

On the stability of MPC with a Finite Input Alphabet

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Abstract: This paper studies stability of Model Predictive Control for systems with a finite input alphabet. Since this kind of systems may present a steady-state error under closed-loop control, the focus is on stability in the sense of ultimate boundedness of solutions. To derive sufficient conditions for stability, two different approaches are presented. The first one approximates the finite input alphabet via saturation-control allowing us to analyze the problem from a robust control perspective. In the second approach, a direct analysis of the problem is carried out. The results thus obtained are shown to be less conservative regarding ultimate bounded set than those obtained via the robust control approach.

Keywords: Quantization; Model predictive control; Stability; Ultimate boundedness

1. INTRODUCTION

Model predictive control (MPC) is a control technique capable to deal with state and input constraints. Most work has focused on convex constraints (Mayne et al. (2000)). However, in several control problems, the control input is characterized by a finite alphabet of possible control actions. One of the most studied cases is related to on-off system, e.g., power electronics converters (see Goodwin et al. (2010)), where each input is restricted to take just two values. Digital control systems also belong to this class of controller since they are affected by a quantization process (Quevedo and Goodwin (2003)).

MPC has been extensively applied to system with finite input alphabet. In Cortés et al. (2008), a survey about different predictive strategies applied to power electronics has been presented. One of the most widely used is the so-called Finite Control Set-MPC (FCS-MPC) which considers the switching elements as control inputs. The main advantage of such strategies comes from the fact that switching actions are explicitly tacked into account. Consequently, modulation stages (to handle the switches) are not required.

Whilst stability of convex MPC formulations is relatively well understood, less is known in the finite alphabet case. In Picasso et al. (2003) a receding-horizon formulation for LTI systems with quantized input has been presented. Here, the stabilization problem is based on the search of invariant sets (without relying on Lyapunov techniques). In Quevedo et al. (2004), some stability results for linear systems with finite constraint sets have been presented. However, analysis is restricted to open-loop stable systems. Moreover, the origin must be an element of the finite control set.

A key observation is that when system inputs are restricted to belong to a finite set, in general, the best one can hope

for, is that state trajectories be bounded. Therefore, in the present work we will focus on practical stability or ultimate boundedness. Here, we will study two analysis methods. The first approach approximates the input via saturation control whereupon the elements of the finite control set are treated as the quantization levels. This gives rise to a quantization error which can be considered as a bounded disturbance. Thus, FCS-MPC can be analyzed as a robust problem; see, e.g., (Lazar et al. (2008), Raković (2009) and Raimondo et al. (2009)).

The second approach is based on a more direct analysis of the system. Here, we present an MPC stability analysis method for the original system, with a suitable local controller. The latter acts in the terminal region and is used to establish stability and also to define the final invariant set. In this case, as it will be shown in this work, the obtained final invariant set is smaller than the obtained one via ISS, due to the fact that no-approximations are considered over the system state predictions.

NOTATION AND BASIC DEFINITIONS

Let \mathbb{R} and $\mathbb{R}_{\geq 0}$ denote the real and non-negative real number sets. The difference between two given sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$ is denoted by $\mathcal{A} \setminus \mathcal{B} = \{x : x \in \mathcal{A}, x \notin \mathcal{B}\}$. We represent the Euclidean norm by $|\cdot|$ and the weighted Euclidean norm by $|x|_P^2 = x'Px$. We define a neighborhood of the origin as a ball via $\mathcal{B}_\delta = \{x \in \mathbb{R}^n : |x| < \delta\}$, where δ is a positive number. We represent the transpose of a given matrix A via A' .

Definition 1. (\mathcal{K} Functions). A function $\sigma: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a \mathcal{K} function if it is continuous, strictly increasing and $\sigma(0) = 0$; σ is a \mathcal{K}_∞ function if it is a \mathcal{K} -function and unbounded ($\sigma(s) \rightarrow \infty$ as $s \rightarrow \infty$); a function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a \mathcal{KL} -function if it is continuous and if, for each $t \geq 0$, the function $\beta(\cdot, t)$ is a \mathcal{K} -function and for each $s \geq 0$ the function $\beta(s, \cdot)$ is nonincreasing and satisfies that $\beta(s, t) \rightarrow 0$ as $t \rightarrow \infty$.

