MATHEMATICAL THINKING AND MATHEMATICS ACHIEVEMENT OF STUDENTS IN THE YEAR 11 SCIENTIFIC STREAM IN JORDAN

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DECLARATION

I herby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution

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Abstract

Mathematical Thinking and Mathematics Achievement of Students in the Year 11 Scientific Stream in Jordan

The first aim of this study was to identify important aspects of mathematical thinking, and to investigate the relationships between the different aspects of mathematical thinking and mathematics achievement. The second aim was to examine possible gender and school location (urban, suburban, and rural) differences related to aspects of mathematical thinking and mathematics achievement.

Two assessments were developed that were suitable for students in the Year 11 scientific stream in Jordan. One test was for aspects of mathematical thinking and the other for mathematics achievement, the latter being consistent with typical school achievement tests for these students in Jordan. The researcher chose and developed items to test mathematical thinking and mathematics achievement from the Third International Mathematics and Science Study (TIMSS), the internet, research literature, specialist books in mathematics and his own experience.

The data were collected in the 2003-2004 academic year from over 500 Year 11 scientific stream students (both male and female) at 20 randomly selected schools from six directorates in the Irbid Governorate, Jordan. In addition, 13 teachers were individually interviewed, and four groups of students were interviewed in focus groups to obtain information about their opinions and about different methods of thinking in mathematics.

The teacher interviews were used to identify consistencies and inconsistencies between the test results and the respondents' opinions of difficulty and importance. In addition, information was obtained about the classroom time teachers devoted to

the different aspects of mathematical thinking and the teaching strategies they employed.

Six aspects of mathematical thinking were identified by the study: Generalization, Induction, Deduction, Use of Symbols, Logical thinking and Mathematical proof. Mathematical proof was also the most difficult aspect, while Logical thinking was the least difficult. Female students had significantly higher mean scores than males on three of the six aspects of mathematical thinking and on the total test scores. Students attending suburban schools had significantly higher mean scores than students at urban and rural schools on four aspects, and on the total scores. Using multiple regression analysis, all six aspects were found to be important for mathematics achievement. Mathematical proof and Generalization were the most important aspects, Use of symbols and Logical thinking were next in importance, and Deduction and Induction were the least important aspects. Approximately 70 per cent of the variance in mathematics achievement was explained by the six aspects of mathematical thinking, gender, and school location.

There was a high level of consistency between teacher opinions of the relative importance of aspects of mathematical thinking and the test results. However, there were some inconsistencies between the teacher opinions and test results with respect to relative difficulty levels of the six aspects.

By clarifying the importance for mathematics achievement of the six aspects of mathematical thinking identified, this study has relevance for the teaching of mathematics to Year 11, scientific stream students in Jordan.

CHAPTER ONE

INTRODUCTION

1.1 MATHEMATICAL THINKING

It is in the nature of a person or student to think, and thinking promoted across a range of disciplines is practised in our schools through education. Therefore, day by day, a sense of the importance of thinking in the educational field generally and more particularly in specific disciplines has grown among educators. Mathematics is an important branch of cognition and the development of mathematical thinking is a fundamental pillar in the orientation of educational development within a new, advanced educational system.

Thinking is an extremely complex process, which is little understood. Bruner (1960) distinguishes between two types of complementary thinking; intuitive thinking and analytic thinking. Intuitive thinking tends to include “maneuvers based seemingly on an implicit perception of the total problem” (p.58) and does not include any careful planning. In contrast, analytic thinking may include careful and deductive reasoning, “often using mathematics or logic and an explicit plan of attack. It may involve a step by step process of induction and experiment, utilizing principles of research design and statistical analysis” (pp.57-58). Mathematical thinking, while mainly utilising analytic thinking, also involves intuitive thinking.

Learning how to think mathematically is an extremely important issue in mathematics education. According to Petocz and Petocz (1994) there are two types of mathematical thinking, inductive and deductive. Inductive thinking involves the search for patterns which according to the National Council of Teachers of Mathematics (NCTM, 1971) is a way of thinking that enables you to arrive at generalizations. This method involves the observation of individual cases, and finding a pattern among these cases, then conjecturing that the pattern will be the same for all similar cases. In contrast, deductive thinking involves the proof of results (Petocz & Petocz, 1994). Similarly, Huetinck and Munshin (2004) stated that there are two different kinds of mathematical thinking, inductive and deductive reasoning. Inductive thinking is observing a number of cases, and then making conjectures. However, deductive reasoning "is a process that starts with statements that are considered true and shows that other statements logically follow from them" (p.28).

The intuitive thinking referred to by Bruner (1960) and mathematical inductive thinking are related to each other as both include implicit perception of the total problem followed by generalizations. The generalizations then need to be checked as to whether they are correct by using analytical or deductive thinking to prove them. When drawing conclusions based on inductive thinking or thinking by analogy, the results should be verified by observation, experiment, consultation of authority, or by deductive thinking. In contrast, a generalization is harder to verify. A generalization may hold true time and time again and may seem convincing. However to be sure, one needs to know why the generalization is true. Deductive thinking can be utilised to achieve this and begins with accepted statements which are known to be true and then a conclusion is formed that is based on the accepted statements. These two methods (induction and deduction) are totally different, complementary and comprise the power of mathematics (Huetinck & Munshin, 2004; NCTM, 1971).

These two types of mathematical thinking, inductive and deductive, are related to two of the aspects of mathematical thinking that are examined as part of this study. Inductive thinking is related to generalization as both of them involve a search for patterns from specific cases, to identifying a pattern and arriving at a general law. In addition, induction involves the two properties of necessitation and generalization. Smith (2002), and Poincaré (1902, 1905, as cited in Smith (2002, p.7)) defined induction in mathematics as an "instrument of transformation [which] contains, condensed, so to speak in a single formula, an infinite number of syllogisms". Dependent on this, generalization is considered a specific case of induction. That means, induction includes two steps, firstly searching for patterns from special cases which is the same as generalization and inductive thinking. Secondly, by proving these patterns to be true for any number, induction is also related to deductive thinking, because both induction and deductive thinking involve proving the patterns by certain kinds of proof, that is, “proof by mathematical induction” (Lucas, 2001; Polya, 1990).

Two examples of items from the mathematical thinking test are now used to distinguish between Generalization and Induction.

Example 1: Notice the two numbers on the right of the equals mark and their totals to its left in the following, and then discuss any generalizations that can be made.

6=3+3   8=5+3   10=5+5   12=5+7   14=7+7   14=3+11

16=11+5   16=13+3.

The purpose of this item is to search for pattern or generalization from specific cases as which is related to inductive thinking “any even number can be expressed as the sum of two odd prime numbers”. However, this example is not suited to induction, because induction must be true for all numbers after proving the generalization. For this example, no mathematician has ever been able to prove it is correct for all numbers, or correct up to a certain even number.

Example 2: Complete the last statement.

1=1

1+3=4

1+3+5=9

1+3+5+7=16

1+3+5+7+---+ (2n-1) =---------.

This example could be included as a test of either the generalization or induction aspects of mathematical thinking, because the conclusion here is “the summation of the first n odd numbers is equal to n^2 (this conclusion is considered to be an example of generalization), and after proving that conclusion is verified for all numbers by mathematical induction (this is considered as induction). Again, generalization is a specific case of induction.

However, deductive thinking is related to mathematical proof, because deductive thinking involves proving the patterns using mathematical induction, and one type of proof is proof by mathematical induction. In addition, induction is considered as the process of overgeneralization and overspecializations (Ben-zeev, 1996). These are the ways in which this study measured these two major aspects of mathematical thinking.

Most able students are more likely to think algebraically and to be proficient in the use of symbols as a means of communicating mathematically. Thinking abstractly is a necessary precursor to all other aspects of mathematical thinking that are considered in this study. The six scales of mathematical thinking that are examined in this study are: Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof. They will be discussed in more detail later in this thesis.

This study is premised on the importance of mathematical thinking for mathematics achievement. If this is accepted, it is important that mathematical thinking be developed through mathematics teaching. Various approaches to developing mathematical thinking have been suggested, including that of Mason, Burton, and Stacey (1991) who based their approach on five important assumptions:-

1) You can think mathematically.
2) Mathematical thinking can be improved by tackling questions and practice with reflection.
3) Mathematical thinking can be provoked by surprise, tension and contradiction.
4) Mathematical thinking can be supported by an atmosphere of questioning, challenge and reflection.
5) Sustaining mathematical thinking helps in increasing our understanding of the world. (pp.146-159).

Ben-Zeev (1996) referred to a specific type of inductive thinking as analogical thinking, and Butler, Wren and Banks (1970) and Dreyfus and Eisenberg (1996) as thinking by analogy. This is considered as an absolutely important aspect in developing mathematical thinking. In different ways, Howard and Sonia (2002) stated that the development of mathematical thinking by practical mathematical modelling and learning metacognitively was more effective than thinking by analogy.

These researchers have used different approaches to develop the mathematical thinking in their students (Butler et al, 1970; Dreyfus & Eisenberg, 1996; Mason et al, 1991). Mason et al. (1991) assumed every student could think mathematically and develop their mathematical thinking through the practice of thinking and challenging, such that the student could understand the environment and the world. Similarly, Butler et al (1970); Dreyfus and Eisenberg (1996) were concerned with the development of mathematical thinking through thinking by analogy, which leads to an understanding of the environment. However, this study aims to find the possible relationships between the different aspects of mathematical thinking and mathematics achievement. In addition, it is hypothesised that if the students have developed mathematical thinking, then they will have a high performance in mathematics achievement.

1.2 THE CONCEPT OF MATHEMATICAL THINKING AND THE JORDANIAN MATHEMATICS CURRICULUM

As discussed in Section 1.1, mathematical thinking is fundamental to mathematics. One hypothesis is that if we intend to achieve a high level of mathematics achievement on the part of our students, they must have a high level of mathematical thinking. Although there are two broad kinds of mathematical thinking, as mentioned in Section 1.1, there is no consensus on the definition of what mathematical thinking is. For example, Mason et al. (1991) defined mathematical thinking generally as “A dynamic process which, by enabling us to increase the complexity of ideas we can handle, expands our understanding” (p.158). More specifically, there are many aspects and features of mathematical thinking and Schielack, Chancellor, and Childs (2000) considered these aspects as Symbolism, Logical analysis, Inference, Optimizations, and Abstraction. Moreover, it was shown by Krutetskii (1976) that able and less able students used different ways of mathematical thinking in their problem solving. Able students outperformed the less able students in using generalization in their mathematical problems.

As Bransford, Zech, Schwartz, Barron, and Vye (1996) point out, mathematical thinking is often limited for students in the middle schools to the dominance of mathematical calculation and following a formula. They assume that “different views of what counts as mathematical thinking can have strong effects on the length and quality of students ‘mathematical careers’” (p.203).

In Jordan, modern mathematical curricula are concerned with the development of the students’ mathematical thinking, and in the mathematical curricula for primary and secondary school levels, the following is mentioned:

1) Using mathematical thinking in scientific areas and general life.
2) Improving and developing mathematical curricula to develop students’ ability in both mathematical thinking and critical thinking, and use of this ability in understanding and solving problems.

These are the aims of the Ministry of Education (2000, p.31-62). Therefore, mathematical thinking is recognised as a central feature of schooling in Jordan, and mathematics curricula are designed to facilitate the development of mathematical thinking.

There are many possible aspects of mathematical thinking, however, in Jordan they have been confined to six fundamental forms based on the views of a group of mathematics education specialists in the Jordanian University, Yarmouk University, and the Ministry of Education, because of their appropriateness to secondary level students and their possibility of measurement (Shatnawi, 1982). According to these scholars the six aspects of mathematical thinking include: Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof.

