Calculating the extremal number ex($v; \{C_3, C_4, \ldots, C_n\}$)

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Abstract

By the *extremal number* $ex(v; \{C_3, C_4, \ldots, C_n\})$ we denote the maximum number of edges in a graph of order v and girth at least $g \geq n + 1$. The set of such graphs is denoted by $EX(v; \{C_3, C_4, \ldots, C_n\})$. In 1975, Erdős mentioned the problem of determining extremal numbers $ex(v; \{C_3, C_4\})$ in a graph of order v and girth at least 5. In this paper, we consider a generalized version of the problem for any value of girth by using the hybrid simulated annealing and genetic algorithm (HSAGA). Using this algorithm, some new results for $n \geq 5$ have been obtained. In particular, we generate some graphs of girth $6, 7$ and 8 which in some cases have more edges than corresponding cages. Furthermore, future work will be described regarding the investigation of structural properties of such extremal graphs and the implementation of HSAGA using parallel computing.

Key Words: extremal graph, cages, extremal number.

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1 Definitions

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [12] for terminology and definitions.

The vertex set (respectively, edge set) of a graph G is denoted by $V(G)$ (respectively, $E(G)$). The set of vertices adjacent to a vertex v is denoted by $N(v)$. The *degree* of a vertex v is $deg(v) = |N(v)|$, and a graph is called *k-regular* when all the vertices have the same degree k. We denote by $\delta(G)$ the minimum degree in G and by $\Delta(G)$ the maximum degree of G. The *distance* $d(u, v)$ of two vertices u and v in $V(G)$ is the length of a shortest path between u and v. We also use the notion of a *distance between a vertex* v *and a set of vertices* X, written $d(v, X)$, which is the distance from v to a closest vertex in X.

The length of a shortest cycle in a graph G is called the *girth* and denoted by $g = g(G)$. A k-regular graph with girth g is called a (k, g)*-graph.* A (k, g)-graph is called a (k, g)*-cage* if it has the least possible number of vertices.

By $ex(v; \{C_3, C_4, \ldots, C_n\})$ we denote the maximum number of edges in a graph of order v and girth at least $g \geq n+1$, and by $EX(v; \{C_3, C_4, C_5, \ldots, C_n\})$ we denote the set of all graphs of order v, girth at least $n + 1$, having number of edges equal to $ex(v; \{C_3, C_4, C_5, \ldots, C_n\})$.

2 Introduction

For a graph, let $C(G)$ denote the set of integers whose elements are lengths of cycles in G. Investigating the properties of graphs that guarantee the existence of given cycle length has received much attention by many authors. For example, the results obtained by Bondy [10] show that, for a graph G with v vertices, if the minimum degree of G is larger than $v/2$, then the graph is pancyclic which means $C(G) = \{3, 4, \ldots, v\}$. However, if the minimum degree of a graph is not more than $v/2$, then we cannot guarantee any odd cycles, as the graph may be bipartite. Subsequent research went on to investigate the problem for particular range of the cycle length. For example, when does $C(G)$ contain all even integers up to the longest even cycle of G? Bollobás and Thomason [9] showed that when G has order v and size at least $\lfloor v^2/4 \rfloor - v + 59$, then $C(G)$ contains all even integers up to $2l$, where $2l$ is the length of a longest even cycle of G. Other interesting results are due to Gould, Haxell and Scott [19], who proved that if a graph G with v vertices has minimum degree at least cv, where $c > 0$ is a constant, then G contains all even integers up to $2l - k$ for some constant k, depending only on c. Many other ranges of cycle lengths have been considered and concepts such as weakly pancyclic graph, acyclic graph etc. have been introduced and extensively studied.

In 1975, Erdős [14] mentioned the problem of determining the values of $ex(v;\ \{C_3, C_4\})$, the maximum number of edges in a graph of order v with girth at least 5. He also conjectured that $ex(v; \{C_3, C_4\}) = (1/2 + o(1))^{3/2} v^{3/2}$. Until now, the current best known result [18] regarding this problem is

$$\frac{1}{2\sqrt{2}} \leq \limsup_{v\to\infty} \frac{ex(v; \{C_3, C_4\})}{v^{3/2}} \leq \frac{1}{2}.$$

It is known that $ex(v; C_3) = \lfloor v^2/4 \rfloor$, and the extremal graph is $K_{\lfloor v/2\rfloor, \lceil v/2\rceil}$, and the exact value of $ex(v; C_4) = (1/2 + o(1))v^{3/2}$ [11, 13] for some specific v. Füredi *et al.* [17] determined the current best known bounds for $ex(v; C_6)$.