2. PROBLEM DESCRIPTION

Consider the following system

$$x^+ = f(x, u), \quad (1)$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the system state and $u \in \mathbb{U} \subset \mathbb{R}^m$ is the control input vector. Each element u_i is restricted to belong to a finite set of p -elements such that

$$u_i \in \mathbb{U} = \{u_1, \dots, u_p\}. \quad (2)$$

It is necessary to find a suitable control law $u = \kappa(x)$ which can lead the system state to the origin. To do this, one can use a MPC strategy, in where the system behavior can be forecast by using a cost function $V(x)$.

In this case, the optimal control problem, $\mathbb{P}_N(x)$, for the system (1), can be stated as

$$\mathbb{P}_N(x) : V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} \quad (3)$$

where $\mathcal{U}_N(x)$ is the set of inputs $\mathbf{u} = \{u(0), \dots, u(N-1)\}$ which satisfies

$$x(k) \in \mathbb{X}, \quad \forall k \in \{0, \dots, N\}, \quad (4)$$

$$u(k) \in \mathbb{U}, \quad \forall k \in \{0, \dots, N-1\}, \quad (5)$$

$$x(N) \in \mathbb{X}_f. \quad (6)$$

The cost function has been defined as

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)). \quad (7)$$

Due to the fact that elements of the set \mathbb{U} can only be considered in the minimization, one can evaluate each element of this set (2) in the proposed cost function and then obtain an optimal input sequence, $\mathbf{u} = \{u(k), \dots, u(N-1)\}$ which minimizes the cost function and satisfies constraints (4)-(6). This methodology is called Finite-Control-Set MPC (FCS-MPC). Therefore, this strategy directly provides a valid input. Nevertheless, establishing stability of this methodology still remains as an open problem. In Quevedo et al. (2004), stability of LTI systems, $x^+ = Ax + Bu$, under quadratic MPC has been presented. However, this analysis is limited to those LTI systems in which the matrix A is Hurwitz, i.e. the plant model is open-loop stable. Additionally, the finite control set \mathbb{U} must include the origin in its interior.

To overcome these limitations, it is necessary to take into account the nature of the input. Since this is restricted to belong to a finite set \mathbb{U} which could not include the origin in its interior, it is not possible to guarantee convergence of the system to the origin due to the fact that oscillations around it (steady-state error) can be presented. For this reason, the performance of these control strategies are addressed in terms of practical stability which is also referred to as ultimate boundedness Khalil (2001).

Definition 2. (Uniform Practical Stability) Let δ be a positive number. The ball $\mathcal{B}_\delta \subset \mathcal{A}$ is said to be Uniformly Practically Asymptotically Stable (UpAS) for (1) if there exists a \mathcal{KL} -function β such that the solution of (1) from any initial state $x_0 \in \mathcal{A}$ satisfies

$$|x(k, x_0)| \leq \beta(|x_0|, k) + \delta, \quad \forall k \geq 0, \quad \forall x_0 \in \mathcal{A}.$$

Additionally, the ball \mathcal{B}_δ is said to be Uniformly Practically Exponentially Stable (UpES) for (1) if there exist a positive number c and a constant $\rho \in (0, 1)$ such that

$$\beta(|x_0|, k) = c \cdot |x_0| \cdot \rho^k,$$

To establish this kind of stability, we investigate in this work two methods. The first one is based on approximating the input via saturation control and uses results from robust control, in particular input-to state stability. The second one presents a direct analysis which gives some conditions to be satisfied for the cost function in the terminal region \mathbb{X}_f .

3. ROBUST CONTROL ANALYSIS

In order to consider FCS-MPC strategy as a robust control problem, we first approximate the system considering the finite input alphabet in (1) as a saturated control \bar{u} , which is defined as:

$$\bar{u} = \{ \bar{u} \in \bar{\mathbb{U}} : \bar{u}_j \in [u_{min}, u_{max}], \quad \forall j \in \{1, \dots, m\} \},$$

in which $\bar{u}_{min} < u_1$ and $\bar{u}_{max} > u_p$.

Now we can consider that real input u as defined in (2) is a quantization of the saturated input \bar{u} . This allow us to represent the real input as $u = \bar{u} + v$, where v stands for the quantization error which is defined via

$$v = \{ v \in \mathbb{V} : |v_j| \leq \mu, \quad \forall j \in \{1, \dots, m\} \}. \quad (8)$$

Now, the quantized input system (1) can be defined by

$$x^+ = f(x, \bar{u}) + w, \quad (9)$$

where $w = f(x, u) - f(x, \bar{u})$ represents the state disturbance produced by the input quantization noise v which is expressed by

$$w = \{ w \in \mathbb{W} : |w_j| \leq \rho_w \cdot \mu, \quad \forall j \in \{1, \dots, m\} \},$$

in which $\rho_w > 0$.

