Input-to-state stability of packetized predictive control over unreliable networks affected by packet-dropouts

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Input-to-State Stability of Packetized Predictive Control over Unreliable Networks Affected by Packet-Dropouts

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***Index Terms*— Control over networks, packet drops, predictive control, input-to-state stability (ISS).**

***Abstract*— This work studies a predictive networked control scheme in which packets containing optimizing sequences of control inputs are sent over an unreliable communication network affected by data-loss. We show that, provided the number of consecutive packet losses is bounded, input-to-state stability can be ensured by appropriate choice of design parameters. Our results apply to the general case of constrained nonlinear plants which are affected by uncertain and unmeasured disturbances.**

I. INTRODUCTION

In a Networked Control System (NCS), plant and controller are typically connected via a communication network which may be shared with other applications; see, e.g., articles in the special issues [1], [2]. When compared with direct point-to-point analog wired connections, the sharing of a network simplifies the cabling and, thus, increases overall system reliability. Traditionally NCS's have relied upon special purpose network protocols such as FIP, Profibus, CAN and variants thereof. However, to increase portability, interoperability, flexibility and maintainability, there has been a growing trend to move towards general purpose protocols and technologies. Indeed, TCP/IP over (wired) Ethernet has become the most widely used network technology in industry. Since general purpose network platforms were not originally designed for applications with critical timing requirements, their use for closed-loop control presents some serious challenges. In addition to being quantized, transmitted data may be affected by time delays and data-dropouts. Thus, in a NCS, links are not transparent, often constituting a significant bottleneck in the achievable performance.

One interesting feature of modern communication protocols is that data is sent in large time-stamped packets. For example, in Ethernet the frame format allows for a data-packet of 46-1500 bytes, the overhead being 26 bytes. Thus, quantization effects are often negligible. Furthermore, the large data-packet size can be used in order to seek robustness with respect to time delays and packet-dropouts. To be more precise, one can conceive controllers which at each (discrete) time instant, transmit tentative (or *suggested*) controls for a finite number of future instants. Depending upon transmission outcomes and through appropriate buffering and selection logic at the receiver node, some of the control values received are passed on to the plant actuator. The idea is related to that used in predictive interfaces and was proposed in [3] for the teleoperation of prestabilized constrained nonlinear systems. The concept also underlies more recent NCS configurations for LTI plants described, e.g., in [4]–[8]. We note that, within this context, predictive control methods are a natural choice, since they inherently provide predictions of the plant input over a finite horizon.

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Whilst experimental results of NCS's which use packetized predictive control ideas are promising, there exist only few supporting theoretical results. For example, in the above mentioned works, stabilizing properties have been mainly studied only for LTI plants. To the best of the authors' knowledge, in relation to networked control of general nonlinear plants and where signal predictions are sent in packets, the only published works which provide direct guidelines for choosing design parameters so as to ensure closed-loop stability of the resulting NCS are [9]–[12]. Unfortunately, in [9]–[11] only systems with no disturbances are treated. On the other hand, in the approach presented in [12] buffer contents need to be known at the controller side (This requires reliable acknowledgments of receipts.) and signal predictions to be transmitted are artificially restricted to belong to a reduced set, thus, limiting achievable closed-loop performance.

In the present note, we describe a packetized predictive networked control scheme in which an optimizing sequence of control inputs is sent over a communication network affected by packet-dropouts. A key aspect of our work lies in the fact that we will treat general nonlinear systems, which are subject to disturbances and input and/or state constraints, and that we do not require receipt acknowledgments. For that purpose, we embellish the approach presented in our conference contribution [9] to take into account disturbances. We show that input-to-state stability (see, e.g., [13], [14]) of the resulting NCS can be imposed directly in the design through rather mild conditions on the plant model and tuning parameters of the controller. These conditions are akin to those encountered in non-networked model predictive control (MPC); see, e.g., [15].

The remainder of this note is organized as follows: In Section II we present the NCS under study. Associated stability properties are established in Section III. Section IV draws conclusions.

Notation and preliminaries

Throughout this work, $|\cdot|$ denotes the Euclidean norm, the superscript ^{T} refers to transposition, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers, $\mathbb{N}_0 = \{0, 1, \dots\}$. For any sequence $\{a_\ell\}_{\ell \in \mathbb{N}_0}$, we write $a_k^i = \{a_k, a_{k+1}, \dots a_i\}$, where $i \geq k \geq 0$; if $i < k$, then $a_k^i \triangleq \{\ \}$, the empty set.

A function $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class-*$\mathcal{K}$ ($\varphi \in \mathcal{K}$), if it is continuous, zero at zero and strictly increasing. It is of class $\mathcal{K}_\infty$, if it is of class-$\mathcal{K}$ and is unbounded. Every $\varphi \in \mathcal{K}_\infty$ is invertible with $\varphi^{-1} \in \mathcal{K}_\infty$. A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of *class-*$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for each fixed t and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$; see, e.g., [13].