$$0.5338 v^{4/3} \leq ex(v; C_6) \leq 0.627 v^{4/3} + O(v^{7/6}).$$

It is known (see page 158 of the book by Bollobas [8]) that if $e > 90kv^{1+1/k}$ then the graph contains a cycle of length $2k$. Therefore, $ex(v; \{C_3, C_4, \ldots, C_{2k}\}) \leq 90kv^{1+1/k}$. A result proved implicity by Erdős [24] gives the lower bound $ex(v; \{C_3, C_4, \ldots, C_n\}) \geq c_n v^{1+1/(n-1)}$, for some positive constant c_n. Lazebnik *et al.* [23] improved this lower bound constructing a family of graphs which shows that for an infinite sequence of values of v the extremal number is lower bounded, $ex(v; \{C_3, C_4, \ldots, C_{2s+1}\}) \geq d_s v^{1+2/(3s-3+\epsilon)}$, where $\epsilon = 0$ if $s \geq 3$ is odd and $\epsilon = 1$ if $s \geq 2$ is even. To our knowledge, this is the best asymptotic lower bound, for the greatest number of edges in graphs of order v and girth g at least $g \geq 5$, $g \neq 11, 12$. For $g = 11, 12$, a better bound is given by the regular generalized hexagon.

Regarding structural properties of extremal graphs, some theorems, which are summarised below, have been obtained by several authors.

Theorem 1 *Let $G \in EX(v; \{C_3, C_4, \ldots, C_n\})$, $n \geq 3$ and $v \geq n + 1$. Then*

(i) [22] There exists an extremal graph G of girth $n+1$; and if $v \neq n+2$, there exists an extremal graph G with minimum degree $\delta \geq 2$ and girth $n + 1$.

(ii) [2, 18, 21] For $v \geq 5$, the girth of $G \in EX(v;\{C_3\})$ is 4 and, for $v \geq 9$, the girth of $G \in EX(v;\{C_3, C_4\})$ is 5.

(iii) [2, 22] For $v \geq 8$, the girth of $G \in EX(v;\{C_3, C_4, C_5\})$ is 6.

(iv) [1] For $v \geq 12$, $v \notin \{15, 30, 80, 170\}$, the girth of $G \in EX(v;\{C_3, C_4, C_5, C_6\})$ is 7, and there exists an extremal graph G of 15 vertices having girth 8.

(v) [22] For $n \geq 12$, $ex(2n+2;\{C_3, C_4, \ldots, C_n\}) = 2n+4$, and there exists an extremal graph G with $g(G) = n+2$.

(vi) [1] The diameter of an extremal graph G is $D(G) \leq n-1$.

(vii) [22] If $\Delta(G) \geq n$ then the girth of G is $g(G) = n+1$.

(viii) [2] If $\Delta(G) \geq \lceil (n+1)/2 \rceil$ and $\delta(G) \geq 2$ then the girth of G is $g(G) \leq n+2$.

(ix) [2] For $n \geq 7$ and $v \geq \Big(2(n-2)^{n-2} + n - 5\Big)/(n-3) + 1$, the girth of G is $g(G) = n+1$.

(x) [2] Let $t = \lceil (n+1)/2 \rceil$. For $n \geq 7$ and $v \geq \Big(2(t-2)^{n-2} + t - 5\Big)/(t-3) + 1$, the girth of G is $g(G) \leq n+2$.

The same kind of structural properties as contained in points $(vi) - (x)$ of the above theorem for bipartite graphs are stated in [4].

By applying Erdős' deletion method on a random graph, it is easy to see that a graph with v vertices and average degree q must have girth at least $log_{q-1}v$. This result implies that the maximum degree is larger than $v^{1/q}+1$. On the other hand, not much is known about the minimum degree of a random graph. We know that, for particular values of girth and order, there do exist some graphs with largest number of edges and minimum degree 1, for example, graphs on 11 vertices have degree sequences $\{1_1, 3_9, 4_1\}$, $\{2_1, 3_{10}\}$ or $\{2_2, 3_8, 4_1\}$. However, in general, it is believed that the degrees are distributed as evenly as possible [26]. This observation relates the problem of constructing extremal graphs of the family $EX(v;\{C_3, C_4, C_5, \ldots, C_n\})$ to the problem of constructing cages. However, as pointed out in many papers, these two classes of graphs are not the same. For example, for the graph with 30 vertices, the Wegner graph is the (5,5)-cage and the known extremal graph in $EX(v;\{C_3, C_4\})$ has 76 edges instead of 75 as in Wegner graph. The degree sequence of this extremal graph is $\{4_4, 5_{20}, 6_6\}$ [26].