Notice that, since sets \mathbb{U} and $\bar{\mathbb{U}}$ are bounded, the input quantization noise set \mathbb{V} and system disturbance set \mathbb{W} are also bounded.

Recently, researches related to MPC have shown that this useful technique can also guarantee robust stability in some cases. In this area input-to state stability (ISS) concepts (Jiang et al. (2001), Jiang and Wang (2002)) have shown to be a good framework to determine robustness of nominally stable MPC controller specially for the predictive technique called min-max MPC, (see Magni et al. (2006), Limón et al. (2006), Lazar et al. (2008) and Raimondo et al. (2009)). Other methodology to obtain robust MPC controller is the so-called tube based MPC (Raković (2009), Raković et al. (2006)). This technique has recently emerged and is based on determining sequence of reachable sets which form a tube considering the nominal system trajectory as its center.

The following assumption on the system is considered:

Assumption 3. The nominal system in (9) is such that the origin is an equilibrium point $f(0, 0) = 0$ and there is a CLF as a cost function $V(x)$ which makes that the origin is asymptotically (or exponentially) stable.

Remark 4. It is important to emphasize that unlike in Quevedo et al. (2004), here we do not require that the origin belongs to the finite set \mathbb{U} . However, to satisfy Assumption 3, the origin must be included in the interior of the associated nominal input set $\bar{\mathbb{U}}$.

Due to the fact that quantization error is bounded and nominal system is considered stable, ISS is a suitable tool to establish stability for this kind of systems.

3.1 Regional ISS

ISS concepts have been extensively used to establish robust stability for discrete-time nonlinear systems, based on the works presented in Jiang et al. (2001) and Jiang and Wang (2002). Recently, this technique has been extended to work with constrained states and inputs, see Magni et al. (2006). This is called regional ISS.

Next, some necessary definitions are presented in order to define the regional ISS theorem.

Definition 5. (Robust control invariant set). A set \mathcal{A} is robust control invariant for the system (9) if for every $x \in \mathcal{A}$ there exists an input $u \in \mathbb{U}$ such that $x^+ \in \mathcal{A} \forall w \in \mathbb{W}$.

Definition 6. (Regional ISS (Magni et al. (2006))). Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ including the origin in its interior, the system (9) is said to be ISS in \mathcal{A} if \mathcal{A} is robust control invariant for (9) and if there exists a \mathcal{KL} -function β and a \mathcal{K} -function γ such that

$$|x(k, x_0, w)| \leq \beta(|x_0|, k) + \gamma(\mu), \quad \forall k \geq 0, \quad \forall x \in \mathcal{A}.$$

This ISS property allows one to define an associated Lyapunov function as follows.

Definition 7. (Regional ISS-Lyapunov Function (Magni et al. (2006))). A function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function in \mathcal{A} for the system (9) if \mathcal{A} is a robust control invariant set and the origin is in its interior and if there exists a compact set Ω which also includes the origin in its interior, some \mathcal{K}_∞ -functions α_1, α_2 and α_3 , some \mathcal{K} -functions σ_i and ε_i such that

$$\begin{aligned} V(x) &\geq \alpha_1(|x|), \quad \forall x \in \mathcal{A} \\ V(x) &\leq \alpha_2(|x|) + \varepsilon_i(\mu), \quad \forall x \in \Omega \\ \Delta V(x) &= V(f(x, \bar{u}) + w) - V(x) \\ &\leq -\alpha_3(|x|) + \sigma_i(\mu), \quad \forall x \in \mathcal{A}, \quad \forall w \in \mathbb{W} \\ \mathcal{D} &= \{x \in \Omega : V(x) \leq \alpha_4(\mu) \leq b\} \subset \Omega. \end{aligned} \quad (10)$$

The following result was presented in Magni et al. (2006) as Theorem 2:

Theorem 8. (Regional ISS-Lyapunov Function implies Regional ISS). If the system (9) admits an ISS-Lyapunov function in \mathcal{A} , then (9) is ISS in \mathcal{A} . Moreover, $\Delta V(x) < 0$ for all $x \in \mathcal{A} \setminus \mathcal{D}$. It implies that the system will be steered to the final positive invariant set \mathcal{D} where system will remain once it is reached.

