j-\alpha_j)n+1)}{\Gamma(\alpha_0 n+1)} z^{\alpha_0^* n+1}\sum_{t=0}^{\infty}\prod_{j=0}^{s}\frac{\Gamma(t+\alpha_j n+1)}{\Gamma(t+\beta_j n+2)} z^t \end{aligned}$$

and use the techniques as in the second proof of Lemma 3 in [1]. We deduce that

$$\begin{aligned} &\lim_{n\to\infty}\frac{\log|I(\boldsymbol{a},\boldsymbol{b},z)|}{n} \\ &\quad = \alpha_0^*\log z + \sum_{j=1}^{s}(\beta_j-\alpha_j)\log(\beta_j-\alpha_j) - \alpha_0\log\alpha_0 \\ &\qquad + \max_{\tau>0}\biggl\{\tau\log z + \sum_{j=0}^{s}\bigl((\tau+\alpha_j)\log(\tau+\alpha_j) - (\tau+\beta_j)\log(\tau+\beta_j)\bigr)\biggr\} \\ &\quad = \operatorname{Re} f_0(\tau_1) \end{aligned}$$

by Lemma 7, thus completing the proof. □

4 Proof of the main result

We shall apply the arithmetic and asymptotic results to the linear forms

$$L(\boldsymbol{a},\boldsymbol{b};z) = I(\boldsymbol{a},\boldsymbol{b};z) - \operatorname{Li}_1(z)J(\boldsymbol{a},\boldsymbol{b};z) = \sum_{j=2}^{s} P_j(z^{-1})f_j(z) - P_0(z^{-1})$$

in the case $s = 3$. It happens that an optimal choice for the integer parameters $\boldsymbol{\alpha},\boldsymbol{\beta}$ depends on $z \in \{1/2, 2/3, 3/4, 4/5\}$, but in all examples below the technical conditions of Lemmas 7 and 8 are fulfilled, and we also have $f_0(\tau_0) \ne \operatorname{Re} f_0(\tau_1)$ ensuring

$$\lim_{n\to\infty}\frac{\log|L(\boldsymbol{a},\boldsymbol{b};z)|}{n} = \max\{f_0(\tau_0), \operatorname{Re} f_0(\tau_1)\},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are taken in accordance with (12). By Corollary of Lemma 3 the non-zero linear forms

$$d^{\nu_0 n} D_{\nu_0 n} D_{\nu n}^2 \cdot \Phi(\boldsymbol{a},\boldsymbol{b})^{-1}\cdot L(\boldsymbol{a},\boldsymbol{b};z), \qquad d \text{ is the denominator of } z^{-1},$$

have integer coefficients; therefore, if

$$\nu_0(1+\log d)+2\nu-\left(\int_0^1 \varphi(x)\,\mathrm{d}\psi(x)-\int_0^{1/\nu}\varphi(x)\frac{\mathrm{d}x}{x^2}\right)+\max\{f_0(\tau_0),\operatorname{Re} f_0(\tau_1)\}<0,$$

then these forms tend to 0 as $n\to\infty$, which implies that at least one of the numbers $f_2(z)$ and $f_3(z)$ must be irrational.

If $z=1/2$ we take

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2,\beta_3)=(3,4,5,6;17,16,15).$$

Then

$$\begin{aligned} f_0(\tau_0)&=f_0(-54.774962\ldots)=-30.978821\ldots,\\ \operatorname{Re} f_0(\tau_1)&=\operatorname{Re} f_0(0.045669\ldots)=-31.342224\ldots \end{aligned}$$

and

$$\nu_0+2\nu-\left(\int_0^1 \varphi(x)\,\mathrm{d}\psi(x)-\int_0^{1/\nu}\varphi(x)\frac{\mathrm{d}x}{x^2}\right)=14+2\cdot 13-9.416718\ldots=30.583281\ldots,$$

since

$$\varphi(x)=\begin{cases} 2 & \text{if } x\in\left[\frac{3}{13},\frac{1}{4}\right)\cup\left[\frac{3}{11},\frac{2}{7}\right)\cup\left[\frac{5}{9},\frac{7}{12}\right)\cup\left[\frac{7}{11},\frac{9}{14}\right)\cup\left[\frac{7}{9},\frac{11}{14}\right)\cup\left[\frac{10}{11},\frac{11}{12}\right),\\ 1 & \text{if } x\in\left[\frac{1}{13},\frac{1}{10}\right)\cup\left[\frac{2}{13},\frac{3}{13}\right)\cup\left[\frac{1}{4},\frac{3}{11}\right)\cup\left[\frac{2}{7},\frac{5}{12}\right)\cup\left[\frac{4}{9},\frac{1}{2}\right)\cup\left[\frac{7}{13},\frac{5}{9}\right)\\ & \qquad\cup\left[\frac{7}{12},\frac{7}{11}\right)\cup\left[\frac{9}{14},\frac{3}{4}\right)\cup\left[\frac{10}{13},\frac{7}{9}\right)\cup\left[\frac{11}{14},\frac{6}{7}\right)\cup\left[\frac{8}{9},\frac{10}{11}\right)\cup\left[\frac{11}{12},\frac{13}{14}\right),\\ 0 & \text{otherwise.} \end{cases}$$