3 Experimental Data

In [21] Garnic *et al.* developed algorithms which combined hill-climbing and backtracking techniques to generate graphs with order up to 201 for $ex(v; \{C_3, C_4\})$, and Wang *et al.* [26] used simulated annealing to generate graphs for several values of v and n; these results resulted in improvements to lower bounds for $ex(v; \{C_3, C_4\})$.

3.1 Hybrid Simulated Annealing and Genetic Algorithm

We have developed an optimization algorithm method [25]: the *Hybrid Simulated Annealing and Genetic Algorithm* (HSAGA). The general idea of HSAGA is that an initial graph is created at the beginning, and used as the initial graph input into *Simulated Annealing*, denoted by SA. If the iteration reaches the maximum generation of attempted moves at the last step of reaching the Maximum frozen, then SA will terminate and the current population will be transferred to the *Genetic Algorithms*, called GA. Otherwise, a candidate solution will be obtained and be saved into the population. Furthermore, the set of elite individuals of the population is chosen by a selection procedure of GA according to their evaluation fitness values, following genetic operations consisting of crossover and mutation. The basic processes of HSAGA are shown in Figure 1, and the details of each process are described below.

(a) Input parameters into our program, such as the number of vertices, required free cycles, as well as *cooling rate*, which controls the decreasing of temperature, and *population size*, that is, the number of chromosomes, and so on.

(b) Create an initial base graph in terms of given number of vertices without any edges. Every graph is represented by an adjacency matrix.

(c) Put the current graph into the method called SA. During its processing, SA will execute moves to improve the current graph. We have a stopping condition whether or not the iteration reaches the maximum generation of attempted moves at the last step of reaching the *maximum frozen*, which is the maximum number of consecutive iterations allowed for *frozen*, normally, the initial value of the frozen starts from 0. If yes, then the process will stop and go to Step d. Otherwise, it will create a chromosome, based on its fitness value, which is represented by the number of edges of the current generated graph, then store each