3.2 Regional ISS applied to FCS-MPC

Here we will show how to establish stability of MPC when the quantization error is propagated over the prediction horizon. Moreover, if the system and the cost function are locally Lipschitz-continuous then exponential stability can be guaranteed.

Assumption 9. (Locally Lipschitz continuity of the Model). System $x^+ = f(x, u)$ is locally Lipschitz in x in the domain $\mathbb{X} \times \mathbb{U}$ with a Lipschitz constant L_f such that

$$|f(x_1, u) - f(x_2, u)| \leq L_f |x_1 - x_2|, \quad \forall x \in \mathbb{X}, \quad \forall u \in \mathbb{U}.$$

Assumption 10. (Locally Lipschitz continuity of the stage cost). Stage cost function in (7) is locally Lipschitz in x in the domain $\mathbb{X} \times \mathbb{U}$ with a Lipschitz constant L_l such that

$$|\ell(x_1, u) - \ell(x_2, u)| \leq L_l |x_1 - x_2|, \quad \forall x \in \mathbb{X}, \quad \forall u \in \mathbb{U}.$$

Assumption 11. (Locally Lipschitz continuity of the final cost). Final cost function in (7) is locally Lipschitz in x in the domain $\mathbb{X} \times \mathbb{U}$ with a Lipschitz constant L_{V_f} such that $|V_f(x_1, u) - V_f(x_2, u)| \leq L_{V_f} |x_1 - x_2|, \quad \forall x \in \mathbb{X}_f, \quad \forall u \in \mathbb{U}.$

Assumption 12. (Nominal Exponential Stability). In Definition 7, $\alpha_i = c_i |x|^a$ where $c_i > 0 \forall i \in \{1, 2, 3\}$ and $a > 0$, the nominal system $\bar{x}^+ = f(\bar{x}, u)$ under MPC, the origin is exponentially stable with a region of attraction X_N . That is, there exists a \mathcal{KL} -function β , a constant $c > 0$ and a constant $\rho_x \in (0, 1)$ such that

$$|x(k, x_0)| \leq \beta(|x_0|, k) = c \cdot |x_0| \cdot \rho_x^k,$$

and a constant $\rho_v \in (0, 1)$ such that

$$V_N^0(f(x, \kappa_N(x))) \leq \rho_v \cdot V_N^0(x), \quad \forall x \in X_N. \quad (11)$$

Now we can define the following Lemma:

Lemma 13. (Locally Lipschitz continuity of the cost function (Limón et al. (2002))). Considering that Assumptions 9-11 hold. Then, cost function (7) is locally Lipschitz in x in the domain $\mathbb{X} \times \mathbb{U}$ with a Lipschitz constant L_J such that

$$|V(x^+, u) - V(\bar{x}^+, u)| \leq L_J \cdot \rho_w \cdot \mu,$$

where

$$L_J = L_l \frac{L_f^{N-1} - 1}{L_f - 1} + L_{V_f} \cdot L_f^{N-1}. \quad (12)$$

Theorem 14. (ISS of FCS-MPC). Let suppose that nominal system $x^+ = f(x, \bar{u})$ is exponentially stable and the quantization error in (8) is bounded by

$$\mu \leq \frac{b \cdot (1 - \rho_v)}{L_J \cdot \rho_w}.$$

Then system (9) controlled by FCS-MPC is UpEs.

Proof. Since Assumption 12 holds, from Definition 7 one can see that (15) and (16) are satisfied when $\alpha_1 = c_1 |x|^a$, $\alpha_2 = c_2 |x|^a$, $\mathcal{A} = X_N$ and $\Omega = \mathbb{X}_f$.

Now we can see that the cost function evolves along the uncertain system trajectories according to

$$\begin{aligned} \Delta V(x) &= V(f(x, \kappa_N(x)) + w) - V_N^0(x) \\ &\leq V_N^0(f(x, \kappa_N(x))) + L_J \cdot \rho_w \cdot \mu - V_N^0(x) \\ &\leq -\alpha_3(|x|) + L_J \cdot \rho_w \cdot \mu, \quad \forall x \in X_N, \end{aligned} \quad (13)$$

where $\alpha_3 = c_3 |x|^a$ and $\sigma_i(\mu) = L_J \cdot \rho_w \cdot \mu$.