Each of these mathematical thinking aspects is briefly described here[1].

1) Generalization:

Polya (1990, p.108) defined generalization as leading "from one observation to a remarkable general law. Many results were found by lucky generalizations in mathematics, physics, and natural sciences, and it may be useful in the solution of problems". Also, Mason et al. (1991) consider specializing and generalizing as two sides of the same coin, and defined the process of generalization as "moving from a few instances to making guesses about a wide class of cases" (p.8). Also, they considered "generalizations are the life-blood of mathematics. Whereas specific results may in themselves be useful, the characteristically mathematical result is the general one" (p.8). In addition, Stacey (1986) described generalization as the process whereby "general rules are discovered by articulating the patterns observed in many particular cases" (p.72). Generalization, therefore, can be considered as a central mathematical process that allows specific observations to be expanded and applied, as a rule, to all similar cases.

2) Induction:

Polya (1990) defined induction as "the process of discovering general laws by the observation and combination of particular instances. It is used in all science, even in mathematics. Mathematical induction is used in mathematics alone to prove theorems of certain kind" (p.114). Induction therefore, is the process of arriving at general laws from specific cases.

Mathematical induction has two distinct characteristics. It firstly involves inference and secondly this inference is a generalization from specific to general (generalization forms part of process of induction). This means that mathematical induction "is similar to both logical deduction and empirical induction" (Smith, 2002, p. 3). As earlier mentioned, induction involves two properties of necessitation and generalization; logical deduction has one

[1] There is the researcher's definition and an example of each aspect of mathematical thinking in Chapter Three, Section 3.1.

property of necessitation that means if the premises are true, then the conclusion must (necessarily) be true and vice versa. However, empirical induction has one property of generalization, but the generalization (conclusion) need not be necessarily true even if the premises are true. Mathematical induction is considered “a special case” (Smith, 2002, p. 3-5). Induction is a method of Mathematical proof that clearly establishes that a mathematical statement is true for all members of a set (which may be all natural numbers for example). Induction may be used to prove a generalization, but the process of arriving at the generalization in the first place is not induction, it is generalization.

3) Deduction:

Sainsbury (1991) defined deduction as when “valid arguments are necessarily truth-preserving” (p.15). Johnson-Laird (1999) later defined deduction in a similar way to Sainsbury as a process that “yields valid conclusion, which must be true given that their premises are true” (p.110). Deduction means to arrive at valid conclusion from truth premises. In addition, deductive thinking is required to prove the general laws (results) that were made by mathematical induction. It can also be thought of as the opposite process to generalization, as deductive thinking can be used to proceed from the general case to a particular case.

4) Use of Symbols.

A symbol may be a letter, relationship or abbreviation representing an expression, quantity, idea, concept or mathematical process. Expression through the symbols means the use of symbols to communicate mathematical ideas or verbal problems. The use of symbols allows for the process of mathematical generalization to be expressed in a concise way. Also, mathematical symbols can be manipulated using deductive thinking, allowing further general results to be determined.

5) Logical thinking:

Macdonald (1986) described Logical thinking as “the idea that there are certain basic rules of grammar with which we can organise our discussion in mathematics is what makes it possible to establish that certain things are “true” in mathematics, Also, logic “is the grammar that makes the conversation possible and holds it together” (p.337). Logical thinking is the ability to work clearly by justification: to work step by step, each step being justified through the previous steps.

6) Mathematical proof:

Milton and Reeves (2003) described mathematical proof as that which includes “the formation of a chain of ‘valid’ reasoning that leads to a conclusion. It is a process of ‘authentication’ or a process wherein the truth or fallacy of a claim is established” (p.384).

Proofs play a fundamental role in the practice of mathematics. Proofs make it possible to establish propositions as results. Understanding relations between propositions and concepts can be achieved through proofs. Proofs in mathematics, therefore, are similar to observations and experiments in science, because they provide evidence to back claims of knowledge. It is therefore crucial for students learning mathematics to appreciate the role of proof. (Schoenfeld, 1994, p 274).

There are three fundamental types of mathematical proofs; Direct Proof, Indirect Proof and Proof by Recursion (Mathematical Induction). “Mathematical proof is such a magnificent thing and nothing can be accepted as mathematically true without being rigorously proven” (Macdonald, 1986, p.359). In addition, according to Baker and Campbell (2004), Mathematical proof involves three main steps: reading the statement, developing an understanding of the problem and then beginning the construction of the mathematical proof.

Different classifications of mathematical thinking have been put forward and have been the topic of much research (Al-Hassan, 2001; Battista, 1990; Cox, 2000; Low & Over, 1993; Ma, 1995; Mills, Ablard, & Stumpf, 1993; Stites, Kennison, & Horton, 2004). The six aspects described above are perhaps the most common classifications used to describe mathematical thinking in Jordan. Although these six aspects of mathematical thinking can be distinguished from each other, as suggested above, there are also clear overlaps. However, the extent of overlaps between aspects of mathematical thinking does not preclude performance on each aspect from being assessed separately.

1.3 ACHIEVEMENT IN MATHEMATICS

Many studies have been conducted to measure the mathematical achievement of school students (for example, see Ai, 2002; Alkhateeb, 2001; Baya'a, 1990; Dennis, 1993; Hanna, 1986; Low & Over, 1993; Ma, 1995; Randhawa, 1988; Uekawa & Lange, 1998; Young, 1998). Mathematics achievement is the measure of overall performance across all mathematical abilities typically found in comprehensive school tests, as compared with mathematical thinking which measures the specific mathematical abilities such as geometrical abilities, ability to generalize, logical abilities, etc. It is hypothesised that mathematical thinking contributes to overall mathematics achievement.

Cox's (1994) study emphasises that student results on tests measuring components of mathematics (Algebra, Functions, Trigonometry, Differentiation, and Integration), generally correlated with overall scores on mathematics achievement tests. However, the results in mathematics here related to the dependent variable of mathematics achievement that could describe the total scores that represented definitions, concepts and skills, generalizations and theorems, and proofs acquired by high school students through their study of the mathematics curriculum. In addition, Cox (1994) and the current study tested the most able students in mathematics achievement, and they covered similar topics. Cox's (1994) study emphasises that student results on tests measuring components of mathematics, generally correlated with overall scores on mathematics achievement tests.

1.4 STUDY AIMS

This study examines relationships between mathematical thinking and mathematics achievement through the use of both quantitative and qualitative data. The quantitative data is concerned with studying the direct relationships between performance in mathematical thinking and mathematics achievement of 17 year old students in Jordan. The study also examines the differences between the scores of males and females in the various aspects of mathematical thinking and mathematics achievement, and investigates the differences in student performance between urban, suburban, and rural schools. The qualitative data are derived from teacher and student interviews and provide more individual perceptions of the relationships between mathematical thinking and mathematics achievement. The teachers' interviews include discussion of how they teach mathematical thinking and its aspects in their schools, as well as how much mathematics class time they spend teaching mathematical thinking. The students' interviews include discussion of the strategies they used to answer questions in the test of mathematical thinking, and how they reached their answers.

The students chosen for this study were selected from the Year 11 scientific stream in Jordan. Mathematics is a particularly important subject for the Year 11 scientific stream as these students need to have high achievement in mathematics. In many cases these students intend to follow scientific careers. It will be important if this study can identify which aspects of mathematical thinking are the most important contributors to high levels of mathematics achievement as measured by regular school achievement tests. Teachers can then be assisted to emphasise and teach these aspects of mathematical thinking in order to improve the achievement of their students.

1.5 EDUCATIONAL SYSTEM IN JORDAN

The Hashemite Kingdom of Jordan is located in the Middle East. It is a country with a population of 5.3 million (Personal email, Department of Statistics, Jan, 24, 2005) and an area of 89,000 km^2. It is divided into three regions, north, middle and south. Each region comprises four governorates. Most of the population is Muslim (approximately 98%), and the educational system is based on single–sex schooling. In order to place the study in context, this section will describe and discuss the national educational development plan of 1987, followed by stages of education in Jordan. In addition, future renewal projects that are intended to develop the educational system and national educational process projects are included in an Appendix 1.1 and 1.2.

1.5.1 Educational System Aims

The educational system in Jordan aims, in general, to achieve the four main following objectives:

1) Using mathematical thinking in scientific areas and general life.
2) Using Arabic language in communication.
3) Collecting data and then processing and applying results in all areas.
4) Using scientific methods in research and problem solving
(Ministry of Education, 2000, p.31).

The national educational development plan introduced in 1987 achieved many developments between the time of its inception in the 1987/1988 academic year until 1999/2000. The most important developments relate to the number of students, number of teachers, number of schools, proportion of students in vocational education (both males and females), number of schools for vocational learning, the proportion of students in private schools, the proportion of teachers with Bachelor degrees or higher education level, the average number of students per teacher, the average number of teachers per supervisor, the number of educational supervisors, and the proportion of the Ministry of Education budget to the overall budget (see Table 1.1 following).

The system grew substantially in overall numbers of students and teachers, but teacher/student ratios changed only slightly. However, the teachers' qualifications improved markedly in terms of the proportion of teachers with bachelor degrees or higher. The proportion of students in private schools almost doubled. For the academic year 1999/2000, there were four authorities providing education, the Ministry of Education (Public schools enrolled almost 70% of the student population), private schools approximately (19%), refugee schools approximately (10%), and other governments (1.5%). The Ministry of Education schools numbered 2823 (58.7%), significant number of private school numbered 1771 (36.8%), a small number of refugee schools 188 (4%), and other government schools, 26 (.5%). It should be noted that the highest class sizes are in refugee schools, whereas the lowest class sizes are at private schools.

TABLE 1.1. EDUCATIONAL SYSTEM DEVELOPMENT FROM 1987/1988 to1999/2000[2]

Developments	Academic year	
	1987/1988	1999/2000
Number of students	940,281	1,407,729
Number of teachers	39,445	63,587
Number of educational supervisors	504	924
Number of schools	3,478	4,808
Proportion of students in vocational learning: males females	 17% 13%	 45% 27%
Number of schools for vocational learning	152	308
Proportion of students in private schools to overall students	10.45%	18.9%
Proportion of teachers with Bachelor degree or above	29.6%	56.7%
Average number of students per teacher	24:1	22:1
Average number of teachers per supervisor	78:1	69:1
Proportion of Ministry of Education budget to overall budget	7.5%	10.4%

[2] This table was extracted from the Year Book, 1999-2000, Ministry of Education, Jordan p.18.

1.5.2 Educational Stages[3]

There are four stages of education in Jordan - kindergarten, primary schools, secondary schools, and informal education.

1) Kindergarten stage (Private stage): This is an optional stage and students can enter kindergarten at age 4, and attend from age 4-6. Kindergarten usually involves four hours of classes daily.

2) Primary schools[4]: This is a compulsory stage (age 6-16 years) and the student must enter the school at six years old, and spend 10 years in this stage.

[3]Extracted from the *YearBook*, 1999-2000, Ministry of Education, Jordan, p.44.

This stage is intended to achieve the general aims of education. Almost all public primary schools are single-sex throughout Jordan, however, sometimes in small areas these schools are coeducational up to 6th grade (age 6-12 years).

3) Secondary schools[5]: This stage comprises two years (first secondary (Year 11) and second secondary (Year 12)), and has two fields i). Comprehensive education and ii). Applied education. Comprehensive education contains two fields: academic and vocational. The academic field contains: a scientific stream, a humanities stream, and an Al-Shari stream which focuses on Islamic studies. The vocational field contains: Theoretical and practical treatment of industry, nursing, agriculture, commerce, and home administration. Applied education is concerned solely with vocational training. All secondary public schools are single-sex throughout Jordan (16-18 years).