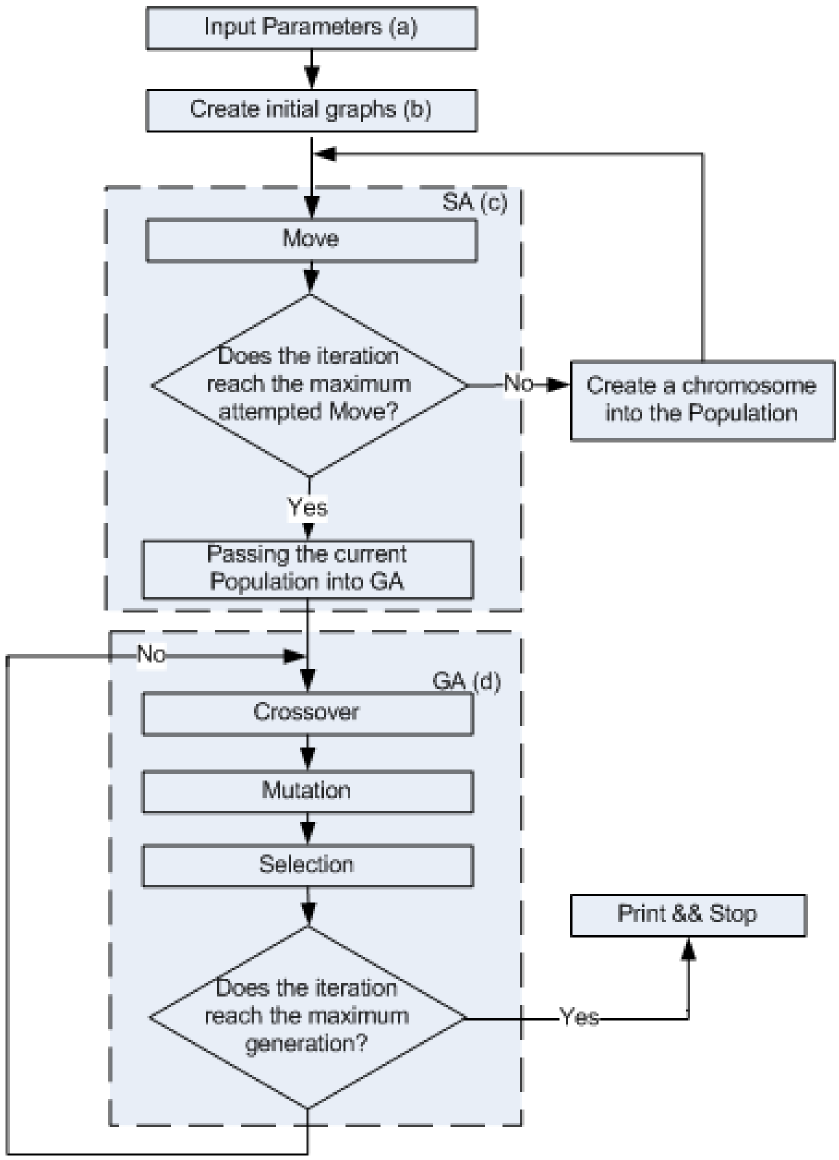

Figure 1: A basic structure of HSAGA.

chromosome into the population. If we fix the population size as 200, HSAGA will obtain a population of the first 200 best chromosomes, based on their fitness values.

(d) Input the current population into GA functions consisting of crossover, mutation and selection, in order to obtain an improved solution. After a new population is created, we determine whether the iteration reaches the maximum generation or not. The maximum generation to run the GA optimizer is set as the number of vertices times the size of the current population. If yes, our whole algorithm terminates. Otherwise, GA needs to check the following operations: If the number of *counter*, which is a parameter for calculating how many times when the best individual in the currently evolved population is not updated consecutively, reaches 50, then 50% of the individuals located at the bottom of the current population will be replaced with individuals newly generated at random. Additionally, the two individuals with the highest fitness value in the new population are passed on to the next generation without being altered by genetic operations.

3.2 Output Results

Each result provided by HSAGA consists of three parts, namely, the maximum number of edges, the adjacency list and the degree sequence; see an example of output in Figure 2. Tables 1, 2 and 3 give the newly found lower bounds for $\mathrm{ex}(v;\{C_3,C_4,C_5\})$, $\mathrm{ex}(v;\{C_3,C_4,C_5,C_6\})$ and $\mathrm{ex}(v;\{C_3,C_4,C_5,C_6,C_7\})$ for $v \leq 39$, denoted by e. They also give the degree sequence $\mathcal{D}$ of the corresponding generated extremal graphs. Thus, in Table 1 the girth of a graph $G \in EX(v;\{C_3,C_4,C_5\})$ obtained by our program is 6, for $v \geq 8$, as Theorem 1 (iii) claims. In Table 2, the girth of the corresponding graphs is 7. Moreover, HSAGA gives a graph of $v = 30$ vertices, degree sequence $\mathcal{D} = \{3_{26}, 4_4\}$, size $e = 47$ having girth 7 (see Figure 3). Then a question asked in [2] as to whether the $(3,8)$-cage does or does not belong to $\mathrm{EX}(30;\{C_3,C_4,C_5,C_6\})$ is solved (negatively) in the following theorem.

The final graph of the 30 vertices with free girth 6 has these matching edges :
The Maximum Edges is : 47

vertex : 0 --- vertex : 1
vertex : 0 --- vertex : 10
vertex : 0 --- vertex : 23
vertex : 1 --- vertex : 2
vertex : 1 --- vertex : 29
vertex : 2 --- vertex : 3
vertex : 2 --- vertex : 8
vertex : 2 --- vertex : 16
vertex : 3 --- vertex : 11
vertex : 3 --- vertex : 13
vertex : 4 --- vertex : 12
vertex : 4 --- vertex : 20
vertex : 4 --- vertex : 21
vertex : 5 --- vertex : 9
vertex : 5 --- vertex : 24
vertex : 5 --- vertex : 28
vertex : 6 --- vertex : 9
vertex : 6 --- vertex : 10
vertex : 6 --- vertex : 18
vertex : 6 --- vertex : 27
vertex : 7 --- vertex : 15
vertex : 7 --- vertex : 22
vertex : 7 --- vertex : 23
vertex : 8 --- vertex : 9
vertex : 8 --- vertex : 14
vertex : 10 --- vertex : 19
vertex : 11 --- vertex : 24
vertex : 11 --- vertex : 26
vertex : 12 --- vertex : 14
vertex : 12 --- vertex : 23
vertex : 13 --- vertex : 15
vertex : 13 --- vertex : 18
vertex : 14 --- vertex : 17
vertex : 15 --- vertex : 25
vertex : 16 --- vertex : 21
vertex : 16 --- vertex : 22
vertex : 17 --- vertex : 25
vertex : 17 --- vertex : 26
vertex : 18 --- vertex : 20
vertex : 19 --- vertex : 21
vertex : 19 --- vertex : 25
vertex : 20 --- vertex : 29
vertex : 22 --- vertex : 27
vertex : 23 --- vertex : 24
vertex : 25 --- vertex : 28
vertex : 26 --- vertex : 27
vertex : 28 --- vertex : 29

***** The graph also has the degree sequence *****
$D = \{ 3_{26}, 4_4 \}$

Figure 2: An adjacency list of a graph of $v = 30$, $g = 7$ and $e = 47$.