II. PACKETIZED PREDICTIVE NETWORKED CONTROL

As foreshadowed in the introduction, we consider discrete-time nonlinear MIMO plant models:

$$x(k+1) = f(x(k), u(k), w(k)), \tag{1}$$

where $x(0) = x_0$ and with $f(0,0,0) = 0$. The plant inputs and states are constrained as per:

$$u(k) \in \mathbb{U} \subseteq \mathbb{R}^p, \quad x(k) \in \mathbb{X} \subseteq \mathbb{R}^n, \qquad \forall k \in \mathbb{N}_0, \tag{2}$$

where $p, n \geq 1$ and with $0 \in \mathbb{U}$, $0 \in \mathbb{X}$. The uncertain disturbance belongs to a compact set $\mathbb{W} \subset \mathbb{R}^r$, where $r \geq 1$ and with $0 \in \mathbb{W}$. We, thus, have:

$$w(k) \in \mathbb{W}, \quad \forall k \in \mathbb{N}_0$$

so that $|w(k)| \leq |\mathbb{W}|$, $\forall k \in \mathbb{N}_0$, where

$$|\mathbb{W}| \triangleq \max_{w \in \mathbb{W}} (|w|) < \infty.$$

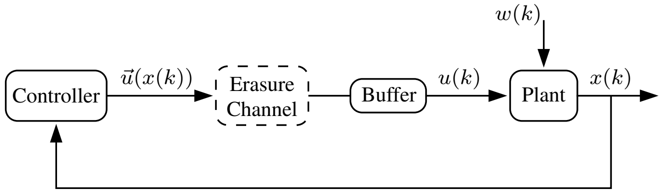

Fig. 1. NCS Configuration

A. Network Model

Our interest lies in clock-driven Ethernet-like communication networks situated between controller output and plant inputs. Data is sent in large time-stamped packets. Time-delays and packet-dropouts are likely to occur. From a closed-loop control perspective, transmission errors are the most serious network effect. This motivates us to model the network as an erasure channel, which operates at the same sampling rate as the plant model (1), see also [16], [17].[1] We, thus, model transmission effects via the discrete dropout process $\{d(k)\}_{k\in\mathbb{N}_0}$ in:

$$d(k) \triangleq \begin{cases} 1 & \text{if packet-dropout occurs at instant } k, \\ 0 & \text{if packet-dropout does not occur at instant } k. \end{cases} \tag{3}$$

In our proposal, at each time instant k, the controller sends a control packet

$$\vec{u}(x(k)) \triangleq \begin{bmatrix} u_0(x(k))^T & u_1(x(k))^T & \dots & u_{N-1}(x(k)) \end{bmatrix}^T \tag{4}$$

to the plant input node. To achieve good performance, despite unreliable communication, $\vec{u}(x(k))$ contains possible control inputs for a finite number of N future time instants. At the plant input side, the received packets are buffered, providing the plant inputs, see Fig. 1. In what follows, we will first describe the buffering procedure and then the control packet design.

B. Buffering

As depicted in Fig. 1, the plant input is connected to a buffer. Its state is overwritten whenever a valid (i.e., uncorrupted and undelayed) control packet arrives. Actuator values are then passed on to the plant sequentially until the next valid control packet is received.

More formally, suppose that at a time k, the controller sends data packet $\vec{u}(x(k))$. The state of the buffer, say $b(k) \in \mathbb{R}^{pN}$, is then given by

$$b(k) = d(k)Sb(k-1) + (1-d(k))\vec{u}(x(k)), \tag{5}$$

where

$$S \triangleq \begin{bmatrix} 0_p & I_p & 0_p & \cdot\,\cdot & 0_p \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_p & \dots & 0_p & I_p & 0_p \\ 0_p & \multicolumn{2}{c}{\dots\dots\dots} & 0_p & I_p \\ 0_p & \multicolumn{3}{c}{\dots\dots\dots\dots\dots} & 0_p \end{bmatrix} \in \mathbb{R}^{pN\times pN} \tag{6}$$

and with initial state $b(0) = 0$.

The buffer states ultimately give rise to the plant inputs in (1) via:

$$u(k) = \begin{bmatrix} I_p & 0_p & \dots & 0_p \end{bmatrix} b(k). \tag{7}$$

Thus, the buffering mechanism amounts to a *parallel-in serial-out shift register*, which acts as a safeguard against packet-dropouts.[2]

[1]Note that small time-delays up to a fixed threshold can be incorporated in the plant model (1). Signals, which are delayed more than the threshold, are then considered as "lost."

[2]The choice in (6) corresponds to setting the buffer state to zero if no data is received over N consecutive instants at the node. Alternatively, if one wished to hold the latest value, one could set the bottom right hand element of S equal to I_p.

C. Control Packet Design

The control packets $\vec{u}(x(k))$, see (4), are formed by adapting the ideas underpinning MPC; see, e.g., [15]. Thus, at each time instant k and for a given plant state $x(k)$, the following cost function is minimized:

$$J(\vec{u}\,', x(k)) \triangleq F(x'_N) + \sum_{\ell=0}^{N-1} L(x'_\ell, u'_\ell). \tag{8}$$

The cost function in (8) examines predictions of the nominal system over a finite horizon of length N. In particular, the predicted state trajectories are generated by the following model:

$$x'_{\ell+1} = f(x'_\ell, u'_\ell, 0), \quad x'_0 = x(k) \tag{9}$$

whilst the entries in

$$\vec{u}\,' = \begin{bmatrix} (u'_0)^T & \dots & (u'_{N-1})^T \end{bmatrix}^T$$

are the associated plant inputs. Predicted plant states and inputs are penalized via the per-stage weighting function $L(\cdot,\cdot)$ and the terminal weighting $F(\cdot)$. (More details on how to choose the weighting functions are included in Section III.)