All schools are regulated by the Ministry of Education. The Ministry of Education publishes and approves textbooks, and all schools are required to follow a national curriculum set by the Ministry. However, decisions about instructional methods and classroom processes are made by teachers and supervisors. All students follow a compulsory mathematics program until the end of Grade 12 (second secondary). The scientific stream includes more complex mathematics, with less advanced mathematics for the humanities stream and applied mathematics for the vocational stream.

4) Informal education: This includes elderly learning programs, evening studies, and home studies.

Based on the Ministry guidelines, this researcher believes it is essential for mathematics teachers across all stages to stimulate their students to discover principles and generalizations in order to solve problems, because this approach can contribute to higher levels of achievement in mathematics. In

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[4] Apply new learning starting for academic year 1989/1990 to change the primary education stage to 10 years instead of 9 years to increase students' achievement in terms of skills and knowledge.

[5] Apply new learning starting for academic year 1989/1990 to change the secondary education stage to 2 years instead of 3 years, because basic education (i.e. primary) is compulsory and perceived as a higher priority.

Jordan, students are not permitted to use calculators in mathematics, especially in primary schools to facilitate this aim. By the end of 12th Grade, all students take the National High School General Examination. This examination is government controlled, being prepared and administered by the Ministry of Education in all subjects taught at this level in high school. The examination is based on the Jordanian national curriculum. This examination is a requirement in applying for admission to universities and colleges inside or outside Jordan. Many students in 12th Grade take tutoring after school in some subjects, especially in mathematics, science and English. There is no difference in the curriculum of male and female schools. More details about the educational system in Jordan can be found in Appendix 1.

1.6 IRBID GOVERNORATE

Irbid, where this study was conducted is one of the four governorates in the northern region of the Hashemite Kingdom of Jordan. It is famous for agriculture and the population is nearly 1 million. The researcher chose this important area of Jordan, because it is the second largest governorate after the capital, and it is where the researcher has experience as a teacher. The number of schools throughout the Irbid governorate is 241(26%) male schools, 183 (20%) female schools, and 507(54%)[6] co-educational schools (from 1st grade to 6th grade), totalling 931 schools (19% of the total number of schools in Jordan). The number of students is 138,443 (51%) males and 133,822 (49%) females, totalling 272,265 students (19% of all students in the country). The number of teachers is 5,082 (40%) male teachers and 7,523 (60%) female teachers, a total of 12,605 teachers (20% of all teachers in Jordan). See Table 1.2.

[6] This information indicates that there are more co-educational schools than single-sex schools in the Irbid governorate. However, in Jordan small and rural schools are likely to be co-educational but only up to year 6.

TABLE 1.2. EDUCATIONAL STATISTICS FOR THE IRBID GOVERNORATE.

	Males	Females	Co-educational
Number of schools	241	183	507
Number of students	138,443	133,822	-----
Number of teachers	5,082	7,523	-----

1.7 STUDY DESIGN

This study links some aspects of student background, that is gender, location and school attended, with mathematical thinking and achievement in mathematics. It is best illustrated in diagrammatic form in Figure 1.1. The figure, based on Shatnawi's classification, indicates the study hypothesises that student gender and school location affect each the of six aspects of mathematical thinking and mathematics achievement. The affect on mathematics achievement may be either direct or indirect, acting through mathematical thinking. With respect to school location, it will be noted that suburban is the omitted category with both urban and rural schools being to compare with it. Finally, the figure also hypothesises that each of the aspects of mathematical thinking has an individual effect on mathematics achievement.

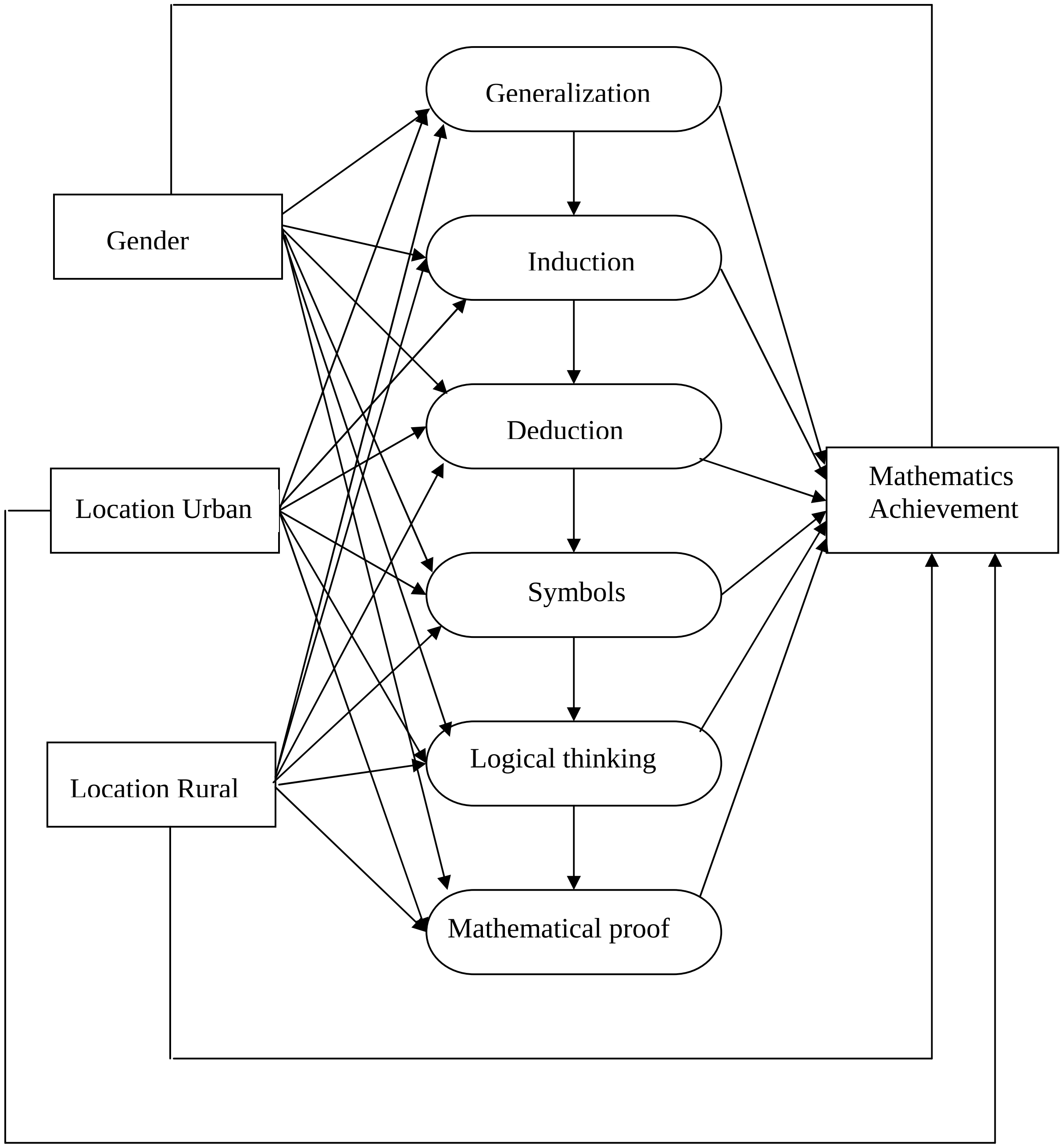

FIGURE 1.1. THE COMPLETE MODEL SHOWING POTENTIAL LINKS BETWEEN THE BACKGROUND VARIABLES AND MATHEMATICAL THINKING AND MATHEMATICS ACHIEVEMENT

1.8 RESEARCH QUESTIONS

This study aims to investigate the relationship between mathematical thinking and mathematics achievement, in order to identify which aspects of mathematical thinking are most important for mathematics achievement. This study also considered possible differences between male and female students in the six aspects of mathematical thinking and mathematics achievement, and investigates potential differences in mathematical thinking and mathematics achievement between students at urban, suburban, and rural schools.

In order to achieve these aims and the qualitative data aims the following research questions are proposed. This study will attempt to answer the following questions which are reported here.

1) What is the relationship between mathematical thinking and mathematics achievement for the Year 11 scientific stream in Jordan?

2) Do male and female students differ in mathematical thinking and mathematics achievement?

3) Is there an interaction effect of individual schools on mathematical thinking and mathematics achievement?

4) Do urban, suburban, and rural students differ in mathematical thinking and mathematics achievement?

5) Is there an interaction effect of gender and school location on mathematical thinking scales and mathematics achievement?

6) What are the significant effects on mathematics achievement, and what is the relative importance of these effects?

7) What the inconsistencies and consistencies between the teachers' opinions about the importance and difficulty of the aspects of mathematical thinking?

8) Are there differences in mathematical thinking for students of different ability and of different gender? Are the students familiar with solving specific problems (such as rice problem) in scientific ways like searching for patterns rather than

more classical methods? Are the students using the fourth step in problem solving according to Polya (1990) (i.e., a checking the answer)?

1.9 SUMMARY

This chapter included descriptions of the two major types of mathematical thinking in general, intuitive thinking (inductive thinking) and analytic thinking (deductive thinking). Subsequently, the concept of mathematical thinking was broken into several of its component aspects which were described individually, followed by a description of mathematics achievement. However, to distinguish between the two, mathematical thinking refers to the specific aspects that overall mathematics achievement depends upon. Mathematics achievement is largely what schools measure when assessing overall performance. The researcher then outlined the study aims and problem, and described the educational system in Jordan to provide a context for the study. Finally, the specific study area was outlined, the study design described and the hypotheses stated.

Following on from this introductory chapter, a literature review is presented in the next chapter. Chapter 2 shows the studies that have linked mathematical thinking with gender, and other studies that have linked mathematics achievement with gender and school location. The instruments and samples are presented and described in Chapter 3. Chapter 4, then includes the mathematical tests, their scoring, reliability, and validity. Results of the quantitative analyses are presented in Chapter 5 and the qualitative analyses in Chapter 6.

For the final two chapters, Chapter 7 provides a discussion of the results presented. Following this chapter of discussion, Chapter 8, provides conclusions, study recommendations and limitations, and suggestions for future studies.

CHAPTER TWO

REVIEW OF LITERATURE

The objective of this chapter is to provide a review of the scholarly literature focusing on mathematical thinking, in relation to geometry, algebra, and reasoning. The review will be concerned with the literature associated with the various aspects of mathematical thinking which will be used in this study. The various aspects are: Mathematical proof, Generalization, Use of Symbols, Logical thinking, Induction and Deduction. As many of the existing studies do not make specific mention of these six aspects, the literature will be grouped as follows. Mathematical proof will be considered with studies related to geometry, as many studies in this area involve geometrical proofs. Generalization and Use of Symbols will be considered with studies related to algebra, and finally, Logical thinking will be related to studies of mathematical reasoning. Gender differences for the six aspects of mathematical thinking will be discussed first in this chapter, which initially had gender as a focus for the review. However, specific studies of Deduction and Induction aspects with gender were not found and therefore, will not be considered in this chapter. The literature as it pertains to achievement levels in mathematics in relation to gender alone, geographical location alone, and both gender and location will then be reviewed.

