Appendix A

Solution of the integral in Section 4.3

We were not able to find a full analytical solution of the integral in (4.42) and (4.51) in standard references [Zwi96, Bur73, Jef94]. Therefore, we attempt to obtain the solution of the integral here.

The integral can be divided into two parts:

$$\begin{aligned} L &= \int_{-\omega_c}^{\omega_c} \frac{\omega_i^2 - \omega^2}{(\omega_i^2 - \omega^2)^2 + 4\zeta_i^2\omega_i^2\omega^2} d\omega \\ &= \omega_i^2 L_1 - L_2 \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} L_1 &= 2\int_0^{\omega_c} \frac{1}{(\omega_i^2 - \omega^2)^2 + 4\zeta_i^2\omega_i^2\omega^2} d\omega \\ L_2 &= 2\int_0^{\omega_c} \frac{\omega^2}{(\omega_i^2 - \omega^2)^2 + 4\zeta_i^2\omega_i^2\omega^2} d\omega. \end{aligned} \tag{A.2}$$

Since both integrands are even functions of ω, it is sufficient to integrate them only from 0 to ω_c as in (A.2).

The denominator can be written as:

$$\begin{aligned} (\omega_i^2 - \omega^2)^2 + 4\zeta_i^2\omega_i^2\omega^2 &= \omega^4 + 2\omega_i^2(2\zeta_i^2 - 1)\omega^2 + \omega_i^4 \\ &= (\omega^2 - \omega_{r1}^2)(\omega^2 - \omega_{r2}^2) \end{aligned} \tag{A.3}$$

where

$$\begin{aligned}\omega_{r1}^2 &= \omega_i^2\left((1-2\zeta_i^2)-2\zeta_i\sqrt{1-\zeta_i^2}\,j\right)\\ \omega_{r2}^2 &= \omega_i^2\left((1-2\zeta_i^2)+2\zeta_i\sqrt{1-\zeta_i^2}\,j\right)\end{aligned} \tag{A.4}$$

and j is $\sqrt{-1}$.

We consider only the under damped case ($\zeta_i < 1$) since the case captures the majority of resonant systems of interest. However, the solution for critically damped and over damped cases can also be obtained in a more straightforward manner since ω_{r1}^2 and ω_{r2}^2 will be real numbers.

A.1 First integral, L_1

The integrand of L_1 can be written as partial fractions:

$$\frac{1}{(\omega_i^2-\omega^2)^2+4\zeta_i^2\omega_i^2\omega^2} = \frac{1}{\Omega_r}\left(\frac{1}{\omega^2-\omega_{r1}^2}-\frac{1}{\omega^2-\omega_{r2}^2}\right) \tag{A.5}$$

where

$$\begin{aligned}\Omega_r &= \omega_{r1}^2-\omega_{r2}^2\\ &= -4\zeta_i\sqrt{1-\zeta_i^2}\omega_i^2 j.\end{aligned} \tag{A.6}$$

Consider the following indefinite integral with a complex constant a. The solution to the indefinite integral is [Spi81]

$$\int \frac{dz}{z^2-a^2} = \frac{1}{2a}\ln\left(\frac{z-a}{z+a}\right)+c \tag{A.7}$$

where c is a constant. The integral L_1 (A.2) can be solved by incorporating (A.5) and (A.7) to give

$$L_1 = L_1^{\omega_c}-L_1^0 \tag{A.8}$$

where

$$L_1^{\omega} = \frac{1}{\Omega_r}\left\{\frac{1}{\omega_{r1}}\ln\left(\frac{\omega-\omega_{r1}}{\omega+\omega_{r1}}\right)-\frac{1}{\omega_{r2}}\ln\left(\frac{\omega-\omega_{r2}}{\omega+\omega_{r2}}\right)\right\}. \tag{A.9}$$

Define:

$$\cos\alpha \;=\; 1 - 2\zeta_i^2. \tag{A.10}$$

Consequently,

$$\sin\alpha \;=\; 2\zeta_i\sqrt{1-\zeta_i^2}. \tag{A.11}$$

From (A.10) and (A.11), the expressions in (A.4) and (A.6) can be re-written as:

$$\begin{aligned}
\omega_{r1} &= \omega_i \mathrm{e}^{-j\frac{\alpha}{2}} \\
\omega_{r2} &= \omega_i \mathrm{e}^{j\frac{\alpha}{2}} \\
\Omega_r &= -2\sin\alpha\,\omega_i^2 j.
\end{aligned} \tag{A.12}$$