Minimization of J in (8) is carried out under constraints akin to those in (2), namely:

$$u'_\ell \in \mathbb{U}, \quad x'_{\ell+1} \in \mathbb{X}, \quad \forall \ell \in \{0, \dots, N-1\}. \tag{10}$$

In addition, the *final state* x'_N is restricted to belong to a given set $\mathbb{X}_f \subseteq \mathbb{X}$, where $0 \in \mathbb{X}_f$:

$$x'_N \in \mathbb{X}_f. \tag{11}$$

The control packet $\vec{u}(x(k))$ is set equal to the constrained optimizer,

$$\vec{u}(x(k)) \triangleq \arg\min_{\vec{u}\,'} J(\vec{u}\,', x(k)), \qquad \text{s.t. (9)–(11)}, \tag{12}$$

and is sent through the network to the actuator node, see Fig. 1.

Following the receding horizon optimization idea, at the next sampling step, i.e., at $k+1$ and given $x(k+1)$, the horizon is shifted by one and another optimization is carried out. The resultant optimizer $\vec{u}(x(k+1))$ is then transmitted. This procedure is repeated *ad infinitum*.

Note that, $\vec{u}(x(k))$ contains possible plant input values for instants $\{k, \dots, k+N-1\}$. If $\vec{u}(x(k))$ is received at time k, then these values are written into the buffer and implemented sequentially until some future (valid) control packet arrives. Simulation studies, such as those documented in [9], illustrate that packetized predictive control has the potential to make the resulting NCS robust with respect to packet-dropouts. In Section III we develop analytical results which guarantee input-to-state stability of the NCS.

In the NCS architecture under study, the plant input design is done dynamically such as to optimize performance. It is important to note that whilst $\vec{u}(x(k))$ is found by evaluating open-loop predictions (and not closed-loop policies, see also [10]), the resultant control policy is a closed-loop one. Indeed, the loop is closed whenever no dropouts occur, i.e., $d(k) = 0$. Simulation studies, including those documented in [9], suggest that sending the control packets $\vec{u}(x(k))$ will in general give better performance than only transmitting the value $u_0(x(k))$.

D. Resultant Control Law

The (time-varying and nonlinear) control law which results from bringing together the control packets, network and buffering procedure is characterised by equations (5), (7) and (12). It can also be written in compact form via:

$$u(k) = \kappa(d^k_{k-N+1}, x^k_{k-N+1}), \tag{13}$$

$$\kappa(d_{k-N+1}^{k}, x_{k-N+1}^{k}) = \begin{cases} u_0(x(k)), & \text{if } d(k)=0 \\ u_1(x(k-1)), & \text{if } d(k)=1 \text{ and } d(k-1)=0 \\ u_2(x(k-2)), & \text{if } d(k)d(k-1)=1 \text{ and } d(k-2)=0 \\ \quad\vdots & \quad\vdots \\ u_{N-1}(x(k-N+1)), & \text{if } \prod_{0\le\ell\le N-2} d(k-\ell)=1 \text{ and } d(k-N+1)=0. \end{cases}$$

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where $\kappa(d_{k-N+1}^{k}, x_{k-N+1}^{k})$ is given at the top of the page.

Note that d_{k-N+1}^{k} takes different values for each of the 2^N possible transmission outcomes over the last N instants. However, different values of d_{k-N+1}^{k} may give the same $\kappa(d_{k-N+1}^{k}, x_{k-N+1}^{k})$.

III. Stability of Packetized Predictive Networked Control

We will next establish sufficient conditions for input-to-state stability (ISS) of the plant model (1) when controlled within the NCS described in the previous section. For further reference, we will denote the associated plant state trajectories via:

$$x(k) = \phi(k, x_0, d_0^{k-1}, w_0^{k-1}), \quad k \in \mathbb{N}_0. \tag{14}$$

We also introduce the (open-loop) iterated mapping $f^i(\cdot,\cdot)$ with implicit input $\vec{u}(x)$, see (12):[3]

$$f^i(x, \varpi_0^{i-1}) \triangleq \begin{cases} x, & \text{if } i=0, \\ f(f^{i-1}(x,\varpi_0^{i-2}), u_{i-1}(x), \varpi(i-1)), & \text{if } i \le N, \end{cases} \tag{15}$$

where $\varpi_0^{i-1} = \{\varpi(0), \varpi(1), \dots, \varpi(i-1)\}$.

Many stability results for standard non-networked MPC loops without disturbances require that the cost function in (8) be such that the following assumption holds, see, e.g., [15], [18]:

Assumption 1 (Tuning Parameters): The terms $F(\cdot)$ and $L(\cdot,\cdot)$ satisfy:

$$\begin{gathered} F(x) \ge 0, \quad \forall x \in \mathbb{X}_f, \qquad F(0)=0, \\ L(0,0)=0,\ L(x,u) \ge \alpha(|x|),\ \forall x \in \mathbb{X}_N,\ \forall u \in \mathbb{U}, \end{gathered} \tag{16}$$

where α is a class $\mathcal{K}_\infty$ function and $\mathbb{X}_N \subseteq \mathbb{X}$ denotes the set of all *feasible* initial states, i.e., plant states such that the optimization problem (12) has a solution.

There exists a *terminal control law* $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that, for all $\xi \in \mathbb{X}_f$:

$$\begin{gathered} F(f(\xi, \kappa_f(\xi), 0)) - F(\xi) + L(\xi, \kappa_f(\xi)) \le 0, \\ f(\xi, \kappa_f(\xi), 0) \in \mathbb{X}_f. \end{gathered} \tag{17}$$

Remark 1: It is worth emphasizing that $\kappa_f(\cdot)$ above constitutes a locally stabilizing control law for the nominal plant model. More precisely, (17) establishes that $F(\cdot)$ is a local control Lyapunov function and that $\mathbb{X}_f$ is controlled invariant. A key point here is that κ_f is not necessarily used on the plant. It is simply a construct which has been adopted to establish stability results for non-networked MPC formulations; see, e.g., [15], [18]. In the present work we will adapt it for networked control over erasure channels. □

Similar to [19], [20], where non-networked MPC of uncertain nonlinear systems was studied, for the networked case, we will assume that some additional properties hold. These involve the plant model, the feasible set $\mathbb{X}_N$, and the (optimal) value function, see (12):

$$V(x) \triangleq J(\vec{u}(x), x).$$

[3]For example, we have $f^1(x,\varpi(0)) = f(x, u_0(x), \varpi(0))$, and $f^2(x, \varpi_0^1) = f(f(x, u_0(x), \varpi(0)), u_1(x), \varpi(1))$.