2.1 GENDER AND MATHEMATICS LEARNING

Gender differences in mathematics learning continue to attract much attention from practitioners, administrators, school systems, government initiatives and researchers (Leder & Forgasz, 1992), and have been studied intensely for about 20 years (Ai, 2002; Alkhateeb, 2001; Baya'a; 1990; Dennis, 1993; Hanna, 1986; Low & Over 1993; Ma, 1995; Randhawa, 1988; Uekawa & Lange 1998; Young 1998). "In many countries, the incentive for such study has been recognition that the lack of mathematical learning and negative beliefs about themselves and mathematics hampers females from achieving equity with males" (Fennema, 1993, p.1).

2.2 STUDIES LINKING MATHEMATICAL THINKING AND GENDER

This section will examine the studies that have examined the link between mathematical thinking and gender. It will focus on the studies that have measured geometrical abilities, algebra and generalization, and reasoning abilities.

Mathematical thinking will be discussed in relation to:

1) Geometry related to Mathematical proof.
2) Algebra related to Use of Symbols and generalization.
3) Reasoning related to Logical thinking.

2.2.1 Geometry and Mathematical proof

2.2.1.1 Introduction

Throughout the history of education, learning proof writing is an important aim of geometry curricula for students. The practice of writing proofs is recognised as one of the most difficult processes for students to learn (Senk, 1985). Senk (1985) in the United States, showed that when students were tested on six geometry problems that involved proof, only 30 percent of the students attending a 12 month geometry course achieved a 75 percent mastery. The study undertaken for this thesis will examine students' mathematical thinking with regard to Mathematical proof. The items chosen generally relate mathematical thinking with regard to proof to the topic of geometry. Mathematics curricula are designed to assist students to increase their understanding of proof through the use of explanation and reasoning in their approach to solving mathematical problems (Callingham & Falle, 2004).

High school mathematics teaching should aim to encourage the development of understanding and knowledge of geometrical properties and the application of “conjecture, deductive reasoning and proof” in using these geometrical properties (Brown, Jones, Taylor, & Hirst, 2004). Given the importance of proof and its prevalence in the topic of geometry, it was chosen as one of the six types of mathematical thinking studied for this thesis.

2.2.1.2 Gender Differences in Performance in Geometry

This section will consider studies that have linked geometry and proof in relation to gender. Consideration will be given first to the extent to which studies suggest that mathematical thinking in geometry favours males. Attention will be given second to the remaining studies that found evidence of superior performance by females.

2.2.1.2.1 Evidence of Superior Male Performance in Geometry

Students’ ability to understand Mathematical proof has been studied for many years by a large number of researchers. With regard to gender differences in student understanding of geometrical concepts, many researchers have found significant distinctions between males and females. Hanna (1986), for example studied gender differences in the mathematics achievement of eighth graders in Ontario. The test items covered five broad topics: arithmetic, algebra, geometry, probability and statistics, and measurement. There were significant gender differences favouring males on two topics: geometry and measurement. Also, El -Hassan (2001) in Lebanon found that, at the 13th grade in operation and geometry topics, males performed better than females.

Battista (1990) conducted a study concerned with the spatial and geometrical thinking of students. High school geometry students were tested in four areas; spatial visualization, logical reasoning, geometrical knowledge, and geometrical problem solving. Battista (1990) found that for geometrical knowledge, males significantly outperformed females. In addition, Huntley (1990), in his study regarding gender differences in geometry items, tested students with 32 geometrical problems with two versions. One version provided a related diagram and one version did not provide a diagram. The results showed that for the overall test, males outperformed females either in problems that provided or did not provide a related diagram.

Senk and Usiskin (1983) studied the gender differences in the understanding of geometrical proof for senior high students ranging from 7th grade to 12th grade, with approximately two thirds of students in 10th grade. The students were tested on their knowledge of geometry at the beginning of the year and their understanding of three types of standard geometry proofs at the end of the year. They found that males achieved significantly higher than females on the knowledge of geometry. This finding is consistent with Battista (1990), who had the same findings in terms of geometrical knowledge.

Ma (1995) conducted a study designed to analyze the variability of mathematics achievement between male and female students and to examine gender differences between two mathematical areas namely, algebra and geometry, using Canadian and Asian students. The study was based on the mathematics achievement data of population A (13- year-old) and B (high school seniors) from the Second International Mathematics and Science Study (SIMSS). There were four education systems (British Columbia, Ontario, Hong Kong, and Japan) involved in the study. Hong Kong and Japan were selected for comparison with British Columbia and Ontario because of their similar economic status. Overall, there was a total sample size of 960 students, with 120 students from each educational system. The results showed gender differences were statistically significant in population B, where the males outperformed the females on the geometry subtest. This finding supports that found by Hanna (1986), because she found that for the geometry subtest, significantly more males gave correct responses. In addition, in TIMSS (2003) in 8th grade, among 49 participating countries, males scored a significantly higher average than females in 11 countries in geometry.

This section has discussed the studies that have found that males outperformed females in geometry in relation to Mathematical proof (Battista, 1990; El Hassan, 2001; Hanna, 1986; Ma, 1995; TIMSS, 2003).

2.2.1.2.2 Evidence of Superior Female Performance in Geometry

In an early study, Senk and Usiskin (1983) found that males outperformed females on the knowledge of geometry. In contrast, on the proof subtest within their study, gender performance was nearly identical. However, the females performed significantly higher on the last proof, when their scores were adjusted for knowledge of the geometry test. The study emphasised that females learned more than males during the year, but there were no significant difficulties on the other proofs.

As earlier mentioned in TIMSS (2003) in 8^{th} grade, among 49 participating countries, males scored a significantly higher average than females. In contrast, females scored a significantly higher average than males in 8 countries for the geometry section with Jordan being one of these countries (TIMSS, 2003, pp.102-123).

2.2.1.3 Conclusion

Most commonly, the differences between gender and the understanding of geometry (which is related to Mathematical proof) favour males (El hassan, 2001; Battista, 1990; Hanna, 1986; Ma, 1995; Huntley, 1990) or there are no differences (Senk & Usiskin; 1983; TIMSS, 2003). There were a few studies that favoured females (Senk & Usiskin; TIMSS, 2003). Significantly, the only study involving Jordan (TIMSS) found that females performed at a higher level than males.

2.2.2 Algebra, Generalization, and Use of Symbols

2.2.2.1 Introduction

The study undertaken for this thesis examines students' mathematical thinking with regard to Use of Symbols and Generalization. The items chosen generally relate mathematical Use of Symbols and Generalization to the topic of algebra. Algebraic thinking describes generalizations succinctly by focusing on the structure of a mathematical statement (MacGregor, 1993). Mason (1980) discussed algebraic thinking by examining "the roles of symbols, the generally absent icons that should support those symbols, and the mathematical processes of conjecturing and proving, or in a slightly more refined form, specialization, generalization and reasoning" (p.8). Given the importance of Use of Symbols and Generalization and their fundamental role in the topic of algebra, they were chosen as two of six types of mathematical thinking studied for this thesis.

2.2.2.2 Studies of Algebra and Gender

Several studies have not found any significant relationship between gender and student performance on algebra tests. An early study by Hanna (1986) found that males outperformed females in geometry; but she found also that no statistically significant gender differences were found for the topics of arithmetic, algebra, and probability and statistics. Also, Low and Over (1993) conducted a study of gender differences in the solution of algebraic problems in Melbourne. Tenth grade students were tested on text-editing skills by being asked to classify 36 algebraic word problems and to determine whether the problems contained missing, sufficient, or irrelevant information required for the solution. Text-editing scores were found to correlate with general mathematical ability using the Australian Mathematics Competition (AMC) to denote general mathematical achievement. Low and Over (1993) found for 11th grade students, that the interactive influence of ability level and gender was not significant. Similarly, in the USA, Armstrong (1981) conducted a study for year 7 and secondary schools students to examine any gender differences in various mathematical areas. The researcher used the women in mathematics survey and the National Assessment of Educational Progress (NAEP) to collect the data. In regards to algebra subtest there was no significant differences between males and females in year 7 and 12 in both surveys.

Furthermore, Stites, Kennison, and Horton (2004) conducted a study of gender differences in solving word problems which is required in algebraic solutions. Stites et al. (2004) tested 96 college students in the United States on twelve word problems. Each problem had four versions. The scenario for two versions was selected to be more familiar to males; one version contained additional information and one version did not. The other two versions were chosen to be more familiar to females, with one version containing additional information and one version not. The results showed that no gender differences were found in solving algebraic word problems which contained, or did not contain, additional information.

In contrast, several other studies have found a relationship between gender and performance on an algebra test, Ma (1995) found that the only statistically significant result appeared on the algebra subtest for high school seniors in Hong Kong. The performance of females was significantly more variable than that of males on this subtest. Also, Jordanian females in 8th grade had a significantly higher average than males in (TIMSS, 2003) in algebra (TIMSS, 2003, chap, 3.pp.102-123)

2.2.2.3 Conclusion

Most commonly in algebra, either no gender differences are found (Armstrong, 1981; Hanna 1986; Low & Over, 1993; Stites, Kennison, & Horton, 2004) or the differences that are found favour females (TIMSS, 2003). This contrasts with the literature on geometry and proof which tends to favour males. Although, it should be noted that for the only study from Jordan (TIMSS, 2003) females were favoured on both geometry and algebra.

2.2.3 Reasoning and Logical thinking

2.2.3.1 Introduction

The study undertaken for this thesis examines students' mathematical thinking with regard to Logical thinking. The items chosen generally relate mathematical reasoning thinking to the topic of Logical thinking. Johnson-Laird (1999) defined reasoning as "a process of thought that yields a conclusion from percepts, thoughts, or assertions" (p.110). Ediger and Rao 2000 (cited in Ediger, 2002) stated that the development of logical thought is a fundamental tool in mathematics education. Given the importance of logic and its fundamental role in the topic of reasoning, it was chosen as one of six types of mathematical thinking studied for this thesis.

2.2.3.2 Studies of Reasoning and Gender

Several studies have found there is no significant relationship between gender and reasoning in mathematics. Battista (1990) used an experimentally constructed test to measure logical reasoning and reported that for reasoning, no gender differences were found.

In contrast, other studies have found a significant relationship between gender and reasoning in mathematics. In Cox's (2000) study, taking into account that Common Assessment Task CAT1 was in investigative project, CAT2 was a challenging problem, CAT3 involved facts and skills, and CAT4 was an analysis task, females scored significantly higher in four areas for CAT1 (extensions space and number, change and approximation, extensions change and approximation and reasoning and data), and three of the subjects for CAT2 (extensions space and number, extensions change and approximation and reasoning and data), whereas males were significantly higher in four of the subjects for CAT3 (space and number, change and approximation, extensions change and approximation and reasoning and data), and three of the subjects for CAT4 (space and number, extensions change and approximation and reasoning and data). This study emphasised that females did better than males in terms of reasoning on investigation and challenging problems, however, males did better than females in terms of reasoning on facts and skills and analysis of problems.

Also, Bitner-Corvin (1987) in a study focused on Logical thinking, ranging from 7th grade to 12th grade, with six reasoning modes (conservation, proportional reasoning, controlling variables, probabilistic reasoning, correlational reasoning, and combinatorial reasoning) found that males scored significantly higher than females on conservation, and probabilistic reasoning, whereas, females scored significantly higher than males on combinatorial reasoning. Also, Mills, Ablard, and Stumpf (1993) reported that in a study of gender differences of younger children (7-11 years of age) who were academically gifted, males outperformed females in mathematical reasoning. There was no interaction between gender and grade level. This point emphases that academically gifted males performed significantly higher than similar females for children in the 7-11 age band.