Using the previous expressions, we obtain

$$\begin{aligned}
\omega - \omega_{r1} &= r_a \mathrm{e}^{j\theta_a} \\
\omega + \omega_{r1} &= r_b \mathrm{e}^{-j\theta_b} \\
\omega - \omega_{r2} &= r_a \mathrm{e}^{-j\theta_a} \\
\omega + \omega_{r2} &= r_b \mathrm{e}^{j\theta_b}
\end{aligned} \tag{A.13}$$

where

$$\begin{aligned}
r_a &= \sqrt{\omega^2 - 2\omega\omega_i\cos\frac{\alpha}{2} + \omega_i^2} \\
r_b &= \sqrt{\omega^2 + 2\omega\omega_i\cos\frac{\alpha}{2} + \omega_i^2} \\
\theta_a &= \cot^{-1}\left(\frac{\omega - \omega_i\cos\frac{\alpha}{2}}{\omega_i\sin\frac{\alpha}{2}}\right) \\
\theta_b &= \cot^{-1}\left(\frac{\omega + \omega_i\cos\frac{\alpha}{2}}{\omega_i\sin\frac{\alpha}{2}}\right)
\end{aligned} \tag{A.14}$$

and $\cot^{-1}(\Gamma)$ denotes the inverse cotangent of Γ.

The following expressions can be obtained from (A.13):

$$\begin{aligned}
\ln\left(\frac{\omega - \omega_{r1}}{\omega + \omega_{r1}}\right) &= \ln\left(\frac{r_a}{r_b}\right) + j(\theta_a + \theta_b) \\
\ln\left(\frac{\omega - \omega_{r2}}{\omega + \omega_{r2}}\right) &= \ln\left(\frac{r_a}{r_b}\right) - j(\theta_a + \theta_b).
\end{aligned} \tag{A.15}$$

After some algebraic manipulation, it can be shown that L_1^ω (A.9) is real valued. This is as expected since the optimal feedthrough term will be real valued:

$$L_1^\omega = \frac{-1}{\omega_i^3 \sin\alpha}\left\{\sin\frac{\alpha}{2}\ln\left(\frac{r_a}{r_b}\right) + \cos\frac{\alpha}{2}(\theta_a+\theta_b)\right\}. \tag{A.16}$$

The next task is to change the variables to the original variables. For this case, $\theta_a + \theta_b$ can be found from trigonometric identities [Zwi96]:

$$\begin{aligned} \cot(\theta_a+\theta_b) &= \frac{\cot\theta_a\cot\theta_b - 1}{\cot\theta_a + \cot\theta_b} \\ &= \frac{\omega^2-\omega_i^2}{2\omega\omega_i\sin\frac{\alpha}{2}}. \end{aligned} \tag{A.17}$$

Writing $\ln(r_a/r_b)$ as $-0.5\ln(r_b^2/r_a^2)$ and using (A.17), the indefinite integral (A.9) becomes

$$L_1^\omega = \frac{1}{2\omega_i^3\sin\alpha}\left\{\sin\frac{\alpha}{2}\ln\left(\frac{r_b^2}{r_a^2}\right) - 2\cos\frac{\alpha}{2}\cot^{-1}\left(\frac{\omega^2-\omega_i^2}{2\omega\omega_i\sin\frac{\alpha}{2}}\right)\right\}. \tag{A.18}$$

Now, L_1 (A.8) can be evaluated from (A.18) by substituting ω with ω_c and 0 respectively:

$$\begin{aligned} L_1 = \frac{1}{2\omega_i^3\sin\alpha}\Bigg\{&\sin\frac{\alpha}{2}\ln\left(\frac{\omega_c^2+2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}{\omega_c^2-2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}\right) \\ &- 2\cos\frac{\alpha}{2}\cot^{-1}\left(\frac{\omega_c^2-\omega_i^2}{2\omega_c\omega_i\sin\frac{\alpha}{2}}\right) + 2\pi\cos\frac{\alpha}{2}\Bigg\}. \end{aligned} \tag{A.19}$$