It implies that the system will be steered by the controller towards the set \mathbb{X}_f and then into the set $\mathcal{D} \subset \mathbb{X}_f$. Once \mathcal{D} is reached ($V(x) \leq b$), and taking into account Assumption 12, one can expressed the cost function as follows

$$\begin{aligned} V(f(x, \kappa_N(x)) + w) &\leq \rho_v V_N^0(x) + L_J \cdot \rho_w \cdot \mu \\ &\leq \rho_v \cdot b + L_J \cdot \rho_w \cdot \frac{b \cdot (1 - \rho_v)}{L_J \cdot \rho_w} \\ &\leq b. \end{aligned}$$

This result proves, as established in Definition 5, that the final set \mathcal{D} is a robust control invariant set and it is ,by Definition 2, UpES. □

4. DIRECT ANALYSIS

In the analysis depicted in Section 3, robust stability has been established taking into account only bounds of

the quantization effect (worst case scenario), determining the propagation of the disturbance effect over the cost function. In this section, we present a direct analysis to establish a uniform practical stability for FCS-MPC strategy where no approximations are considered over the system predictions except for the last element.

Assumption 15. (Terminal Region Condition) For a \mathcal{K} -function σ_p , the cost function satisfies the following condition:

$$\min_{u \in \mathbb{U}} \{V_f(f(x, u) + \ell(x, u) - V_f(x))\} \leq \sigma_p(\mu), \quad (14)$$

for all $x \in \mathbb{X}_f$ and $f(x, u) \in \mathbb{X}_f$.

This assumption implies that for all $x \in \mathbb{X}_f$ there exists an input $u \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}_f$. This means that, by Definition 5, \mathbb{X}_f will be a robust control invariant set.

Theorem 16. (Uniform Practical Stability of FCS-MPC). Let $\mathcal{D}_{d_p} = \{x \in \mathbb{X}_f : |x| < d_p\}$ be a neighborhood of the origin. If

- (1) there exist two \mathcal{K}_∞ -functions α_1 , and α_2 , and some \mathcal{K} -function α_3 , ε_p and σ_p such that the cost function satisfies that

$$V(x) \geq \alpha_1(|x|), \quad \forall x \in X_N \quad (15)$$

$$V(x) \leq \alpha_2(|x|) + \varepsilon_p(\mu), \quad \forall x \in \mathbb{X}_f \quad (16)$$

$$\begin{aligned} \Delta V(x) &= V(f(x, u)) - V(x) \\ &\leq -\alpha_3(|x|) + \sigma_p(\mu), \quad \forall x \in X_N. \end{aligned} \quad (17)$$

- (2) for all $x \in X_N \setminus \mathcal{D}$ the condition $\ell(x, u) > \sigma(\mu)$ is satisfied.

Then, \mathcal{D}_{d_p} is UpAS for system (1) under FCS-MPC strategy presented in (3)-(7).

The proof follows the shifted sequence technique as used, e.g. in Rawlings, J. B. and Mayne, D. Q. (2009). It is included in Appendix A.

Remark 17. It is important to emphasize that u in (14) only considers elements from the finite set \mathbb{U} defined in (2). Therefore, the cost function $V(f(x, u))$ in (17) is not approximated unlike in (10) where the system error w is propagated ahead (considering the worst case) over the predictions in the cost function $V(f(x, \bar{u}) + w)$, as presented in Lemma 13.

Corollary 18. Let consider that in Theorem 16 $\alpha_i = c_i|x|^a$, where $c_i > 0 \forall i \in \{1, 2, 3\}$ and $a > 0$. Then, \mathcal{D}_{d_p} is UpES for system (1) under FCS-MPC strategy.

5. EXAMPLE: LTI SYSTEM WITH QUANTIZED-INPUT

As an illustrative example we apply the ideas presented in the previous section to a linear time-invariant (LTI) system. Therefore, the system to be controlled will be represented by

$$x^+ = f(x, u) = Ax + Bu, \quad (18)$$

where matrix A is not necessary stable. The state x is subject to the constraint $x \in \mathbb{X} \subset \mathbb{R}^n$ and the control u is constrained to belong to a finite set, $u \in \mathbb{U} \subset \mathbb{R}^m$ as shown in (2).

Regarding to the cost function $V(x)$ presented in (7), the stage cost is defined as $\ell(x, u) = |x|_Q^2 + |u|_R^2$ where Q and

R are positive definite. In addition, the final cost is defined as $V_f = |x|_P^2$ in which P is positive definite.