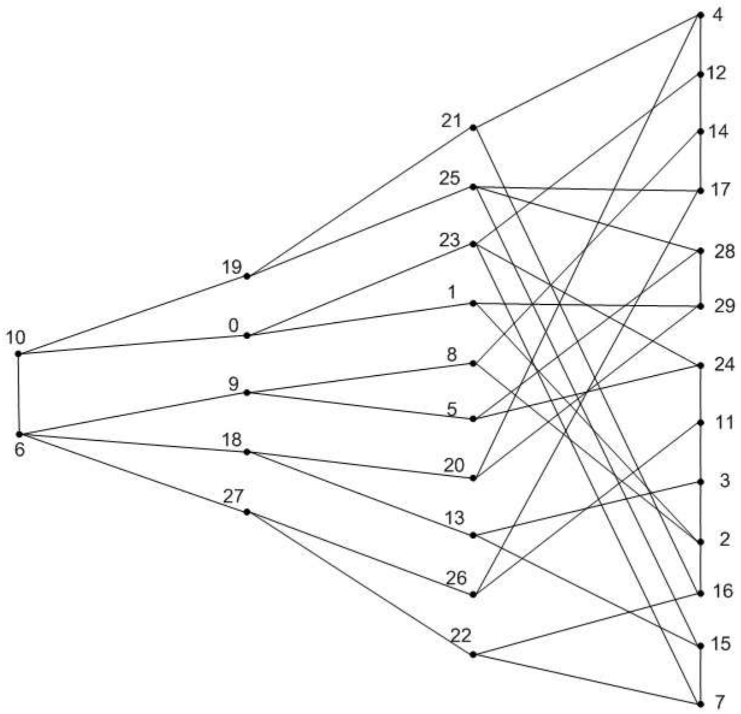

Figure 3: A graph of $v = 30$, $g = 7$ and $e = 47$.

N. of vertices	0	10	20	30
0	\	$e = 12$ $\mathcal{D} = \{2_6, 3_4\}$	$e = 34$ $\mathcal{D} = \{3_{12}, 4_8\}$	$e = 61$ $\mathcal{D} = \{3_2, 4_{24}, 5_4\}$
1	\	$e = 14$ $\mathcal{D} = \{2_5, 3_6\}$	$e = 36$ $\mathcal{D} = \{2_1, 3_{10}, 4_{10}\}$	$e = 63$ $\mathcal{D} = \{3_3, 4_{23}, 5_5\}$
2	\	$e = 16$ $\mathcal{D} = \{2_4, 3_8\}$	$e = 39$ $\mathcal{D} = \{3_{10}, 4_{12}\}$	$e = 67$ $\mathcal{D} = \{4_{26}, 5_6\}$
3	\	$e = 18$ $\mathcal{D} = \{2_3, 3_{10}\}$	$e = 42$ $\mathcal{D} = \{3_8, 4_{15}\}$	$e = 69$ $\mathcal{D} = \{3_3, 4_{21}, 5_9\}$
4	\	$e = 21$ $\mathcal{D} = \{3_{14}\}$	$e = 45$ $\mathcal{D} = \{3_6, 4_{18}\}$	$e = 73$ $\mathcal{D} = \{3_3, 4_{18}, 5_{13}\}$
5	\	$e = 22$ $\mathcal{D} = \{2_1, 3_{14}\}$	$e = 48$ $\mathcal{D} = \{3_4, 4_{21}\}$	$e = 74$ $\mathcal{D} = \{3_3, 4_{21}, 5_{11}\}$
6	$e = 6$ $\mathcal{D} = \{2_6\}$	$e = 24$ $\mathcal{D} = \{3_{16}\}$	$e = 52$ $\mathcal{D} = \{4_{26}\}$	$e = 76$ $\mathcal{D} = \{2_1, 3_2, 4_{23}, 5_8, 6_2\}$
7	$e = 7$ $\mathcal{D} = \{1_1, 2_5, 3_1\}$	$e = 26$ $\mathcal{D} = \{2_2, 3_{12}, 4_3\}$	$e = 53$ $\mathcal{D} = \{2_1, 3_1, 4_{24}, 5_1\}$	$e = 82$ $\mathcal{D} = \{3_1, 4_{19}, 5_{17}\}$
8	$e = 9$ $\mathcal{D} = \{2_6, 3_2\}$	$e = 29$ $\mathcal{D} = \{3_{14}, 4_4\}$	$e = 55$ $\mathcal{D} = \{2_1, 3_2, 4_{23}, 5_2\}$	$e = 85$ $\mathcal{D} = \{2_1, 3_1, 4_{16}, 5_{19}, 6_1\}$
9	$e = 10$ $\mathcal{D} = \{2_7, 3_2\}$	$e = 31$ $\mathcal{D} = \{2_1, 3_{12}, 4_6\}$	$e = 57$ $\mathcal{D} = \{3_5, 4_{21}, 5_3\}$	$e = 89$ $\mathcal{D} = \{3_1, 4_{15}, 5_{23}\}$