2.2.3.3 Conclusion

In general, most of these studies comparing male and female reasoning give mixed results (Bitner-Corvin, 1987; Cox, 2000). However, there is one study that favours males (Mills, et al, 1993) and another that shows no difference (Battista; 1990).

2.3 MATHEMATICS ACHIEVEMENT

2.3.1 Introduction

This section will examine the studies that linked mathematics achievement with gender and location. It will focus on the studies that measured general mathematical abilities. This section is different to the first section which focused on specific mathematical abilities, such as geometrical abilities, algebra and generalization, and reasoning abilities. It will focus on the standard tests that measure overall mathematics achievement such as Third International Mathematics and Science Study (TIMSS), National Assessment of Education Progress tests (NAEP), Longitudinal Study of American Youth (LSAY), and Monitoring Learning Achievement (MLA).

Mathematics achievement will be discussed in relation to:

1) Gender differences (including mathematics achievement and gender, gender differences in TIMSS, possible explanations for these differences, and conclusion).
2) Location (urban, suburban, rural).
3) Gender and location.
4) Jordan and TIMSS.

2.3.2 Studies of Mathematics Achievement and Gender

As cited in Friedman (1989), other researchers have found until high school either there are no differences between gender and mathematics achievement, or the differences that are found favour females. Similarly, throughout the high school years, differences favouring males are common (Ai, 2000; El hassan, 2001; Leder & Forgasz; 1992; Uekawa & Lange; 1998; Young, 1994). Hensel (1989) studied the differences in mathematics achievement related to gender and found that educators and researchers observed some gender differences on standardized tests. These differences were initially thought to be caused by genetic differences.

2.3.2.1 Studies that Show no Relationship between Gender and Mathematics Achievement

While there is ample evidence of differences in mathematics achievement and gender, several studies have not found any significant distinctions, for example, Ai (2002), in his study of gender differences in the growth in mathematics achievement in relation to various social and psychological factors, such as attitude toward mathematics and self-esteem. Ai (2002) used the Longitudinal

Study of American Youth Instrument (LSAY) for students from grade 7 to grade 10 in Los Angeles. The mathematics test consisted of 60 items from the National Assessment of Education Progress tests (NAEP). All students were tested by NAEP tests from grade 7 to grade 10, and the students score at grade 7 represented their initial status. The results showed that for students who started with high achievement, there were no gender differences in their initial status and growth rate in terms of mathematics scores. However, for those who started at a low level in a initial status, a significant gender gap was found in average mathematics achievement for grade 7 which favoured females. In contrast, Zabel and Nigro (2001) examined academic achievement in reading, language, and mathematics for juvenile offenders, where one third of the students had had special education experience, including those with disabilities, especially emotional or behavioral disorders, learning disabilities, and mild to moderate mental retardation. The students were 130 youths aged between 12 and 18 who were confined to a regional juvenile detention facility in Kansas. The results showed that there were no significant differences between males and females in the areas of computation and applied mathematics.

Other researchers have also found no significant relationship between gender and mathematics achievement. Young (1994) found that year 3 and year 7 females outperformed year 3 and year 7 males in mathematics achievement, but that this was not statistically significant. Leder (1990b cited in Leder & Forgasz, 1992) also studied gender differences in mathematics achievement involving students in years 3, 6, 7, and 10 in metropolitan schools in Melbourne, Victoria. The results showed no significant difference between males and females in mathematics achievement in years 3, 6, and 7. However, there were significant differences between males and females favouring males in mathematics achievement in year 10.

Uekawa and Lange (1998) reported on a comparative study between United States and Korea using the Third International Mathematics and Science Study (TIMSS) on students in the eighth grade. No significant difference was found between US males and females in mathematics achievement. However, there were significant differences within eighth grade for Koreans favouring males.

Similarly, using TIMSS (1995) in Canada, Lauzon (1999) found no differences between the sexes in mathematics achievement in grades 3 and 4, and a slight difference in grade 7 and 8 favouring males. Also, Lauzon found in the TIMSS repeat (1999) with 8th grade that, no gender differences existed in mathematics achievement. Moreover, El Hassan (2001) studied gender differences in achievement in Arabic, foreign language (English or French), mathematics, sciences, and life skills for elementary 9th grade and intermediate 13th grade students in Lebanon, using the Monitoring Learning Achievement (MLA). The Mathematics test covered five topics: numbers, operations, geometry, measurement, and problem solving at the 9th grade, and three topics: operation, geometry, and algebra at the 13th grade. The results showed that overall achievement in mathematics was not statistically different for males and females in 9th and 13th grades.

2.3.2.2 Studies that Show a Relationship between Gender and Mathematics Achievement

This section will focus on the studies that linked mathematics achievement to gender. Firstly this section will discuss the mathematics achievement studies that favour males, and in the second section, the studies that favour females will be discussed.

2.3.2.2.1 Studies that Show Males do better than Females in Mathematics Achievement

In this section, studies that find that males achieve more highly than females in mathematics are considered. As stated earlier, Young (1994) found no significant differences for years 3 and year 7, similar to Leder (1990b, cited Leder & Frgasz, 1992) for years 3, 6, and 7, however, both Young (1994) and Leder (1990b) found differences in mathematics achievement for year 10 students, where year 10 males outperformed year 10 females. Uekawa and Lange (1998) found eighth grade Korean males outperformed females in TIMSS for mathematics. The findings from these studies would suggest that the gender gap may increase with age in favour of males. Ai (2002) also found that the effect of a positive mathematics attitude was stronger for males than for females, meaning that attitude and achievement were more strongly linked for boys.

Baya'a (1990) studied 9th grade to 12th grade Arab students at the private Terra-Santa Arab high school, which presented a high socio-economic status (SES), and an Arab comprehensive high school, which presented a low SES in Acre in Israel. This study aimed to see whether male and female differences in mathematics achievement were independent of socio-economic status. The sample size was 418 students; 167 students from the comprehensive school and 231 students from Terra-Santa school (214 male students and 204 female students). The average grades in the mathematics final examinations for the last four trimesters were used to measure achievement in mathematics. The results showed there was a significant difference between the mathematics achievement of males and females, in favour of males. The difference, however, was significant in favour of males for only the Arabic comprehensive school students (low SES), and no significant difference between males and females was found for Terra-Santa school students (high SES). On this evidence it would seem that SES was the deciding factor, although no information about teaching practices at the two schools was provided.

Low and Over (1993) found that for 10th grade students, the number of problems that were correctly classified varied significantly with the ability level of students and gender in favour of males. They also found that males had higher mean text editing scores than did females on the classification of problems that had missing and irrelevant information. The researchers found in their second experiment that males performed significantly better than females on the solution of problems with relevant information as well as on problems with irrelevant information.

As stated earlier, Ma (1995) found that, on the overall test, no gender differences in either mathematical area were found to be statistically significant in either population group within each education system, due to the small sample size in each country. Although, significant gender differences may be found globally, they may not be detectable within each education system. The only statistically significant result appeared on the algebra subtest for population B in Hong Kong where the performance of females was significantly more variable than that of males on this subtest.

2.3.2.2.2 Studies that Show Females do better than Males in Mathematics Achievement

Randhawa and Hunt (1987) conducted a study to investigate any gender and location (rural-urban) differences in various subjects, particularly in mathematics (mathematics concepts and computation) on standardized tests. A random sample of grades 3, 7, and 10 were chosen from the mid-western province in Canada. In terms of grade 10, Randhawa and Hunt (1987) found that females scored better than males on mathematics computation subtests, whereas males scored better than females only on mathematics concepts. Also, Cook (2000) tested 164 male and female students at Brown University, all of whom had similar mathematics scores, in three different groups comprising women in one single-sex group, men in one single-sex group, and a combination of both sexes in the other group. Cook (2000) found that females do 12% better on mathematics achievement when tested alone. On the other hand, males did not perform any differently when tested alone or with females. This suggests that gender based context of the testing was important for females, but not for males.

Alkhateeb (2001) investigated gender differences in mathematics achievement of 12th grade high school students in United Arab Emirates (UAE) over a 10-year period. The sample was two thousand senior high students (1000 males and 1000 females) comprised of one hundred males and one hundred females from each of the 10 academic years, 1990-1991 to 1999-2000. The males and females in UAE receive their education and testing in gender segregated schools. The results of the National High School General Examination at the end of each of 10 academic years were considered as students' mathematics achievement scores in the scientific stream (this stream focuses more on science areas such as mathematics, physics, and biology). The results showed for the total sample of students that females had generally higher scores than male students in mathematics achievement but their performance was not statistically significantly better. Alkhateeb (2001) also found that female students outperformed male students in mathematics achievement, especially during the last 6 years from 1994-95 to 1999-2000, but the differences were significant only in 1998-1999 and 1999-2000. These results are consistent with those of Cook (2000) who found that females perform better when they are tested alone. Ai (2002) reported that for those who started at a low level in initial status, a significant gender gap in average mathematics achievement developed in grade 7 which favoured females. Also, Dennis (1993) in a study at the two-year college level in New York showed that females outperformed males for the academic years 1970,1975,1980,1985, and 1990 in four of the introductory level mathematics courses.

In Jordan, the Ministry of Education (2001) conducted a national test for 9th grade, to measure the achievement level across gender. The sample was chosen randomly and comprised 5% of all 9th grade students from all directorates of education throughout Jordan. The national test measured knowledge and scope of understanding, and scope of higher mental activities such as problem solving and investigation. The test covered numbers and processes, geometry, measurement, trigonometry, algebra, statistics and probability. The results showed that for the total test females achieved 42% with a standard deviation of 7.29 versus 38% (sd 7.36) for males in mathematics achievement, and that this difference was significant.

2.3.3 Gender Differences in TIMSS in Jordan

As it is a major study TIMMS will be considered in detail in this section. The TIMSS (1999) mathematics test covered five topics: fractions, measurement, geometry, algebra and data. In Jordan gender differences in eighth grade mathematics achievement in 1999 were not statistically significant (mathematics achievement mean for females was 431 (standard deviation 4.7) and a mean of 425 (5.9) for males. This result is consistent with other countries, because in most countries the gender difference in TIMSS (1999) was negligible (TIMSS, 1999, pp. 13-26). However, the TIMSS (2003) mathematics test covered five topics similar to the previous TIMSS (1999), test in, numbers, algebra, measurement, geometry, and data analysis. In Jordan, gender differences in eighth grade mathematics achievement were significant for all five topics, and for the overall test, the differences favouring females. The average of mathematics achievement for females was 438 (standard deviation 4.6) and a mean of 411(5.8) for males. This is a largest difference between genders among participating countries after Bahrain (TIMSS, 2003, pp. 30-53 & 102-123).

2.3.4 Possible Explanations for Differences in Mathematics Achievement Related to Gender

Despite the TIMSS studies described in the previous section, other researchers have found males had higher mean scores than females in mathematics achievement. There are a number of possible reasons to explain why researchers have found males most often do better than females. These explanations fall into six main areas: biological and genetic explanations, age explanations, subject differences, social explanations, personality explanations such as level of confidence, and the test instruments themselves. Firstly, some early studies provided biological explanations that maintained that there are innate differences between genders which imply a differential in mathematics achievement (Benbow & Stanley, 1980, Dennis, 1993). However, there has been little support for this type of explanation more recently. Secondly, other explanations maintain that gender differences may increase with age (Leder, 1990b, cited in Leder & Forgasz, 1992; Young, 1994). Thirdly, it has been suggested than females do better in humanities subjects but males do better in mathematics (Randhawa & Hunt, 1987; Randhawa, 1988; Uekawa & Lange, 1998). However, no explanations of why this might be the case was offered. Fourthly, socialization or environmental reasons are provided to explain why female students sometimes took fewer mathematics courses, leading to lower achievement levels or formal training, or quality of teaching. Affective factors involve attitudes toward mathematics, learning behaviors, and motivation. In some cases, teachers gave more attention in the class to the males than to females and this was re-enforced by parents (Baya'a, 1990; Begley, 1988; Dennis, 1993; Hanna, 1986; Randhawa, 1988). Fifthly, level of confidence may be a factor. Some researchers believe that the cause of the gender gap in mathematics is due to the individual characteristics of the gender, that is, males tend to have higher levels of confidence (Leonard, 1995; Manning, 1998). Finally, the tests themselves are another possible reason because in the past, a gender gap may have existed due to the nature of the test questions. They were often geared more to males than females (Begley; 1988). Over time, this text bias has been addressed to better assess all students equally. Since the changes in these tests, the gap has narrowed.