Assumption 2 (Continuity): There exist class $\mathcal{K}$ functions φ_x, φ_w and a $\mathcal{K}_\infty$ function φ_V such that, $\forall x, z \in \mathbb{X}_N$, $\forall u \in \mathbb{U}$, and $\forall w \in \mathbb{W}$ the following are satisfied:

$$\begin{gathered} |f(x,u,w) - f(z,u,0)| \le \varphi_x\big(|x-z|\big) + \varphi_w\big(|w|\big), \\ |V(x) - V(z)| \le \varphi_V\big(|x-z|\big). \end{gathered} \tag{18}$$

Assumption 3 (Robust Positive Invariance): The feasible set $\mathbb{X}_N$ is a compact and robust positively invariant set for the mapping $f^i(\cdot,\cdot)$, $\forall i \in \{1,2,\dots,N\}$, see (15), i.e., it holds that

$$f^i(x, \varpi_0^{i-1}) \in \mathbb{X}_N, \quad \forall x \in \mathbb{X}_N,\ \forall \varpi_0^{i-1} \in \mathbb{W}^i,\ \forall i \in \{1,2,\dots,N\}. \tag{19}$$

Remark 2: Continuity of the optimal value function, see (18), is often used to show robust stability of non-networked MPC for nonlinear constrained systems; see, e.g., [15], [20], [21]. Continuity holds, for example, in the case of linear systems with convex constraints and where $F(\cdot)$ and $L(\cdot,\cdot)$ are continuous.

Remark 3: Assumption 3 is also akin to that encountered in non-networked MPC for constrained systems with bounded uncertainties; see, e.g., [15], [19]–[23], and also [24]–[27] where linear systems with polytopic constraints are studied. The main difference lies in that, in the non-networked case, only robust positive invariance of $\mathbb{X}_N$ for the map $f^1(x, \varpi(0))$ is needed, whereas in the networked case we require invariance for up to N-steps. A central theme in [20], [22]–[27] is that finding robust positively invariant sets is facilitated by tightening constraints on predicted states in the optimization problem. This idea is certainly applicable also in the networked case. □

With the above as a background, we can now state our first technical result in Lemma 1 below.

Lemma 1: Suppose that Assumptions 1–3 are satisfied. Then there exists $\bar{\gamma} \in \mathcal{K}$, such that:

$$\begin{aligned} &V(f^i(x, \varpi_0^{i-1})) - V(x) \le -\alpha(|x|) + \bar{\gamma}(|\mathbb{W}|), \\ &\qquad\qquad \forall x \in \mathbb{X}_N,\ \forall \varpi_0^{i-1} \in \mathbb{W}^i,\ \forall i \in \{1,\dots,N\}, \end{aligned} \tag{20}$$

where $\alpha \in \mathcal{K}_\infty$ is as in (16).

Proof: To prove this result, we introduce the *nominal mapping* (i.e., without disturbances):

$$\bar{f}^i(x) \triangleq f^i(x, \{0,\dots,0\}), \quad \forall i \in \{0,1,\dots,N\}, \tag{21}$$

and separate the left hand side of (20) into one term which considers the case without disturbances and another which solely takes into account the disturbance effects as follows:

$$\begin{aligned} &V(f^i(x, \varpi_0^{i-1})) - V(x) \\ &= \big(V(\bar{f}^i(x)) - V(x)\big) + \big(V(f^i(x, \varpi_0^{i-1})) - V(\bar{f}^i(x))\big). \end{aligned} \tag{22}$$

1) We first analyze $V(\bar{f}^i(x)) - V(x)$ and adapt the shifted sequence technique, which is often used to prove stability of MPC schemes; see, e.g., [18].

Suppose that $i \le N-1$ and consider the sequence

$$\vec{u}^{\,\sharp} = \{u_i(x), u_{i+1}(x), \dots, u_{N-1}(x), u_N^\sharp, u_{N+1}^\sharp \dots, u_{N+i-1}^\sharp\},$$

whose first $N-i$ elements are taken from the optimizer $\vec{u}(x)$, see (12). The remaining i elements of $\vec{u}^{\sharp}$ use the control law κ_f in (17) according to:

$$u^{\sharp}_{N+j} = \kappa_f(x^{\sharp}_{N+j}), \quad j \in \{0,1,\dots,i-1\} \tag{23}$$

where the values $\{x^{\sharp}_{N+j}\}$ are provided by the recursion

$$x^{\sharp}_{N+j} = f(x^{\sharp}_{N+j-1}, u^{\sharp}_{N+j-1}, 0), \quad j \in \{1,2,\dots,i-1\}$$

with initial value $x^{\sharp}_N = \bar{f}^N(x)$.