Table 1: The current lower bounds for $\mathrm{ex}(v; \{C_3, C_4, C_5\})$ for $v \leq 39$.

N. of vertices	0	10	20	30
0	\	$e = 11$ $\mathcal{D} = \{2_8, 3_2\}$	$e = 27$ $\mathcal{D} = \{2_6, 3_{14}\}$	$e = 47$ $\mathcal{D} = \{3_{26}, 4_4\}$
1	\	$e = 12$ $\mathcal{D} = \{1_1, 2_7, 3_3\}$	$e = 29$ $\mathcal{D} = \{2_5, 3_{16}\}$	$e = 48$ $\mathcal{D} = \{2_2, 3_{24}, 4_5\}$
2	\	$e = 14$ $\mathcal{D} = \{2_8, 3_4\}$	$e = 31$ $\mathcal{D} = \{2_4, 3_{18}\}$	$e = 50$ $\mathcal{D} = \{2_2, 3_{24}, 4_6\}$
3	\	$e = 15$ $\mathcal{D} = \{1_1, 2_7, 3_5\}$	$e = 33$ $\mathcal{D} = \{2_3, 3_{20}\}$	$e = 52$ $\mathcal{D} = \{2_2, 3_{24}, 4_7\}$
4	\	$e = 17$ $\mathcal{D} = \{2_8, 3_6\}$	$e = 36$ $\mathcal{D} = \{3_{24}\}$	$e = 54$ $\mathcal{D} = \{3_{28}, 4_6\}$
5	\	$e = 18$ $\mathcal{D} = \{2_{10}, 3_4, 4_1\}$	$e = 37$ $\mathcal{D} = \{2_1, 3_{24}\}$	$e = 55$ $\mathcal{D} = \{2_1, 3_{28}, 4_6\}$
6	\	$e = 20$ $\mathcal{D} = \{2_8, 3_8\}$	$e = 38$ $\mathcal{D} = \{2_5, 3_{18}, 4_3\}$	$e = 58$ $\mathcal{D} = \{2_4, 3_{28}, 4_4\}$
7	$e = 7$ $\mathcal{D} = \{2_7\}$	$e = 22$ $\mathcal{D} = \{2_7, 3_{10}\}$	$e = 41$ $\mathcal{D} = \{2_1, 3_1, 4_{24}, 5_1\}$	$e = 60$ $\mathcal{D} = \{2_3, 3_{30}, 4_4\}$
8	$e = 8$ $\mathcal{D} = \{2_8\}$	$e = 23$ $\mathcal{D} = \{2_9, 3_8, 4_1\}$	$e = 43$ $\mathcal{D} = \{2_1, 3_2, 4_{23}, 5_2\}$	$e = 62$ $\mathcal{D} = \{2_3, 3_{30}, 4_5\}$
9	$e = 9$ $\mathcal{D} = \{1_1, 2_7, 3_1\}$	$e = 25$ $\mathcal{D} = \{2_7, 3_{12}\}$	$e = 44$ $\mathcal{D} = \{3_5, 4_{21}, 5_3\}$	$e = 63$ $\mathcal{D} = \{2_5, 3_{28}, 4_6\}$

Table 2: The current lower bounds for $\mathrm{ex}(v; \{C_3, C_4, C_5, C_6\})$ for $v \leq 39$.