By considering the trigonometric identities $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = 1 - 2\sin^2\frac{\alpha}{2}$ [Bur73] and (A.10):

$$\begin{aligned} \cos\frac{\alpha}{2} &= \sqrt{1-\zeta_i^2} \\ \sin\frac{\alpha}{2} &= \zeta_i. \end{aligned} \tag{A.20}$$

A.2 Second integral, L_2

We consider the second integral L_2 (A.2). The integrand can again be written as partial fractions:

$$\frac{\omega^2}{(\omega_i^2-\omega^2)^2+4\zeta_i^2\omega_i^2\omega^2} = \frac{1}{\Omega_r}\left(\frac{\omega_{r1}^2}{\omega^2-\omega_{r1}^2} - \frac{\omega_{r2}^2}{\omega^2-\omega_{r2}^2}\right). \tag{A.21}$$

Using (A.7), the integral is

$$L_2 \;=\; L_2^{\omega_c} - L_2^0 \tag{A.22}$$

where

$$L_2^{\omega} \;=\; \frac{1}{\Omega_r}\left\{\omega_{r1}\ln\left(\frac{\omega-\omega_{r1}}{\omega+\omega_{r1}}\right) - \omega_{r2}\ln\left(\frac{\omega-\omega_{r2}}{\omega+\omega_{r2}}\right)\right\}. \tag{A.23}$$

L_2^{ω} (A.23) can be shown, after some algebraic simplification, to be real valued as expected:

$$\begin{aligned} L_2^{\omega} &= \frac{-1}{\omega_i\,\sin\alpha}\left\{-\sin\frac{\alpha}{2}\ln\left(\frac{r_a}{r_b}\right) + \cos\frac{\alpha}{2}(\theta_a+\theta_b)\right\} \\ &= \frac{1}{2\omega_i\,\sin\alpha}\left\{-\sin\frac{\alpha}{2}\ln\left(\frac{r_b^2}{r_a^2}\right) - 2\cos\frac{\alpha}{2}\cot^{-1}\left(\frac{\omega^2-\omega_i^2}{2\omega\omega_i\sin\frac{\alpha}{2}}\right)\right\}. \end{aligned} \tag{A.24}$$

Then (A.22) can be evaluated using (A.24) after substitution of ω with ω_c and 0 respectively:

$$\begin{aligned} L_2 \;=\; & \frac{1}{2\omega_i\,\sin\alpha}\left\{-\sin\frac{\alpha}{2}\ln\left(\frac{\omega^2+2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}{\omega_c^2-2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}\right)\right. \\ & \left. -\,2\cos\frac{\alpha}{2}\cot^{-1}\left(\frac{\omega_c^2-\omega_i^2}{2\omega_c\omega_i\sin\frac{\alpha}{2}}\right) + 2\pi\cos\frac{\alpha}{2}\right\}. \end{aligned} \tag{A.25}$$

The integral L (A.1) can be solved using (A.19) and (A.25) as follows:

$$\begin{aligned} L &= \frac{1}{\omega_i\,\sin\alpha}\sin\frac{\alpha}{2}\ln\left(\frac{\omega_c^2+2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}{\omega_c^2-2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}\right) \\ &= \frac{1}{2\omega_i\,\cos\frac{\alpha}{2}}\ln\left(\frac{\omega_c^2+2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}{\omega_c^2-2\omega_c\omega_i\cos\frac{\alpha}{2}+\omega_i^2}\right) \end{aligned} \tag{A.26}$$

where $\cos\frac{\alpha}{2} = \sqrt{1-\zeta_i^2}$ (A.20).