It follows directly from (8) that the associated cost satisfies:

$$\begin{aligned} J(\vec{u}^{\sharp}, \bar{f}^i(x)) &= F(x^{\sharp}_{N+i}) + \sum_{\ell=i}^{N-1} L(\bar{f}^{\ell}(x), u_\ell(x)) + \sum_{\ell=N}^{N+i-1} L(x^{\sharp}_\ell, u^{\sharp}_\ell) \\ &= V(x) - \sum_{\ell=0}^{i-1} L(\bar{f}^{\ell}(x), u_\ell(x)) \\ &\qquad + F(x^{\sharp}_{N+i}) - F(\bar{f}^N(x)) + \sum_{\ell=N}^{N+i-1} L(x^{\sharp}_\ell, u^{\sharp}_\ell) \\ &= V(x) - \sum_{\ell=0}^{i-1} L(\bar{f}^{\ell}(x), u_\ell(x)) \\ &\qquad + \sum_{\ell=N}^{N+i-1} \big(F(x^{\sharp}_{\ell+1}) - F(x^{\sharp}_\ell) + L(x^{\sharp}_\ell, u^{\sharp}_\ell)\big) \\ &\le V(x) - \sum_{\ell=0}^{i-1} L(\bar{f}^{\ell}(x), u_\ell(x)), \end{aligned} \tag{24}$$

where we have used (17).

Since, due to optimality, we have $V(\bar{f}^i(x)) \le J(\vec{u}^{\sharp}, \bar{f}^i(x))$, it follows from (16) that:

$$\begin{aligned} V(\bar{f}^i(x)) - V(x) &\le -\sum_{\ell=0}^{i-1} L(\bar{f}^{\ell}(x), u_\ell(x)) \\ &\le -L(x, u_0(x)) \le -\alpha(|x|). \end{aligned} \tag{25}$$

For the case $i = N$, we consider the sequence

$$\vec{u}^{\sharp} = \{u^{\sharp}_N, u^{\sharp}_{N+1} \dots, u^{\sharp}_{2N-1}\},$$

where now all N elements are as in (23). If we define $\sum_{\ell=N}^{N-1} L(\bar{f}^{\ell}(x), u_\ell(x)) = 0$, then (25) follows as in the case $i \le N-1$ studied above.

2) We will next bound the disturbance effect in (22), namely

$$V(f^i(x, \varpi_0^{i-1})) - V(\bar{f}^i(x)).$$

Here, we use the definitions in (15) and (21) and the system property in (18) to obtain the recursion:

$$\begin{aligned} \big|f^i(x, \varpi_0^{i-1}) - \bar{f}^i(x)\big| &= \big|f(f^{i-1}(x, \varpi_0^{i-2}), u_{i-1}(x), \varpi(i-1)) \\ &\qquad - f(\bar{f}^{i-1}(x), u_{i-1}(x), 0)\big| \\ &\le \varphi_x\big(\big|f^{i-1}(x, \varpi_0^{i-2}) - \bar{f}^{i-1}(x)\big|\big) + \varphi_w(|\varpi(i-1)|) \\ &\le \varphi_x\big(\big|f^{i-1}(x, \varpi_0^{i-2}) - \bar{f}^{i-1}(x)\big|\big) + \varphi_w(|\mathbb{W}|), \end{aligned}$$

for all $i \in \{2,\dots,N\}$. For $i = 1$, we simply have

$$|f^1(x, \varpi(0)) - \bar{f}^1(x))| \le \varphi_w(|\mathbb{W}|).$$

The above expressions lead to

$$\big|f^i(x, \varpi_0^{i-1}) - \bar{f}^i(x))\big| \le \gamma_i(|\mathbb{W}|), \quad \forall i \in \{1,2,\dots,N\}, \tag{26}$$

where $\gamma_i \in \mathcal{K}$ is defined recursively via

$$\gamma_{i+1} = \varphi_x \circ \gamma_i + \varphi_w,$$

with $\gamma_1 = \varphi_w$.

Due to Assumption 3, $V(f^i(x, \varpi_0^{i-1}))$ exists. If we now use (18) in (26) and note that

$$\gamma_\ell(|\mathbb{W}|) \le \gamma_{\ell+1}(|\mathbb{W}|), \quad \forall \ell \in \{1,2,\dots,N-1\} \tag{27}$$

then we obtain:

$$|V(f^i(x, \varpi_0^{i-1})) - V(\bar{f}^i(x))| \le \varphi_V\big(\gamma_i(|\mathbb{W}|)\big) \le \varphi_V\big(\gamma_N(|\mathbb{W}|)\big). \tag{28}$$

The result (20) follows from setting $\bar{\gamma} = \varphi_V \circ \gamma_N$ and using (28) and (25) in (22). □

Lemma 1 is instrumental in establishing sufficient conditions for closed-loop stability of the NCS in the presence of packet-dropouts. To state this result, in the sequel we denote the time instants where there are no packet-dropouts, i.e., where $d(k) = 0$, as

$$K = \{k_i\}_{i\in\mathbb{N}_0} \subseteq \mathbb{N}_0, \quad k_{i+1} > k_i, \ \forall i \in \mathbb{N}_0$$

whereas the number of consecutive packet-dropouts is denoted via:

$$m_i \triangleq k_{i+1} - k_i - 1, \quad i \in \mathbb{N}_0.$$

Note that $m_i \ge 0$, with equality if and only if no dropouts occur between instants k_i and k_{i+1}.