Females and males sometimes learn different kinds of mathematics. This is consistent with Fennema (2000) statement that ″there were differences between females' and males' learning of mathematics, particularly in activities that required complex reasoning ″ (p.4), and with Leder (1993) who stated ″fewer U.S females than males enrol for more advanced mathematics courses such as trigonometry, precalculus, and calculus, and the same is true for intensive and advanced mathematics courses in the United Kingdom and Australia″ (p.1262).

There are a number of possible reasons to explain why researchers have found females do better than males. In some cases, family encouragement of males and female students in the study of science and mathematics subjects may have been important for the differences found (Alkhateeb, 2001; Shatnawi, 1982). Another possible reason may be that if the students are tested alone, the females may do better in mathematics than males (Alkhateeb, 2001; Cook, 200). This is the case throughout Jordan, particularly in high schools, where Education and examinations are separated between males and females.

2.3.5 Conclusion

Several studies have found no significant relationship between gender and mathematics achievement (Ai, 2002; El Hassan, 2001; Hanna, 1986; Lantz & Smith, 1981; Lauzon, 1999; Low & Over, 19983; Uekawa & Lange, 1998; Young, 1994). On the other hand, other studies have found a relationship between theses factors. These studies include those which found males outperformed females (Baya’a, 1990; Cox, 2000; El Hassan 2001; Hanna, 1986; Leder, 1990b, cited in Leder & Frogasz, 1992; Low & Over, 1993; Ma, 1995; Randhawa & Hunt, 1987; Uekawa & Lange, 1998; Young, 1994), and those which found females outperformed males (Ai, 2002; Alkhateeb, 2001; Cook, 2000; Cox, 2000; Dennis, 1993; Ma, 1995; Randhawa & Hunt, 1987).

2.4 MATHEMATICS ACHIEVEMENT AND LOCATION

2.4.1 Introduction

Educational research over the past quarter century has examined location differences in mathematics achievement. According to Young (1994, 1998) there is the general impression among educators, researchers, legislators, and the general public, that students from larger urban or suburban schools receive a better education than that of students from smaller and rural schools. There has been little empirical evidence to challenge that view, however, several studies have not found any significant differences between urban, suburban, and rural or small schools (Alspaugh, 1992; Edington & Martellaro, 1984; Fan & Chen, 1999; Howley, 2003, Lee & McIntire, 2000; Monk & Haller, 1986; Randhawa & Hunt, 1987).

2.4.2 Studies of Mathematics Achievement and Location

2.4.2.1 Studies that Show that no Relationship between Mathematics Achievement and Location

There have been a number of studies that have not found any differences between the location of a school and the mathematics achievement of its students. Monk and Haller (1986) conducted a study in New York State, where they found no significant location school gap in different subjects including mathematics achievement. Two studies in New Mexico found a similar result. Ward and Murray (1985) studied the factors impacting on the achievement on the high school students using the New Mexico achievement test with 375 high school students in New Mexico in 1984. They found that the students from rural areas achieved as well as their peers in urban locations. Edington and Martellaro (1984) studied the relationship between location and mathematics achievement, for 5th, 8th, and 11th in public schools over the years 1978-1981 using school mean on the Comprehensive Tests at Basic Skills. They found overall that, no relationship between urban and rural schools and mathematics achievement.

Other researchers in USA also have not found any differences between the location of a school and mathematics achievement. Howley (2003) found no mathematics achievement gap between rural students and students in nonrural, suburban, or urban classification. In both 1996 and 2000, as reported by Howley (2003) the National Assessment of Educational Progress (NAEP) mathematics scores of students in rural and small schools. There were no statistically different from the national average in all grade level tests. Fan and Chen (1999) examined achievement scores from the National Educational Longitudinal Survey in reading, mathematics, science, and social studies in 1988 for the 8th, 10th and 12thgrades. In terms of mathematics achievement, they found no significant differences between students in rural schools and their counterparts in metropolitan schools. Lee and McIntire in 2000 used the National Assessment of Educational Progress (NAEP) 8th grade data for two studies in 1992 and 1996, to examine any potential differences between rural, nonrural and mathematics achievement. In 1996 they found that overall, rural students had higher mean scores than nonrural. In contrast, in 1992, there was no statistically significant difference between students in rural and nonrural areas in regard to mathematics achievement.

As stated earlier, Randhawa and Hunt (1987) found that females outperformed males on mathematics computation, whereas males outperformed females on mathematics concepts. Randhawa and Hunt (1987) also found that students from rural schools achieved as well as their peers from urban schools in two mathematics subtests. Similarly, Alspaugh (1992) used the Missouri Mastery Achievement Test (MMAT) scores in Missouri State for fifth grade reading and mathematics to examine any potential differences between urban and rural students. Alspaugh (1992) found for both mathematics and reading achievement, there were no statistically significant differences between geographical location and student scores.

2.4.2.2 Studies that Show that a Relationship between Mathematics Achievement and Location

In contrast to several other studies, Young (1994, 1998) found a relationship between the geographical location of the school and the mathematics achievement of their students in favour of regions of upper socio-economic status. Students of upper socio-economic status were often located in urban areas while students of lower socio-economic status were more often located in rural areas. Easton and Ellerbruch 1985 (cited in Young, 1994, 1998) found that students from rural areas scored lower on citizenship and social studies tests than other students from urban areas. Kleinfeld et al. (1985, cited in Young, 1994, 1998) reported that the students who achieved the superior results were affected by the following factors: strong teachers, school administration and community partnership, and school and community consensus on educational programs. In addition, there is a positive relationship between the ability of the staff to work toward an educational partnership with their community and the quality of education programs. Uekawa and Lange (1998) found in their study for 8^{th} grade mathematics performance in rural, urban and suburban schools in United States and Korea, that urban schools in Korea outperformed the rural and suburban schools. In contrast in the US, they found that suburban schools outperformed the urban and rural schools.

Young (1994) investigated the differences in performances between students in metropolitan, rural and remote locations throughout Western Australian schools. The students were in years 3, 7, and 10 and their performance was measured in the areas of mathematics, reading and writing. The results showed that students in the metropolitan schools significantly outperformed those students in rural and remote schools in mathematics in all years. The metropolitan schools also outperformed the rural and remote schools in year 10 in writing. Similarly, students in rural schools outperformed students in remote schools in writing. In contrast, Young (1994) found there were no significant differences between each of the three schools' location for reading. Cox (2000) found that students in metropolitan schools did better than students in the country schools in three of subjects in (CATs), CAT1, CAT2 and CAT3, and one subject in CAT4 (CAT1 was in investigative project, CAT2 was a challenging problem, CAT3 involved facts and skills, and CAT4 was an analysis task). Each CAT contained six distinct sections, i.e. space and number, extensions space and number, change and approximation, extensions change and approximation, reasoning and data, and extensions reasoning and data). Cox (2000) found also that students in the country did better than students in the metropolitan areas in two of subjects in CAT4. In contrast, Young (1994) found that students in the metropolitan schools significantly outperformed those students in rural and remote schools in mathematics in all years up to year 10. The differences in the two student bodies in relationship between mathematics and school location may be due to state and age differences. It is also possible that the time difference between the two data collections was important.

Clarke, Nyberg, and Worth (cited in Randhawa, 1988) found in Alberta, Canada, in their 1980 study of grade three students that children from urban schools performed worse than those from rural schools in certain subjects including mathematics. Randhawa (1988) also found a significant multivariate location effect for four tests reading, mathematics, written expression, and using sources of information, of the Canadian Tests of Basic Skills (CTBS). However, none of the univariate results were significant. In terms of mathematics achievement, students from rural schools outperformed students from urban schools on concepts, and on two of the micro-skills.

Haller, Monk, and Tien (1993) studied mathematics scores of tenth grade students from 1987 to 1989 using the Longitudinal Study of American Youth Instrument (LSAY) in USA. They found a positive relationship between the proportion of students who were enrolled in more advanced courses in mathematics and mathematics achievement. However, students from urban schools appeared to take more advanced mathematics courses than students from rural schools, indicating that urban school students scored significantly higher than rural school students.

2.4.3 Conclusion

This section first described the studies that found no relationship between location and mathematics achievement (Alspaugh, 1992; Edington & Martellaro, 1984; Fan & Chen, 1999; Howley, 2003; Lee & McIntire, 2000; Monk & Haller, 1986, Randhawa & Hunt, 1987). In contrast, other studies have found a relationship between the location of the school and mathematics achievement. These studies include those which found urban or metropolitan outperformed rural or suburban (Cox, 2000; Uekawa and Lange, 1998; Young, 1994; 1998). However, other researchers have found rural or suburban outperformed urban or metropolitan (Lee and McIntire, 2000; Randhawa, 1988; Uekawa & Lange, 1998). These differences described were contributed to by country differences. Most of the studies that conducted in USA and Canada found that no relationship between school location and mathematics achievement. In contrast, the other counties such as Australia and Korea found relationships between these two variables favouring urban schools.

2.5 MATHEMATICS ACHIEVEMENT AND GENDER AND LOCATION

2.5.1 Introduction

A number of studies have examined gender differences in mathematics achievement (Ai, 2002; Alkhateeb, 2001; Baya'a, 1990; Cox, 2000; Dennis, 1993; Hanna, 1986; Leder, 1990b, cited in Leder & Frogasz, 1992; Low & Over; 1993; Ma, 1995; Zabel & Nigro; 2001), and recent studies have examined the possible differences between mathematics achievement and rural, urban and suburban school locations (Alspaugh, 1992; Cox, 2000; Monk & Haller, 1986; Howley, 2003; Uekawa & Lange, 1998; Young, 1994, 1998). Some of these studies reported on the interaction between gender differences and location (Cox, 2000; Randhawa, 1988).

2.5.2 Studies that Show the Interaction between Gender Differences and Location

Randhawa (1988) conducted a study using grade ten students in Canada to examine gender and location differences on academic basic skills and mathematics achievement. The sample was 79 classrooms with 1490 students from urban and rural areas. All students were administered the complete battery of tests including reading, mathematics including computation, concepts and problem solving, written expression, and using sources of information, of the Canadian Tests of Basic Skills (CTBS). Randhawa found females outperformed males on all the tests except mathematics in which males did better than females. In particular, males scored better on mathematical problem solving than females. The univariate local and gender interaction was significant for only the computation subskill. Both genders from rural schools had equivalent scores on this computation (component), however the urban males scored significantly better than the urban females. For the eight micro skill scores on the mathematics test (operations computation, equivalent forms and order concept, concepts, basic mathematical principals concepts, problems involving equivalent forms and order, algebra, and geometry and measurement; common applications problems) males achieved better than females on three of them (computation involving common applications, algebra, geometry, and measurement; common applications problems; and problems involving statistics, graphs, and tables), whereas there were no differences on the other micro skill scores. Students from rural schools performed better than students from urban schools on two of them (equivalent forms and order concept and basic mathematical principals concepts). They also found males and females from rural classrooms had equivalent scores on the computation subskill. However, urban males were significantly better than the urban females on this computation subskill.