The key idea to establish stability of FCS-MPC is based on finding a suitable controller, $u_f = \kappa(x)$, which acts in the terminal region \mathbb{X}_f in order to satisfy the stability conditions presented in the Theorem 16.

It is well known that for a disturbance-free LTI system under linear quadratic MPC one can consider $u_f = Kx$ as a stabilizing controller for the terminal region \mathbb{X}_f (see Section 2.5 in Rawlings, J. B. and Mayne, D. Q. (2009)).

For our system, we propose to use the quantized input of the nominal solution, it means $u_f = q\{Kx\} = Kx + v$. Thus, the system can be expressed via:

$$x^+ = A_K x + w, \quad \forall x \in \mathbb{X}_f. \quad (19)$$

where $A_K = A + BK$ and $w = Bv$ is the state disturbance produced by the input quantization noise v .

Now we will check the Assumption 15 given for the terminal region

$$\begin{aligned} V_f(f(x, u)) + \ell(x, u_f) - V_f(x) &\leq |A_K x + w|_P^2 + |x|_Q^2 + |Kx + v|_R^2 - |x|_P^2, \\ &= |A_K x|_P^2 + |x|_Q^2 + |Kx|_R^2 - |x|_P^2 + |v|_R^2 + |Bv|_P^2 \\ &\quad + 2x'(A'_K PB + K'R)v, \\ &= x'(A'_K PA_K + Q^* - P)x + |Bv|_P^2 + |v|_R^2 \\ &\quad + 2x'(A'_K PB + K'R)v, \end{aligned} \quad (20)$$

where $Q^* = Q + K'RK$. Similarly to the nominal case, matrix P is chosen to be the solution to the discrete Riccati equation

$$A'_K PA_K + Q^* - P = 0, \quad (21)$$

in which $K = -(B'PB + R)^{-1}B'PA$, is the optimal controller gain. In addition, the following relationship can be derived

$$A'_K PB + K'R = A'PB + K'(B'PB + R) = 0.$$

Remark 19. It is important to emphasize that the controller $u_f = q\{Kx\}$ is not implemented. It is only used to establish stability and to define the final invariant set \mathcal{D} . However, matrix K determine the size of the terminal region \mathbb{X}_f which must satisfies that for all $x \in \mathbb{X}_f$, $(A_K x + w) \in \mathbb{X}_f$ for all $w \in \mathbb{W}$ and $K\mathbb{X}_f \subseteq \mathbb{U}$.

Finally, from (20), Assumption 15 for an LTI system with finite input alphabet is satisfied as follows:

$$V_f(f(x, u)) + \ell(x, u_f) - V_f(x) \leq |Bv|_P^2 + |v|_R^2.$$

Consequently, we have that the evolution of the cost function between two consecutive instants (A.5) can be expressed by:

$$\Delta V(x) \leq -\ell(x, u) + |Bv|_P^2 + |v|_R^2, \quad \forall x \in X_N,$$

Since $\ell(x, u) > c_1|x|^2$, one can say that

$$\Delta V(x) \leq -c_1|x|^2 + \sigma_p(\mu), \quad \forall x \in X_N, \quad (22)$$

where $\sigma_p(\mu) = |B'PB + R| \cdot \mu^2$ and $c_1 = \lambda_{\min}(Q)$.

To establish the smallest final invariant set \mathcal{D} , it is convenient to define a region \mathcal{B}_r as follows

$$\mathcal{B}_r = \{x \in \mathbb{X}_f : |x|^2 \leq r^2 = \frac{|B'PB + R|}{\lambda_{\min}(Q)} \cdot \mu^2\}. \quad (23)$$

Remark 20. Notice that from (22), one can guarantee that for all $x \in \mathbb{X}_f \setminus \mathcal{B}_r$, the cost function will decrease monotonically, $V_N^0(x^+) - V_N^0(x) < 0$. Nevertheless, when the system reaches the ball \mathcal{B}_r , the cost function could be increased, steering the system state outwards of this set.

Afterwards, we define the maximum outward movement that the system state can present from the ball \mathcal{B}_r by

$$d_p \leq |f(r, u) + w| = |A_K r + Bv| \leq |A_K| r + |B| |v|,$$

$$d_p \leq \left(|A_K| \sqrt{\frac{|B'PB + R|}{\lambda_{\min}(Q)}} + |B| \right) \mu.$$

Finally, the final control invariant set is defined by

$$\mathcal{D}_{d_p} = \{x \in \mathbb{X}_f : |x| \leq d_p\}, \quad (24)$$

which is UpES for the system (18), for all $x \in X_N \setminus \mathcal{D}_{d_p}$.