When packets are lost, the NCS operates in open-loop. Thus, one can expect that, to ensure desirable properties of the NCS, the number of consecutive packet-dropouts should be bounded. In fact, to establish stability of the NCS, we make the following assumption:

Assumption 4 (Packet-dropouts): The number of consecutive packet-dropouts is uniformly bounded by the prediction horizon minus one, i.e., we have $m_i \le N-1$, $\forall i \in \mathbb{N}_0$. □

Theorem 1 (ISS with dropouts): Suppose that Assumptions 1–4 hold and that, at the first successful transmission instant, we have $x(k_0) \in \mathbb{X}_N$. Then there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the plant states, see (14), are bounded via:

$$|x(k)| \le \beta(|x(k_0)|, k - k_0) + \gamma(|\mathbb{W}|), \quad \forall k \ge k_0.$$

Proof: We first note that, by (16) and Assumption 2, we have $V(0) = 0$ and

$$\alpha(|x|) \le V(x) \le \varphi_V(|x|), \quad \forall x \in \mathbb{X}_N. \tag{29}$$

Consider $x(k_0) \in \mathbb{X}_N$ and any dropout scenario satisfying Assumption 4.

We will first focus on the instants $k_i \in K$ and the plant states $x(k_i) = \phi(k_i - k_0, x(k_0), d_{k_0}^{k_i}, w_{k_0}^{k_i})$. For that purpose, we note that in terms of (15), it holds that:

$$x(k_{i+1}) = f^{k_{i+1}-k_i}(x(k_i), w_{k_i}^{k_{i+1}-1}), \quad \forall k_i \in K \tag{30}$$

and that Assumptions 3 and 4 ensure that $x(k_i) \in \mathbb{X}_N$, for all $k_i \in K$. By Lemma 1, Equation (29) and [15, Lemma B.38] (see also [20, Theorem 2] and [14, Lemma 3.5]) [4], we obtain that there exist functions $\bar{\beta} \in \mathcal{KL}$ and $\bar{\gamma} \in \mathcal{K}$ which give

$$|x(k_i)| \le \bar{\beta}(|x(k_0)|, k_i - k_0) + \bar{\gamma}(|\mathbb{W}|), \quad \forall k_i \in K.$$

The inequality on the right hand side of (29) then establishes that there exist $\tilde{\beta} \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{K}$ such that

$$V(x(k_i)) \le \tilde{\beta}(V(x(k_0)), k_i - k_0) + \tilde{\gamma}(|\mathbb{W}|), \quad \forall k_i \in K. \tag{31}$$

[4]Note that (30) describes a time-varying system, whereas [15, Lemma B.38] is stated for time-invariant systems. Nevertheless, it is easy to see that the latter result can still be applied, since the inequality in (20) holds for all $i \in \{1,\dots,N\}$.

We next consider the instants $k > k_0$, where $k \notin K$, and use (16) to obtain:

$$
\begin{aligned}
V(x(k_i)) &= F(\bar{f}^N(x(k_i))) + \sum_{\ell=0}^{N-1} L(\bar{f}^\ell(x(k_i)), u_\ell(x(k_i))) \\
&\geq \sum_{\ell=0}^{N-1} \alpha\big(|\bar{f}^\ell(x(k_i))|\big), \quad \forall k_i \in K. \quad (32)
\end{aligned}
$$

It is easy to show that, since $\alpha \in \mathcal{K}_\infty$, we have[5]

$$
\sum_{\ell=0}^{M} \alpha(b_\ell) \geq \alpha\left(2^{-M}\sum_{\ell=0}^{M} b_\ell\right), \quad \forall b_\ell \geq 0,\ \forall M \in \mathbb{N}_0. \quad (33)
$$

By using (33) in (32), and applying (26) and (27), we obtain the bound:

$$
\begin{aligned}
2^{N-1}\alpha^{-1}(V(x(k_i))) &\geq \sum_{\ell=0}^{N-1} |\bar{f}^\ell(x(k_i))| \\
&\geq |x(k_i)| + \sum_{\ell=1}^{N-1} \big(|f^\ell(x(k_i), w_{k_i}^{k_i+\ell-1})| - \gamma_\ell(|\mathbb{W}|)\big) \\
&\geq -(N-1)\gamma_{N-1}(|\mathbb{W}|) + |x(k_i)| + \sum_{\ell=1}^{N-1} |f^\ell(x(k_i), w_{k_i}^{k_i+\ell-1})|.
\end{aligned}
\quad (34)
$$

On the other hand, Assumption 4 ensures that $k_{i+1} \leq k_i + N$, yielding:

$$
\begin{aligned}
\sum_{\ell=1}^{N-1} |f^\ell(x(k_i), w_{k_i}^{k_i+\ell-1})| &\geq \sum_{\ell=1}^{k_{i+1}-k_i-1} |f^\ell(x(k_i), w_{k_i}^{k_i+\ell-1})| \\
&= \sum_{k=k_i+1}^{k_{i+1}-1} |\phi(k-k_0, x(k_0), d_{k_0}^k, w_{k_0}^k)|.
\end{aligned}
$$