As stated earlier, Cox (2000) examined the differences in various mathematics subjects in terms of gender, location and the interaction between gender and location at year 12 in Victoria which were assessed identically using four externally set Common Assessment Tasks CATs, CAT1, CAT2, CAT3, and CAT4. In terms of gender, females outperformed males in some mathematics subjects, whereas males outperformed females in the other subjects. More generally, students in urban locations outperformed those in rural locations. The interaction between gender and location indicated that urban males outperformed rural males on half the subjects in CAT1 to CAT3, whereas, rural males outperformed urban males on only two subjects in CAT4. However, urban females outperformed rural females on five subjects in CAT1 to CAT3, but, rural females outperformed urban females on only one subject in CAT4.

2.5.3 Conclusion

This section focused on the studies in relation to mathematics achievement and the interaction between gender and location (Cox, 2000; Randhawa, 1988). The relationship between gender and mathematics achievement that was found in the Randhawa study (1988) entirely occurred only in urban areas. However, in rural areas males and females were identical in mathematics achievement. Hence, geographical location is likely to be a significant factor influencing any gender effect in mathematics achievement.

2.6 JORDAN AND TIMSS IN 1999 AND 2003

Although location was an important variable in the current research, TIMSS in 1999 and 2003 did not include this variable in analysis. Jordan was one of 38 countries that participated in TIMSS (1999). The eighth- grade students in Jordan (14 year olds) participated in TIMSS (1999). For mathematics achievement the international average of 487 was obtained by averaging across the mean scores for each of the 38 participating countries. The results reveal substantial differences in mathematics achievement between the highest- and lowest-performing countries, from an average of 604 for Singapore to 275 for South Africa. The average for Jordan was 428 and this is significantly lower than the international average. The position of Jordan regarding the average was 32 out of 38 countries. Jordan had a higher average performance than Indonesia, Chile, Philippines, Morocco and South Africa, and was not significant different to than Turkey and Iran Islamic Republic, but it had a significantly lower mean achievement than the other participating countries (TIMSS, 1999, pp. 13-26).

In TIMSS 2003, Jordan was one of 49 countries that participated in this test, with eighth-grade students (13.9 year olds) participating. For mathematics achievement the international average of 467 was obtained by averaging across the mean scores for each of the 49 participating countries. The results reveal substantial differences in mathematics achievement between the highest- and lowest-performing countries from an average of 605 for Singapore to 264 for South Africa. The average for Jordan was 424 and this is significantly lower than the international average. The position of Jordan regarding the average was 32 out of 49 countries. Jordan had a higher average performance than Iran, Indonesia, Tunisia, Egypt, Bahrain, Palestinian National Authority, Chile, Morocco, Philippines, Botswana, Saudi Arabia, Ghana, and South Africa, and was not significantly different to Lebanon, but it had a significantly lower mean achievement than the other participating countries (TIMSS, 2003, pp. 30-52).

The results for Jordan were very similar for the two TIMSS studies. Those related to gender were discussed above, but neither study included an analysis related to school location.

2.7 CONCLUSION

The review of gender differences in relation to the aspects of mathematical thinking presented in this chapter focused particularly on the aspects of geometry, reasoning, and algebra. The significant contributions related to gender differences with geometry and reasoning confirmed that there are gender differences in relation to proofs and reasoning which favoured males or there were no gender differences. In contrast, gender differences were evident with algebra where females performed better than males or there was no difference. Unfortunately, there was no direct literature available in relation to the other two aspects of mathematical thinking (Induction and Deduction) with gender.

Mathematics achievement was then examined in relation to gender differences, location, and the interaction between gender and location. The researcher presented studies that linked mathematics achievement and gender. To facilitate description of these studies, the researcher put them into three categories, first studies that show no relationship between gender and mathematics achievement (Ai, 2000; El hassan, 2001; Lauzon, 1999; Leder, 1990b, cited in Leder & Frogasz, 1992; Uekawa & Lange, 1998; Young, 1994). The second category, which followed shows that males outperformed females in mathematics achievement in high school ranging from 8th grade to 12th grade (Baya'a, 1990; Leder, 1990b, cited in Leder & Frogasz, 1992; Low & Over, 1993; Uekawa and Lange, 1998; Young; 1994). The final category shows that females outperformed males in mathematics achievement (Alkhateeb, 2001; Cook, 2000; Randhawa & Hunt, 1987; Ministry of Education (Jordan), 2001; TIMSS (Jordan), 2003).

The studies that linked mathematics achievement and gender in the main suggested that mathematical performance is the same for males and females, particularly in primary schools. In contrast, males generally outperformed females in mathematical performance in high schools. However, females outperformed males in mathematics in Arabic countries such as Jordan and UAE. These results are supported by Cook (2000), whose findings indicated that females performed better when tested alone, as is the case in most Arabic countries. In addition, at the secondary level female students in Arabic countries are also educated in single-sex schools.

The researcher then discussed the studies that linked mathematics achievement and location, which were divided into two categories. The first category focused on the studies that showed no relationship between mathematics achievement and location (Edington & Martellaro, 1984; Fan & Chen, 1999; Haller, Monk, & Tien, 1993; Howley, 2003; Lee & McIntire, 2000; Monk & Haller, 1986). The second category focused on the studies that showed a relationship between mathematics achievement and location. Students from upper socio-economic background tended to exhibit higher levels of achievement in mathematics and vice versa (Cox, 2000, Uekawa & Lange, 1998; Young, 1994, 1998). Given that upper socio economic families are often located in urban and suburban areas, rural students outperformed urban students in mathematics (Clarke, et al, 1980, cited in Randhawa, 1988; Randhawa, 1988).

Finally, studies that focused on mathematics achievement and the interaction between gender and location were presented (Cox, 2000; Randhawa, 1988). Differences between gender and mathematics achievement were evident in urban areas (Randhawa, 1988). However, males and females in rural areas were equal in mathematics achievement. A comparison was then carried out for TIMSS (1999, 2003) between participating countries generally and Jordanian mathematics achievement specifically.

To date, the literature dealing with gender, location, and mathematical thinking or achievement, has been inconclusive. While there is evidence of a gender bias in favour of males in secondary schools, it is not a consistent trend, particularly in Arabic countries. Also, with regard to location differences, some differences are found between urban and rural areas, and when gender is considered, between males and females in urban locations. While many studies have looked at mathematical achievement overall, others have considered the various aspects of mathematical thinking that contribute to it. Mathematical proof (as related to geometry), Use of symbols and Generalization (as related to algebra) and Logical thinking or reasoning and their relationship to gender have all been studied in the literature. Conversely, mathematical thinking as it relates to Induction and Deduction, has not previously been studied with respect to gender differences.

This study will add to the mathematics education literature by providing further insight into the nature of mathematical thinking and its relationship to mathematical achievement in Jordan. The effects of gender and location will be examined with respect to all aspects of mathematical thinking and mathematical achievement. In addition, examination of the mathematical thinking aspects of

Induction and Deduction with respect to gender and location has not been undertaken in the literature previously, but is covered by this study.

CHAPTER THREE

THE INSTRUMENTS AND SAMPLE

In this chapter an overview of the study methodology will be provided. This will involve describing the development of the instruments, the participating schools and students and the procedures used in the study. The researcher administered two tests and two interviews; a test of mathematical thinking, a test of mathematics achievement, teacher interviews and student interviews. More than 500 students participated in the two tests, with 13 teachers participating individually in the teacher interviews and four groups of students being involved in student interviews.

3.1 SCALE DEVELOPMENT: MATHEMATICAL THINKING SCALE

For this study mathematical thinking is taken to be comprised of six sub-scales: Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof. These six scales were based on Shatnawi's (1982) scales. To devise his scales, Shatnawi (1982) distributed one question to a committee of 10 people taken from the staff at Yarmouk University in Jordan, the staff of the national team for the development of mathematics in the Ministry of Education (this team devises the mathematics curricula and prepares national tests), mathematics supervisors and teachers in schools. The question asked was "In your opinion, what are the aspects of mathematical thinking?" Then, based on previous research and the responses of the committee, Shatnawi outlined 14 aspects of mathematical thinking. He then re-asked this same question to the same committee, requesting them to choose from his 14 aspects of mathematical thinking, those aspects that satisfy the following two criteria: aspects that can be quantified or measured and the suitability of those aspects for students at secondary school. As a result of this study, Shatnawi concluded that the six sub-scales named above were the most significant, representative and quantifiable components of mathematical thinking.

Additionally, in a small-scale study undertaken as a masters degree project in 1998, the researcher asked 15 students who were his peers and colleagues to choose the aspects of mathematical thinking from a list of 14 aspects of mathematical thinking and to order them from the most important to least important, The aspects were similar to those used by Shatnawi (1982). Then, the peers and colleagues were asked to choose those in terms of aspects that best satisfied the following two criteria: suitability for year 11 scientific stream-curricula, and those aspects that would have minimal interaction between them. Their responses for the most important aspects overwhelmingly concurred with the six aspects of mathematical thinking identified in Shatnawi's scale. A comparison of the researcher's aspects and Shatnawi's aspects of mathematical thinking are shown in Table 3.1 in descending order of importance as identified by the two sets of peers and colleagues. The first nine aspects are basically the same and the last five aspects are different.

TABLE 3.1. THE COMPERSION BETWEEN RESEARCHER AND SHATNAWI ASPECTS OF MATHEMATICAL THINKING.

Aspects from researcher's 1998 study	Shatnawi's study
GENERALIZATION	GENERALIZATION AND ABSTRACTION
INDUCTION	INDUCTION
DEDUCTION	DEDUCTION
USE OF SYMBOLS	USING SYMBOLS
LOGICAL THINKING	LOGICAL THINKING
MATHEMATICAL PROOF	MATHEMATICAL PROOF
REASONING	REASONING
PROBLEM SOLVING	PROBLEM SOLVING
CREATIVE THINKING	CREATIVE THINKING
Specialization	Systemic thinking
Using patterns	Modeling (Pattern cognition)
Ability to find the optimal solution	Imaginative thinking
Inferences from premises	Critical thinking
Using mathematical expression or the ability to translate from words to equations	Building a concept

Note: The aspects that match between the two studies, written in capital letters, are basically the same.

The Mathematical thinking test used in this study was prepared by the researcher to measure the six scales: Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof, with five items for each scale. The researcher chose items from the TIMSS (1995), specialist books and articles in mathematics education (NCTM, 1971; Petocz & Petocz, 1994; and Zorn, 2000), the internet (British Broadcasting Corporation (BBC), 2001; Challen, undated), researcher experience, and from the Shatnawi scale. Details of the rationale and sources for each scale are given below. A copy of the mathematical thinking test can be found in Appendix 2.1 in both Arabic and English languages.

A brief description of each of the mathematical thinking scales follows.

3.1.1 Generalization

Polya (1990, p.108) defined generalization as leading "from one observation to a remarkable general law. Many results were found by lucky generalizations in mathematics, physics, and natural sciences, and it may be useful in the solution of problems". Generalization involves arriving at general formulas that satisfy all cases from specific cases. It is an important aspect of mathematical thinking, because as Mason et al. (1991, p.8) wrote "generalizations are the life-blood of mathematics", that is, there is no mathematics without generalization. In addition, mathematics is rich in generalizations; there are generalizations in each mathematical area. Also, searching for patterns is one of the aspects of generalization (it actually precedes generalization), and finding patterns helps the students to develop their mathematical thinking (May, 1996).