N. of vertices	0	10	20	30
0	\	$e = 10$ $\mathcal{D} = \{1_2, 2_6, 3_2\}$	$e = 25$ $\mathcal{D} = \{2_{10}, 3_{10}\}$	$e = 45$ $\mathcal{D} = \{3_{30}\}$
1	\	$e = 12$ $\mathcal{D} = \{2_9, 3_2\}$	$e = 27$ $\mathcal{D} = \{2_9, 3_{12}\}$	$e = 46$ $\mathcal{D} = \{2_1, 3_{30}\}$
2	\	$e = 13$ $\mathcal{D} = \{2_{10}, 3_2\}$	$e = 29$ $\mathcal{D} = \{2_8, 3_{14}\}$	$e = 47$ $\mathcal{D} = \{2_2, 3_{30}\}$
3	\	$e = 14$ $\mathcal{D} = \{1_1, 2_9, 3_3\}$	$e = 30$ $\mathcal{D} = \{1_1, 2_7, 3_{15}\}$	$e = 49$ $\mathcal{D} = \{2_3, 3_{28}, 4_2\}$
4	\	$e = 16$ $\mathcal{D} = \{2_{10}, 3_4\}$	$e = 32$ $\mathcal{D} = \{2_8, 3_{16}\}$	$e = 51$ $\mathcal{D} = \{2_3, 3_{28}, 4_3\}$
5	\	$e = 18$ $\mathcal{D} = \{2_9, 3_6\}$	$e = 34$ $\mathcal{D} = \{2_7, 3_{18}\}$	$e = 52$ $\mathcal{D} = \{2_4, 3_{28}, 4_3\}$
6	\	$e = 19$ $\mathcal{D} = \{1_1, 2_8, 3_7\}$	$e = 36$ $\mathcal{D} = \{2_6, 3_{20}\}$	$e = 54$ $\mathcal{D} = \{2_3, 3_{23}, 4_9, 5_1\}$
7	\	$e = 20$ $\mathcal{D} = \{1_1, 2_9, 3_7\}$	$e = 38$ $\mathcal{D} = \{2_5, 3_{22}\}$	$e = 56$ $\mathcal{D} = \{2_1, 3_{26}, 4_{10}\}$
8	$e = 8$ $\mathcal{D} = \{2_8\}$	$e = 22$ $\mathcal{D} = \{2_{10}, 3_8\}$	$e = 40$ $\mathcal{D} = \{2_4, 3_{24}\}$	$e = 58$ $\mathcal{D} = \{2_1, 3_{26}, 4_{11}\}$
9	$e = 9$ $\mathcal{D} = \{1_1, 2_7, 3_1\}$	$e = 24$ $\mathcal{D} = \{2_9, 3_{10}\}$	$e = 42$ $\mathcal{D} = \{2_3, 3_{26}\}$	$e = 59$ $\mathcal{D} = \{2_2, 3_{26}, 4_{11}\}$

Table 3: The current lower bounds for $\mathrm{ex}(v; \{C_3, C_4, C_5, C_6, C_7\})$ for $v \leq 39$.

Theorem 2 *ex*$(30; \{C_3, C_4, C_5, C_6\}) \geq 47$ *and the* $(3, 8)$*-cage does not belong to EX*$(30; \{C_3, C_4, C_5, C_6\})$.

From Theorem 1 (iv) and Theorem 2 the following corollary is immediate.

Corollary 1 *For* $v \geq 12$, $v \notin \{15, 80, 170\}$, *the girth of* $G \in EX(v; \{C_3, C_4, C_5, C_6\})$ *is* 7, *and there exists an extremal graph* G *of 15 vertices and 18 edges having girth 8.*

Some of the obtained extremal graphs are cages. For instance, in Table 1, if $v = 14$, the graph is the $(3, 6)$-cage, and if $v = 26$, the graph is the $(4, 6)$-cage. Furthermore, in Table 2, if $v = 24$, the graph is the $(3, 7)$-cage, and in Table 3, if $v = 30$, the graph is the $(3, 8)$-cage. Further, these extremal graphs are not unique. This shows that the computed lower bounds by using HSAGA are reasonable.

After we produced good results for extremal graphs with small girth, we also ran the program for large girths. For example, it is known that for $n \geq 12$, $ex(n+2; \{C_3, C_4, \ldots, C_n\}) = 2n+4$, and there exist $G \in EX(v; \{C_3, C_4, \ldots, C_n\})$ with $g(G) = n + 2$ (see Theorem 1(v)). Wang *et al.* [26] used pure simulated annealling to generate an optimal solution in the above case for $n \geq 15$ and

$g(G) \geq n+1$. By running our HSAGA, we also generate optimal solutions for $n \geq 12$, $v = n+2$ and $g(G) = n+2$.