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Notations

a	dimension of a plate; coefficient in Rayleigh-Ritz solution
A	cross sectional area; linear transformation matrix; state space matrix
A_s	enclosed area of a hollow section
b	dimension of a plate; coefficient in assumed-modes solution
B	state space matrix
$\mathcal{B}$	differential operator: boundary conditions
c^E	elastic coefficient matrix
C	state space matrix; capacitance; parameter related to sensor outputs
C^v	parameter related to sensor outputs
C_w	FE spatial displacement vector
$\mathcal{C}$	differential operator: damping
d	dimension of a square; controller damping ratio
d_{31}, d_{32}	piezoelectric charge constant
D	flexural rigidity of a plate; electric displacement vector; feedthrough (state space) matrix
$\hat{D}$	proportional damping matrix
e	dielectric permittivity matrix
E	electric field vector; Young's modulus; error system
F	nodal force vector
f	distributed force/torque; parameter related to modal controllability/observability
g_{31}, g_{32}	piezoelectric voltage constant
G	Shear modulus
G, G_r	transfer function
h	thickness; length of an element
H	step function; transfer function gain
$\bar{H}$	Hermite cubic polynomial
I	second moment of area; unit matrix; number of modes; number of transducers

j	imaginary number
J	cost function; number of transducers
J_p	polar moment of inertia
J_r, J_2, J_∞	cost function
J_t	torsional parameter
k	proportional term in a feedthrough term; number of modes in an FE model
k_{31}, k_{32}	electromechanical coupling factor
K	stiffness matrix; constant related to piezoelectric actuator moment; feedthrough term; controller transfer function
$\hat{K}$	global stiffness matrix
$\mathcal{K}$	modal observability
L	length; integral; Lagrangian
L_o	observability Gramian matrix
$\mathcal{L}$	differential operator: stiffness
m	distributed mass
M	bending moment; number of modes
$\hat{M}$	global mass matrix
$\mathcal{M}$	differential operator: mass; modal controllability
N	number of admissible functions; number of elements in an FE model; number of modes
N_c	number of controlled modes
p	perimeter of a section; pressure
P	transfer function gain; solution of Lyapunov inequality
q	generalized coordinate; electric charge
Q	shear force; generalized force; spatial weighting function
r	point coordinate; radius of a circle
R	control weight; controller feedthrough term
$\mathbf{R}$	set of real numbers
S	strain vector; controllability Gramian matrix
$\mathcal{S}_c$	spatial controllability
$\mathcal{S}_o$	spatial observability
$\mathcal{R}$	Rayleigh's quotient; spatial domain
T	tension; transfer function
$\mathcal{T}$	kinetic energy
$\mathcal{T}^*$	reference kinetic energy
t	time
u	displacement; system input
v	displacement; sensor signal
V	transducer voltage
$\mathcal{V}$	potential/strain energy
$\mathcal{V}_{max}$	maximum potential energy

w	displacement; disturbance; low-pass filter
W	width; weighting function
W_r	weighting function
$\delta\overline{W}$	virtual work
x	point coordinate; state vector
y	point coordinate; system output
z	point coordinate; system output
α	general actuator parameter; strain gradient; controller modal gain; transfer function gain; parameter related to modal controllability/observability
α^s	dielectric matrix at constant mechanical strain
β	parameter related to spatial controllability/observability
γ	shear strain; upper bound of $\mathcal{H}_\infty$ norm
Γ	state space matrix
δ	Dirac/Kronecker delta function
ϵ	longitudinal strain
ζ	damping ratio
θ	angular displacement
Θ	nodal force parameter; state space matrix
κ	parameter related to a strain gradient
λ	eigenvalue
Λ	eigenvalue matrix
ν	Poisson's ratio
Ξ	transfer function gain
Π	state space matrix
ρ	density
σ	stress vector; longitudinal stress
τ	shear stress
Υ	transfer function parameter
ϕ	admissible (trial) function; eigenfunction; eigenvector
Φ	eigenvector matrix
Ψ	parameter of piezoelectric transducers
ω	frequency; resonance frequency
ω_c, ω_{co}	cut-off frequency
Ω	piezoelectric sensor parameter

subscripts

a	actuator
b	beam
e	elemental (local)
i	beam mode number
m, n	plate mode numbers
n	n^{th} element
p	piezoelectric patch
s	sensor

acronyms

FE	finite element
LTI	linear time-invariant
ODE	ordinary differential equation
MIIO	multiple-input, infinite-output
MIMO	multiple-input, multiple-output
PDE	partial differential equation
SISO	single-input, single-output
$\mathbf{Re}(F)$	real part of F
$\mathrm{tr}\{M\}$	trace of the matrix M
M^*	conjugate transpose of the matrix M
M^T	transpose of the matrix M
$\lambda_{max}(M)$	maximum eigenvalue of the matrix M

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