Remark 21. Notice that the size of the final invariant set \mathcal{D}_{d_p} depends not only on the quantization error μ , but also on the matrices Q and R which can be adjusted in order to guarantee that $\mathcal{D}_{d_p} \subset \mathbb{X}_f$.

6. SIMULATION RESULTS

Simulation studies were carried out in order to verify the conditions for uniform practical stability of FCS-MPC presented in this work. Let consider the LTI system (18) with finite input alphabet, where

$$A = \begin{bmatrix} 0.6 & 0 \\ 0.7 & -1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0.8 \end{bmatrix}.$$

Here, the input is restricted to belong to the following finite set

$$\mathbb{U} = \{u \in \mathbb{R} : u \in \{-1, -0.6, -0.2, 0.2, 0.6, 1\}\}. \quad (25)$$

In this case, the maximum quantization error defined in (8) is given by $\mu = 0.2$. Thus, the approximated saturated input is expressed by

$$\bar{u} = \{\bar{u} \in \bar{\mathbb{U}} : \bar{u} \in [-1.2, 1.2]\}, \quad (26)$$

Notice that the origin is not an element of the finite control set \mathbb{U} . Nevertheless, as stated in Remark 4, the associated non-quantized input set $\bar{\mathbb{U}}$ does contain the origin.

FCS-MPC strategy was implemented by using a quadratic cost function, where the stage cost, $\ell(x, u) = |x|_Q^2 + |u|_R^2$, is adjusted by setting $Q = I_2$ and $R = 0.2$. With this, the terminal cost $V_f = |x|_P^2$ can be obtained by solving the Ricatti equation (21):

$$P = \begin{bmatrix} 1.5399 & -0.1046 \\ -0.1046 & -1.3963 \end{bmatrix}.$$

In addition, the optimal gain of the controller $u_f = q\{Kx\}$ and the matrix A_K are given by

$$K = [-0.7478 \ 1.2157], \quad A_K = \begin{bmatrix} 0.5252 & 0.1215 \\ 0.1018 & -0.2274 \end{bmatrix}.$$

The prediction horizon was considered as $N = 4$.

Due to the input constraint (26), the terminal region for the approximated system is defined as

$$\mathbb{X}_f = \{x \in \mathbb{R}^2 : |x| \leq a = 0.84\}.$$

The final invariant set, defined in (24), is given by

$$\mathcal{D}_{d_p} = \{x \in \mathbb{X}_f : |x| \leq d_p = 0.28\}.$$

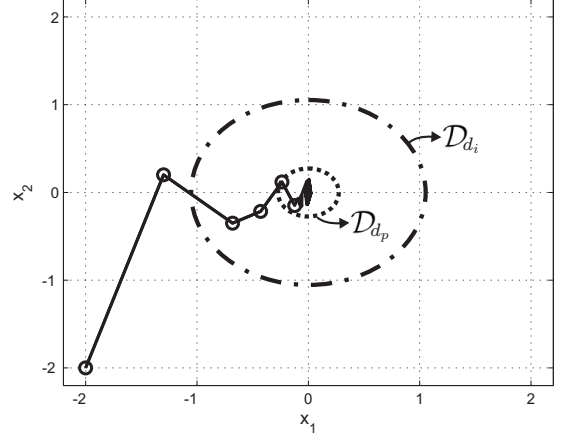


Fig. 1. Evolution of the system state under FCS-MPC.

In order to illustrate the benefits of using the direct analysis we will compare it with the robust control approach (based on regional ISS), presented in Section 3, in term of the design of the final invariant set \mathcal{D} .

To do this, firstly we obtain the following Lipschitz constants

$$L_f = |Ak| = 0.5418,$$

$$L_l = |Q|(\rho_w \cdot \mu + 2a) = 1.9503,$$

$$L_{V_f} = |P|(\rho_w \cdot \mu + 2a) = 3.1107,$$

where $\rho_w = |B| = 0.8062$. Thus, the Lipschitz constant for the cost function (12) is given by $L_J = 4.0740$.

On the other hand, considering that Assumption 12 holds, from (11) we have that $\rho_v = 1 - c_1/c_2 = 0.3730$.