Not only does our algorithm produce reasonably good solutions, but furthermore, HSAGA seems robust with the selection of values for the parameters, such as cooling rate and crossover rate. In either simulated annealling or genetic algorithm, a change in these parameters will affect the results of the program. However, it seems that our algorithm is more tolerant of changes in the values of parameters, in other words, the results are not affected too much if parameters are modified slightly.

From our experiments, we did find some graphs with degree 1, but also some graphs with degrees distributed in small ranges. Again, this is consistent with our belief that for given order and girth, there exist some graphs with largest number of edges and "evenly" distributed degrees.

4 Connectivity of Extremal Graphs

Recall that a graph G is called *connected* if every pair of vertices is joined by a path; that is, the diameter $D = D(G) < \infty$. If $S \subset V$ and $G - S$ is not connected, then S is said to be a *cut set*. A (noncomplete) connected graph is called *k-connected* if every cut set has cardinality at least k. The *connectivity* $\kappa = \kappa(G)$ of a (noncomplete) connected graph G is defined as the maximum integer k such that G is k-connected. The *connectivity* of a complete graph $K_{\delta+1}$ on $\delta+1$ vertices is defined as $\kappa(K_{\delta+1}) = \delta$. The *edge-connectivity* $\lambda = \lambda(G)$ of a graph G is defined analogously. A classical result, due to Whitney, is that $\kappa \leq \lambda \leq \delta$, for every graph G of minimum degree $\delta = \delta(G)$. A graph is *maximally connected* if $\kappa = \delta$, and maximally edge-connected if $\lambda = \delta$. Sufficient conditions for a graph G of minimum degree δ to be maximally connected have been given in terms of its diameter and its girth. In this regard, the following result is contained in [16, 20]:

$$\kappa = \delta \quad \text{if} \quad D \leq 2\lfloor (g-1)/2 \rfloor - 1. \tag{1}$$

The restricted edge connectivity was proposed by Esfahanian and Hakimi [15] who denoted it by $\lambda'(G)$. For a connected graph G, the *restricted edge connectivity* is defined as the minimum cardinality of a set W of edges such that $G - W$ is not connected and W does not contain the set of incident edges of any vertex of the graph, then $G - W$ does not contain isolated vertices. The restricted edge connectivity has been studied under the name of *super edge connectivity*. This is a stronger measure of connectivity than the standard edge connectivity, and was proposed by Boesch [6] and Boesch and Tindell [7]. A graph is *super edge connected*, or *super-*λ, if every minimum edge cut consists of a set of edges incident with one vertex. See [6, 7] for more details. A graph G is *maximally edge connected* if $\lambda(G) = \delta(G)$. Clearly, $\lambda'(G) > \delta(G)$ is a sufficient and necessary condition for G to be super edge connected.

It was shown [15] that $\lambda'(G)$ exists if G is not a star and its order is at least 4, and $\lambda'(G) \leq \xi(G)$, where $\xi = \xi(G)$ denotes the *minimum edge-degree* of G, defined as $\xi(G) = \min\{(d(u) + d(v) - 2 : uv \in E(G)\}$. The following result was proved in [3].

$$\lambda' = \xi \quad \text{if} \quad D \leq g - 2. \tag{2}$$

Applying these results to extremal graphs, we obtain the following result.

Corollary 2 *Every graph* $G \in EX\,(\nu; \{C_3, \ldots, C_n\})$ *has* $\lambda' = \xi$. *Furthermore, if* n *is even then* $\kappa = \delta$.

Proof By Theorem 1(vi), the diameter is $D \leq n - 1 \leq g - 2$, because $g \geq n + 1$. Therefore, from (2) it follows that $\lambda' = \xi$. Moreover, either $n - 1 = g - 2$ which means g is odd because n is even and hence $\kappa = \delta$ by (1), or $n - 1 \leq g - 3$ and $D \leq n - 1 \leq g - 3 \leq 2\lfloor (g - 1)/2 \rfloor - 1$, yielding $\kappa = \delta$, again by (1). ■

5 Future Work

In order to improve its efficiency, we shall next modify our HSAGA for parallel computation. Additionally, we have proved that every $G \in EX(v; \{C_3, C_4, \ldots, C_n\})$ is maximally connected for all even n. Now we propose

Conjecture 1 *Every* $G \in EX(v; \{C_3, C_4, \ldots, C_n\})$ *has* $\kappa = \delta$ *for all odd* n.

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