If $z=2/3$ we take

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2,\beta_3)=(5,6,7,8;17,16,15).$$

Then

$$\begin{aligned} f_0(\tau_0)&=f_0(-67.615443\ldots)=-32.714219\ldots,\\ \operatorname{Re} f_0(\tau_1)&=\operatorname{Re} f_0(0.314519\ldots)=-34.100076\ldots \end{aligned}$$

and

$$\begin{aligned} \nu_0(1+\log 2)+2\nu&-\left(\int_0^1 \varphi(x)\,\mathrm{d}\psi(x)-\int_0^{1/\nu}\varphi(x)\frac{\mathrm{d}x}{x^2}\right)\\ &=12\cdot(1+\log 2)+2\cdot 11-9.957743\ldots=32.360022\ldots, \end{aligned}$$

since

$$\varphi(x)=\begin{cases} 2 & \text{if } x\in\left[\frac{2}{11},\frac{1}{5}\right)\cup\left[\frac{2}{7},\frac{3}{10}\right)\cup\left[\frac{4}{11},\frac{2}{5}\right)\cup\left[\frac{4}{7},\frac{3}{5}\right)\cup\left[\frac{8}{11},\frac{3}{4}\right)\cup\left[\frac{7}{9},\frac{4}{5}\right)\cup\left[\frac{8}{9},\frac{9}{10}\right),\\ 1 & \text{if } x\in\left[\frac{1}{11},\frac{1}{8}\right)\cup\left[\frac{1}{7},\frac{2}{11}\right)\cup\left[\frac{1}{5},\frac{1}{4}\right)\cup\left[\frac{3}{11},\frac{2}{7}\right)\cup\left[\frac{3}{10},\frac{4}{11}\right)\cup\left[\frac{3}{7},\frac{1}{2}\right)\cup\left[\frac{6}{11},\frac{4}{7}\right)\\ & \qquad\cup\left[\frac{3}{5},\frac{5}{8}\right)\cup\left[\frac{7}{11},\frac{7}{10}\right)\cup\left[\frac{5}{7},\frac{8}{11}\right)\cup\left[\frac{3}{4},\frac{7}{9}\right)\cup\left[\frac{6}{7},\frac{8}{9}\right)\cup\left[\frac{9}{10},\frac{11}{12}\right),\\ 0 & \text{otherwise.} \end{cases}$$

If $z = 3/4$ we take

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3) = (7, 8, 9, 10; 20, 19, 18).$$

Then

$$\begin{aligned} f_0(\tau_0) &= f_0(-96.814387\ldots) = -42.031285\ldots, \\ \operatorname{Re} f_0(\tau_1) &= \operatorname{Re} f_0(0.679858\ldots) = -40.372159\ldots \end{aligned}$$

and

$$\begin{aligned} &\nu_0(1+\log 3) + 2\nu - \left(\int_0^1 \varphi(x)\,\mathrm{d}\psi(x) - \int_0^{1/\nu} \varphi(x)\frac{\mathrm{d}x}{x^2}\right) \\ &\quad = 13\cdot(1+\log 3) + 2\cdot 12 - 11.149689\ldots = 40.132270\ldots, \end{aligned}$$