Expressions (31) and (34) then give:

$$
\begin{aligned}
|\phi(k-k_0, x(k_0), d_{k_0}^k, w_{k_0}^k)| &\leq \sum_{k=k_i}^{k_{i+1}-1} |\phi(k-k_0, x(k_0), d_{k_0}^k, w_{k_0}^k)| \\
&\leq 2^{N-1}\alpha^{-1}(V(x(k_i))) + (N-1)\gamma_{N-1}(|\mathbb{W}|) \\
&\leq 2^{N-1}\alpha^{-1}\big(\tilde{\beta}(V(x(k_0)), k_i - k_0) + \tilde{\gamma}(|\mathbb{W}|)\big) \\
&\qquad + (N-1)\gamma_{N-1}(|\mathbb{W}|) \\
&\leq 2^{N-1}\alpha^{-1}\big(\tilde{\beta}\big(\varphi_V(|x(k_0)|), k_i - k_0\big) + \tilde{\gamma}(|\mathbb{W}|)\big) \\
&\qquad + (N-1)\gamma_{N-1}(|\mathbb{W}|) \\
&\leq \beta(|x(k_0)|, k-k_0) + \gamma(|\mathbb{W}|), \quad \forall k \in \{k_i, \dots, k_{i+1}-1\},
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma(|\mathbb{W}|) &= 2^{N-1}\alpha^{-1}\big(2\tilde{\gamma}(|\mathbb{W}|)\big) + (N-1)\gamma_{N-1}(|\mathbb{W}|) \\
\beta(|x(k_0)|, k-k_0) &= 2^{N-1}\alpha^{-1}\big(2\tilde{\beta}(\varphi_V(|x(k_0)|), k_i - k_0)\big),
\end{aligned}
\quad (35)
$$

for all $k \in \{k_i, \dots, k_{i+1}-1\}$, for all $k_i \in K$, and where we have used (29) and (33). Clearly, $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$.[6] □

Theorem 1 constitutes the main contribution of this note. It states that, for the NCS with packet-dropouts (and provided the assumptions hold), plant state trajectories will remain inside a ball of size

$$
\beta(|x(k_0)|, k-k_0) + \gamma(|\mathbb{W}|).
$$

[5]This follows directly from the inequality $\alpha(a+b) \leq \alpha(2a) + \alpha(2b)$, $\forall a, b \geq 0$, used in, e.g., [13, Eq.(12)].

[6]Note that the gains obtained are conservative and that $\beta \in \mathcal{KL}$ is not strictly decreasing in $k - k_0$. A bounding class-$\mathcal{KL}$ function with strict decrease in the second argument can be constructed from β given in (35) by using Lemma 1 in [28].

Our result shows that if the number of consecutive packet-dropouts is bounded, then the design parameters of the cost function in (8) can be chosen following established MPC ideas and are such that ISS is ensured.

Remark 4: It is worth noting that the conditions for stability with packet-dropouts obtained are only slightly more restrictive than those established for non-networked MPC in earlier works. The only additional requirements needed in the networked case are the requirement on the horizon N (see Assumption 4) and the fact that the feasible set $\mathbb{X}_N$ needs to be robustly invariant "over more than one step", see Remark 3. □

Remark 5: It would be useful to remove the restriction on the packet-dropouts made in Assumption 4. Unfortunately, since during periods of packet-dropouts the plant is operated in open-loop, one cannot ensure ISS for unbounded m_i in a general case.[7] One exception corresponds to situations, where there exists $\kappa_f(\cdot)$, which complies with Assumption 1 and can be computed at the plant input side *without explicit knowledge of the current plant state*. Here one can conceive a scheme, where, if $k_{i+1} > k_i + N$, then the values $\kappa_f(x(k_i + N + \ell))$, $\ell \geq 0$, are implemented at the plant input. Stability can then be ensured even for unbounded m_i. In particular, if $f(\cdot, 0, 0)$ has a stable equilibrium point at the origin and $\mathbb{X}_f$ lies within its basin of attraction, then one can simply set $\kappa_f(\xi) = 0$, $\forall \xi \in \mathbb{X}_f$. The latter situation arises, for example, in quantized control of stable LTI systems, see [30]. □

The disturbance-free case, which was to some extent examined in our earlier work [9], can be treated as a direct consequence of Theorem 1 as follows:

Corollary 1 ($\mathcal{KL}$-stability): Suppose that $\mathbb{W} = \{0\}$, so that $|\mathbb{W}| = 0$ and $w(k) = 0$, $\forall k \in \mathbb{N}_0$, and that the assumptions in Theorem 1 hold. Then there exists $\beta \in \mathcal{KL}$ which bounds the plant states via

$$
|x(k)| \leq \beta(|x(k_0)|, k-k_0), \quad \forall k \geq k_0.
$$

IV. Conclusions

This note has studied a NCS architecture where a packetized predictive controller uses an unreliable network affected by packet-dropouts to control a constrained nonlinear plant. It has been shown that, provided plant disturbances belong to a bounded set and the number of consecutive packet-dropouts is bounded, input-to-state stability can be ensured by appropriate choice of tuning parameters. Future work could include the study of more general NCS's where not only plant inputs, but also output measurements are sent over an unreliable network.

References

[1] J. Baillieul and P. Antsaklis, "Control and communication challenges in networked real-time systems," *Proc. IEEE*, vol. 95, no. 1, pp. 9–27, Jan. 2007.

[2] T. Javidi, M. Franceschetti, P. R. Kumar, S. Mitter, and D. Teneketzis, "Guest editorial control and communications," *IEEE J. Select. Areas Commun.*, vol. 26, no. 4, pp. 577–579, May 2008.

[3] A. Bemporad, "Predictive control of teleoperated constrained systems with unbounded communication delays," in *Proc. IEEE Conf. Decis. Contr.*, Tampa, Florida, 1998, pp. 2133–2138.

[4] A. Casavola, E. Mosca, and M. Papini, "Predictive teleoperation of constrained dynamic systems via internet-like channels," *IEEE Trans. Contr. Syst. Technol.*, vol. 14, no. 4, pp. 681–694, July 2006.

[5] P. L. Tang and C. W. de Silva, "Compensation for transmission delays in an Ethernet-based control network using variable-horizon predictive control," *IEEE Trans. Contr. Syst. Technol.*, vol. 14, no. 4, pp. 707–718, July 2006.