Finally, the final invariant set defined by regional ISS is expressed via:

$$\mathcal{D}_{d_i} = \{x \in \mathbb{R}^2 : |x| \leq d_i = 1.0463\}.$$

It is clear that the final invariant set define by the direct analysis, \mathcal{D}_{d_p} , is less conservative than the obtained one by regional ISS, \mathcal{D}_{d_i} .

Figure 1 depicts the system state evolution when FCS-MPC is applied, starting from the initial condition $x(0) = [-2, -2]'$ until reaching the final invariant set \mathcal{D}_{d_p} . It is important to notice that the controller directly provided a valid input to steer the system towards the final set, it means only elements of the finite set (25) are considering in the minimization of the cost function.

7. CONCLUSION

In this work, sufficient conditions to ensure practical stability of systems with finite input alphabet have been presented. It has been shown that robust control can be used to guarantee uniform asymptotical (exponential) practical stability of MPC for this kind of systems. However, since this technique propagates the quantization error ahead over all states predictions and considers the worst case scenario, the analysis becomes unnecessarily conservative. A less conservative method was presented by carrying out a direct analysis for the real system. To illustrate the differences between both approaches, these methodologies where applied to an LTI system using a quadratic FCS-MPC strategy.

The presented analysis in this paper only considers regulation of the system. So, further work must be done for reference tracking. Other issue to be studied is the design of a robust FCS-MPC controller for external disturbances.

Appendix A. PROOF OF THEOREM 16

Proof. Let $x(0) = x \in \mathbb{X} \subset \mathbb{R}^n$ be the initial condition for the state. Then, the cost function for the given initial condition is defined by:

$$V_N^0(x) = V_N(x, \mathbf{u}^0(x)) \quad (\text{A.1})$$

where $\mathbf{u}^0(x) = \{u(k), u(k+1), \dots, u(N-1)\}$, is the optimal minimizing control sequence which satisfies the constraints stated in the optimal problem (3). This will generate the following optimal state sequence

$$\mathbf{x}^0(x) = \{x, x(k+1), \dots, x(N)\}.$$

Now, we can define the cost function for the next step via:

$$V_N^0(x^+) = V_N(x^+, \mathbf{u}^0(x^+)). \quad (\text{A.2})$$

To analyze how the cost function evolves over the time, it is necessary to compare (A.1) with (A.2). To simplify this task, we can use a suboptimal solution for (A.2) such that

$$V_N^0(x^+) = V_N(x^+, \mathbf{u}^0(x^+)) \leq V_N(x^+, \tilde{\mathbf{u}}),$$

where $\tilde{\mathbf{u}}$ is chosen as $\tilde{\mathbf{u}}(x) = \{u(k+1), u(k+2), \dots, u(N-1), u_f\}$. Here, the last element u_f is the control input which is applied in the final set \mathbb{X}_f . Hence, we will obtain the following state sequence

$$\tilde{\mathbf{x}} = \{x(k+1), \dots, x(N), f(x(N), u_f)\}.$$

As define in (7), we know that

$$\begin{aligned} V_N^0(x) &= V_N(x, \mathbf{u}^0) \\ &= \ell(x, \kappa_N(x)) + \sum_{j=0}^{N-1} \ell(x(j), u(j)) + V_f(x(N)). \end{aligned} \quad (\text{A.3})$$

and

$$\begin{aligned} V_N(x^+, \tilde{\mathbf{u}}) &= \sum_{j=1}^{N-1} \ell(x(j), u(j)) + \ell(x(N), u_f) \\ &\quad + V_f(f(x(N), u_f)). \end{aligned} \quad (\text{A.4})$$

Comparing (A.3) with (A.4) we have that

$$\begin{aligned} \Delta V(x) &= V_N^0(x^+) - V_N^0(x) \\ &\leq V_N(x^+, \tilde{\mathbf{u}}) - V_N^0(x) \\ &= -\ell(x, \kappa_N(x)) + V_f(f(x(N), u_f)) \\ &\quad + \ell(x(N), u_f) - V_f(x(N)). \end{aligned}$$

Considering that Assumption 15 holds, it follows that

$$\Delta V(x) \leq -\ell(x, u) + \sigma_p(\mu), \quad \forall x \in X_N. \quad (\text{A.5})$$

Finally, the system will be steered by the controller towards the set \mathbb{X}_f due to (A.5) will be negative when $x \in X_N \setminus \mathcal{D}$ and then into the set $\mathcal{D} \subset \mathbb{X}_f$ where finally the system will be confined. \square

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