since

$$\varphi(x) = \begin{cases} 2 & \text{if } x \in \left[\frac{1}{4}, \frac{3}{11}\right) \cup \left[\frac{2}{5}, \frac{3}{7}\right) \cup \left[\frac{1}{2}, \frac{6}{11}\right) \cup \left[\frac{5}{8}, \frac{7}{11}\right) \cup \left[\frac{7}{10}, \frac{5}{7}\right) \cup \left[\frac{3}{4}, \frac{10}{13}\right) \\ & \qquad \cup \left[\frac{4}{5}, \frac{9}{11}\right) \cup \left[\frac{9}{10}, \frac{10}{11}\right), \\ 1 & \text{if } x \in \left[\frac{1}{12}, \frac{1}{7}\right) \cup \left[\frac{1}{6}, \frac{1}{4}\right) \cup \left[\frac{3}{11}, \frac{4}{11}\right) \cup \left[\frac{3}{8}, \frac{2}{5}\right) \cup \left[\frac{3}{7}, \frac{5}{11}\right) \cup \left[\frac{6}{11}, \frac{4}{7}\right) \\ & \qquad \cup \left[\frac{7}{12}, \frac{5}{8}\right) \cup \left[\frac{7}{11}, \frac{7}{10}\right) \cup \left[\frac{5}{7}, \frac{8}{11}\right) \cup \left[\frac{10}{13}, \frac{4}{5}\right) \cup \left[\frac{9}{11}, \frac{11}{13}\right) \\ & \qquad \cup \left[\frac{7}{8}, \frac{9}{10}\right) \cup \left[\frac{10}{11}, \frac{12}{13}\right), \\ 0 & \text{otherwise.} \end{cases}$$

If $z = 4/5$ we take

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3) = (11, 12, 13, 14; 26, 25, 24).$$

Then

$$\begin{aligned} f_0(\tau_0) &= f_0(-136.248182\ldots) = -51.532175\ldots, \\ \operatorname{Re} f_0(\tau_1) &= \operatorname{Re} f_0(1.686050\ldots) = -52.579778\ldots \end{aligned}$$

and

$$\begin{aligned} &\nu_0(1+\log 4) + 2\nu - \left(\int_0^1 \varphi(x)\,\mathrm{d}\psi(x) - \int_0^{1/\nu} \varphi(x)\frac{\mathrm{d}x}{x^2}\right) \\ &\quad = 15\cdot(1+\log 4) + 2\cdot 14 - 12.384270\ldots = 51.410144\ldots, \end{aligned}$$

since

$$\varphi(x) = \begin{cases} 2 & \text{if } x \in \left[\frac{1}{14}, \frac{1}{13}\right) \cup \left[\frac{1}{12}, \frac{1}{11}\right) \cup \left[\frac{1}{7}, \frac{2}{13}\right) \cup \left[\frac{1}{6}, \frac{2}{11}\right) \cup \left[\frac{3}{14}, \frac{3}{13}\right) \cup \left[\frac{1}{4}, \frac{4}{15}\right) \\ & \qquad \cup \left[\frac{3}{10}, \frac{4}{13}\right) \cup \left[\frac{5}{14}, \frac{4}{11}\right) \cup \left[\frac{3}{7}, \frac{5}{11}\right) \cup \left[\frac{1}{2}, \frac{7}{13}\right) \cup \left[\frac{3}{5}, \frac{8}{13}\right) \cup \left[\frac{5}{7}, \frac{8}{11}\right), \\ 1 & \text{if } x \in \left[\frac{1}{13}, \frac{1}{12}\right) \cup \left[\frac{2}{13}, \frac{1}{6}\right) \cup \left[\frac{3}{13}, \frac{1}{4}\right) \cup \left[\frac{4}{15}, \frac{3}{11}\right) \cup \left[\frac{2}{7}, \frac{3}{10}\right) \cup \left[\frac{4}{13}, \frac{5}{14}\right) \\ & \qquad \cup \left[\frac{4}{11}, \frac{5}{13}\right) \cup \left[\frac{2}{5}, \frac{3}{7}\right) \cup \left[\frac{5}{11}, \frac{6}{13}\right) \cup \left[\frac{7}{13}, \frac{6}{11}\right) \cup \left[\frac{4}{7}, \frac{3}{5}\right) \cup \left[\frac{8}{13}, \frac{7}{11}\right) \\ & \qquad \cup \left[\frac{9}{14}, \frac{5}{7}\right) \cup \left[\frac{8}{11}, \frac{13}{15}\right) \cup \left[\frac{9}{10}, \frac{14}{15}\right), \\ 0 & \text{otherwise.} \end{cases}$$

The presented computations show the truth of Theorem 1.

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Kh. Hessami Pilehrood and T. Hessami Pilehrood:
Mathematics Department,
Shahrekord University,
P.O. Box 115, Shahrekord, IRAN

School of Mathematics,
Institute for Studies in Theoretical Physics and Mathematics,
P.O. Box 19395-5746, Tehran, IRAN
E-mail: hessamik@ipm.ir, hessamit@ipm.ir

W. Zudilin:
Department of Mechanics and Mathematics,
Moscow Lomonosov State University,
Vorobiovy Gory, GSP-2,
119992 Moscow, RUSSIA
E-mail: wadim@ips.ras.ru