[7]In some situations, it is possible to design the system to ensure *stochastic* stability even if m_i is unbounded; see our recent work [29].

[6] ——, "Stability validation of a constrained model predictive networked control system with future input buffering," *Int. J. Contr.*, vol. 80, no. 12, pp. 1954–1970, Dec. 2007.

[7] G. P. Liu, J. X. Mu, D. Rees, and S. C. Chai, "Design and stability analysis of networked control systems with random communication time delay using the modified MPC," *Int. J. Contr.*, vol. 79, no. 4, pp. 288–297, Apr. 2006.

[8] Y.-B. Zhao, G. P. Liu, and D. Rees, "Improved predictive control approach to networked control systems," *IET Control Theory Appl.*, vol. 2, no. 8, pp. 675–681, Aug. 2008.

[9] D. E. Quevedo, E. I. Silva, and G. C. Goodwin, "Packetized predictive control over erasure channels," in *Proc. Amer. Contr. Conf.*, New York, N.Y., 2007.

[10] ——, "Control over unreliable networks affected by packet erasures and variable transmission delays," *IEEE J. Select. Areas Commun.*, vol. 26, no. 4, pp. 672–685, May 2008.

[11] R. Findeisen and P. Varutti, "Stabilizing nonlinear predictive control over nondeterministic networks," in *Nonlinear Model Predictive Control: Towards New Challenging Applications*, ser. LNCIS, L. Magni, D. M. Raimondo, and F. Allgöwer, Eds. Berlin Heidelberg: Springer-Verlag, 2009, vol. 384, pp. 167–179.

[12] G. Pin and T. Parisini, "Stabilization of networked control systems by nonlinear model predictive control: A set invariance approach," in *Nonlinear Model Predictive Control: Towards New Challenging Applications*, ser. LNCIS, L. Magni, D. M. Raimondo, and F. Allgöwer, Eds. Berlin Heidelberg: Springer-Verlag, 2009, vol. 384, pp. 195–204.

[13] E. D. Sontag, "Smooth stabilization implies coprime stabilization," *IEEE Trans. Automat. Contr.*, vol. 34, no. 4, pp. 435–443, Apr. 1989.

[14] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, June 2001.

[15] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*. Madison, WI: Nob Hill Publishing, 2009.

[16] O. C. Imer, S. Yüksel, and T. Başar, "Optimal control of LTI systems over unreliable communication links," *Automatica*, vol. 42, no. 9, pp. 1429–1439, Sept. 2006.

[17] J. Wu and T. Chen, "Design of networked control systems with packet dropouts," *IEEE Trans. Automat. Contr.*, vol. 52, no. 7, pp. 1314–1319, July 2007.

[18] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Optimality and stability," *Automatica*, vol. 36, no. 6, pp. 789–814, June 2000.

[19] D. M. Raimondo, D. Limón, M. Lazar, L. Magni, and E. F. Camacho, "Min-max model predictive control of nonlinear systems: A unifying overview on stability," *European J. Contr.*, vol. 15, no. 1, pp. 5–21, 2009.

[20] L. Magni, D. M. Raimondo, and R. Scattolini, "Regional input-to-state stability for nonlinear model predictive control," *IEEE Trans. Automat. Contr.*, vol. 51, no. 9, pp. 1548–1553, Sept. 2006.

[21] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, "Examples when nonlinear model predictive control is nonrobust," *Automatica*, vol. 40, no. 10, pp. 1729–1738, Oct. 2004.

[22] D. Limón, T. Álamo, and E. F. Camacho, "Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive disturbances," in *Proc. IEEE Conf. Decis. Contr.*, Las Vegas, Nevada, 2002, pp. 4619–4624.

[23] M. Cannon, V. Deshmukh, and B. Kouvaritakis, "Nonlinear model predictive control with polytopic invariant sets," *Automatica*, vol. 39, no. 8, pp. 1487–1494, Aug. 2003.

[24] E. C. Kerrigan and J. M. Maciejowski, "Robust feasibility in model predictive control: necessary and sufficient conditions," in *Proc. IEEE Conf. Decis. Contr.*, vol. 1, Orlando, U.S.A., 2001, pp. 728–733.

[25] M. Cannon and B. Kouvaritakis, "Optimizing prediction dynamics for robust MPC," *IEEE Trans. Automat. Contr.*, vol. 50, no. 11, pp. 1892–1897, Nov. 2005.

[26] D. Q. Mayne, M. M. Serón, and S. V. Raković, "Robust model predictive control of constrained linear systems with bounded distrurbances," *Automatica*, vol. 41, no. 2, pp. 219–224, Feb. 2005.

[27] L. Chisci, J. A. Rossiter, and G. Zappa, "Systems with persistent disturbances: Predictive control with restricted constraints," *Automatica*, vol. 37, no. 7, pp. 1019–1028, July 2001.

[28] D. Nešić, A. R. Teel, and E. D. Sontag, "Formulas relating $\mathcal{KL}$-stability estimates of discrete-time and sampled-data nonlinear systems," *Syst. & Contr. Lett.*, vol. 38, no. 1, pp. 49–60, Sept. 1999.

[29] D. E. Quevedo, J. Østergaard, and D. Nešić, "Packetized predictive control of stochastic systems over bit-rate limited channels with packet loss," *IEEE Trans. Automat. Contr.*, 2010, to be published.

[30] D. E. Quevedo, G. C. Goodwin, and J. A. De Doná, "Finite constraint set receding horizon quadratic control," *Int. J. Robust Nonlin. Contr.*, vol. 14, no. 4, pp. 355–377, Mar. 2004.