Example: Complete the last statement.

1=1

1+3=4

1+3+5=9

1+3+5+7=16

1+3+5+----+ (2n-1) =-----------.

The students were expected to know that the observation of the outcome in each statement is equal to the square number of terms in each statement. Then the generalization for the last statement is equal to n^2 (see p. A-57 for the complete solution). The researcher chose five items to measure generalization; three out of five items (2, 3, and 4) were chosen from National Council of Teachers of Mathematics (NCTM, 1971) (p, 22, 28, 30 respectively). Of the other two items, one was adapted from the Shatnawi (1982) scale (item 5) and the other was derived from the researcher’s experience (item 1). These items were designed to measure the students’ understanding of generalization, because when the students responded to these items, they were required to find a pattern from the specific cases. However, in some cases, students tried to find a general law which would satisfy all given cases. These responses are indicative of the various aspects of generalization.

3.1.2 Induction

Polya (1990, p.114) defined induction as “ the process of discovering general laws by the observation and combination of particular instances. It is used in all science, even in mathematics. Mathematical induction is used in mathematics alone to prove theorems of a certain kind”. However, Shatnawi (1982, p. 6) defined induction[7] as “the arrival at a general result through a number of specific observations”. Induction is an important aspect of mathematical thinking, and it occurs after checking whether the general rule or “generalization” is true for all cases.

Example: The number of bacteria in a colony was growing exponentially. At 1 pm yesterday the number of bacteria was 1000, and at 3 pm yesterday it was 4000. How many bacteria were there in the colony at 6 pm yesterday?

For example, in item 1 shown above, some of the students recognised this item doubled every hour or showed a pattern as 1000, 2000, 4000, and so on.

[7] This definition is a translation from Arabic to English language, as is the case for all of Shatnawi’s definitions.

However, other students tried to find a general relationship (geometric series) which would satisfy all given cases.

The researcher chose five items to measure Induction. The first item was obtained from TIMSS (1995, p.77); the third item was obtained from Petocz and Petocz (1994, p. 3). Items 2 and 5 were from the Shatnawi scale (1982). Item 4 was derived from the researcher's experience as a teacher when he witnessed students consistently applying this analysis to represent induction. These items were designed to measure various aspects of Induction, because when the students responded to these items, they were required to find a pattern. We can use mathematical induction to prove general laws or patterns for these items. Again, generalization forms part of the process of induction. These two aspects are sequential and therefore not entirely independent; generalization precedes induction, which is proof of the truth for any case after which it becomes proof by mathematical induction. Induction leads us to find a pattern from specific cases "particular to general", in contrast, deduction lead us to specific results from general conclusions "general to particular".

3.1.3 Deduction

Johnson-Laird (1999, p.110) defined deduction as a process that "yields valid conclusions, which must be true given that their premises are true". Shatnawi (1982, p.6) defined Deduction as "arriving at a particular result from a known or assumed principle". Deduction and induction are two sides of the one coin. Just as induction is an important aspect of mathematical thinking, deduction is also considered important. Induction involves discovering general laws from specific cases, whereas deduction involves arriving at conclusions from true premises.

Example: All numbers in group A are divisible by 5. The number 20 is divisible by 5, and belongs to group B, therefore, we infer from that:

a) Group A is equal to group B.

b) A is a sub-group of B.

c) B is a sub-group of A.

d) None of the above.

The number 20 (is a member of group B) but 20 may not be a member of A, then the alternatives (a) and (c) are incorrect. Perhaps the number 5 (a member of A) may not be a member of B. Therefore we can not conclude any of the first three options. This conclusion was drawn from valid premises (see p. A-62 for the complete solution). The researcher chose five items to measure deduction. The first and the fourth items were obtained from the TIMSS (1995, p. 64, 70 respectively). The other three items, 2, 3, and 5 were adapted from the Shatnawi scale (1982). These items were designed to measure different aspects of Deduction, because when the students responded to these items, they tried to find valid arguments to arrive at the correct answer. However, some of the students tried to arrive at a particular valid conclusion from previously true premises. In contrast to Induction, Deduction arrives at specific cases from a general law, or arrives at a conclusion from some truth premises. Both types of responses are indicative of aspects of Deduction. Because the nature of Deduction is to arrive at a valid conclusion from the premises, a multiple choice format was considered appropriate. However, the researcher also asked the students to write their explanations and justification for each step they used in determining their answers. In this way it was possible to cover an appropriate range of topics within Deduction as an aspect of mathematical thinking, while obtaining additional information about mathematical process used by students.

3.1.4 Use of Symbols

A symbol may be a letter, relationship or abbreviation representing an expression, quantity, idea, concept or mathematical process. Expression through the symbols means the use of symbols to communicate mathematical ideas or verbal problems. Shatnawi (1982, p6) defined use of symbols as “using symbols as a language to express ideas and mathematical information”. The Use of Symbols scale measures the use of mathematical symbols or translations to solve word problems, as these types of problems are related to algebra. Algebra is an important aspect in mathematics, because it is one of NCTM standards in the primary and secondary school curricula (Burke, Erickson, Lott, & Obert, 2001; Friel et al, 2001, Huetinck & Munshin, 2004; Hynes, 1995, 1996).

Example: Unit circle, its centre (0, 0), a line L through (0, 1), with rational slope (m). Write an expression for the equation of the circle in terms of X.

The equation of the circle is $x^2 + y^2 = 1$, and the equation of the straight line is $y = mx + 1$, because of line cut the circle, then the line equation will satisfy the circle equation, $x^2 + (mx + 1)^2 = 1$ (m is known value). Equations, variables and constants indicate the aspects of Use of Symbols.

The researcher chose five items to measure Use of Symbols. Item 2 was obtained from Zorn (2000). The other four items (items no 1, 3, 4, and 5) were derived from the researcher's experience. These items were designed to measure different aspects of Use of Symbols, because when the students responded to these items, for example, items 1 and 2 which are verbal or written problems, translation from words to a symbolic language or computational form using mathematical Use of Symbols was required. In addition, item 1 may require the use of a model to assist in the translation from words to symbols. However, for items 3 and 5, the students were required to analyse the shape of areas and represent the total area of the shape as an algebraic expression. In addition, item 5 (see p. A-10) required the students to analyse the three shapes of areas and represent the total area for each shape, then find a pattern through which they can analyse any shape in general knowing what the variables and constants are for each shape. For item 4, the students would normally use mathematical symbols to express the geometrical relationship using the following theorem: any two opposite angles in any circular quadrilateral must be 180°, using the symbols for angles, degrees, and equals to arrive at the correct answer. These results indicate responses which apply aspects of Use of Symbols. Symbol items were chosen to measure different aspects of algebra, because algebraic thinking is an important factor in developing mathematical thinking. Some of the Use of Symbols items were designed to ascertain whether the student was able to create a general solution.

3.1.5 Logical thinking

Logical thinking plays a fundamental role in every mathematical area. Macdonald (1986, p.337) described Logical thinking as “The idea that there are certain basic rules of grammar with which we can organise our discussion in mathematics is what makes it possible to establish that certain things are “true“ in mathematics, Also, it is the grammar that makes the conversation possible and holds it together”. Reasoning as related to logic thinking in the current study is considered one of NCTM standards (Hynes, 1995, 1996). This description focuses on grammar rather than arguments. Grammar in mathematics is intended to include organising the discussion to make the possibility of establishing that specific statements are true in mathematics. However, argument in logic is the ability to decide whether some statements are true or not logically. Shatnawi (1982, p.6) defined Logical thinking as “the transition from the known to the unknown guided by objective rules and principles, which are the grammar of logic”.

Example: The symmetric difference of two sets A and B is defined to be.
$A \Delta B = (A-B) \cup (B-A)$.
a) Draw a Venn diagram to illustrate $A \Delta B$.
b) Prove that $(A-B) \cup (B-A) = (A \cup B) - (A \cap B)$.

This example explains the symmetry between two sets using union, intersection, and Venn diagram to prove the symmetry. These concepts indicate the aspects of Logical thinking. (See p. A-67 for the complete the solution)

The researcher chose five items to measure Logical thinking. The first and second items were chosen from Osbaldestin (2000a, b), then items 3 and 4 were adapted from the Shatnawi scale (1982). Item 5 was chosen from TIMSS (1995, p.101). These items were designed to measure aspects of Logical thinking, because the first 2 items were chosen from a test of Logical thinking, and it was clear from the students’ work samples for these items that they were using logical understanding to express symmetry, union, and intersection, and to negate specific statements. These responses indicate these are all aspects of Logical thinking. For items 3 and 4, the students responded to these items using their understanding of the truth table. For example, item 3 addressed the meaning of the negation of union, which means for the card to be correct both values must be false. For item 4 a letter and a number appear on the card and the item is true if the card contains both number and letter which demonstrates understanding of the concept of intersection. For item 5, it was required to find the correct conclusion from the premise ‘if the rug is in the garage, then it is in the car’, and this connective (If --- then) is indicative of Logical thinking. All of these responses indicate aspects of Logical thinking such as word connections (and, or), Venn diagrams, and negating statements, etc.

3.1.6 Mathematical proof

“Mathematical proof is such a magnificent thing and nothing can be accepted as mathematically true without being rigorously proven, you might have got the idea that only formal proofs are worthy of your attention” (Macdonald, 1986.P359). Writing Proofs plays a fundamental role in mathematics; it is necessary on the part of the teachers in promoting geometrical understanding. It is an important aspect in mathematical thinking, because proof is an important part of the mathematics curriculum at any stage of schooling.

Shatnawi (1982, p.6) defined Mathematical proof as “using logical evidence to show the correctness of an expression that follows from the proof of previous expressions”.

Example: Prove that $\sqrt{2}$ can not be expressed as a fraction (in other words a/b, where a and b are integers and $b \neq 0$).

Proof by contradiction: Let $\sqrt{2} = a/b$ where a and b are both integers, and the largest common factor is equal to 1. If we square both sides then $2 = a^2/b^2 \Rightarrow 2b^2 = a^2$, we know that a^2 must be even (because the double any integer number is an even number) and a is even number as well (because if the square of a number is even, then the number itself must be even). Since a is even it can be written in form $a = 2r$, where r is integer number, then $2b^2 = (2r)^2 = 4r^2$, divide both sides by 2 to get: $b^2 = 2r^2$, b^2 and b are both even numbers (the same arguments that a^2 and a are both even numbers). If this is the case, then it can be written as $b = 2k$, where k is an integer number. Then, $\sqrt{2} = a/b = 2r/2k$. However, this contradicts the original assumption that the largest common factor between a and b is equal to 1. This contradicts our assumption, then $\sqrt{2}$ is can not be expressed as a/b.

The researcher chose five items to measure Mathematical proof. The first item was chosen from Osbaldestin (2001c). In this item, the students use an understanding of proof to respond (direct proof). The next item was chosen from the British Broadcasting Corporation (BBC, 2001) website and required the student to use proof by contradiction. Item 3 was chosen from Challen *(*2001) and Item 4 was chosen from the TIMSS (1995, p.89). Items 3 and 4 required the students to use justification and proof for statements. Item 5 from the Shatnawi scale (1982) required the students to use their understanding of Pythagoras' theorem and logical relations. These items were designed to measure aspects of Mathematical proof and clearly measured Mathematical proof, because they required justification and proof, proof by contradiction, the use of axioms, direct proofs, and arguments. Thus, these responses indicate aspects of Mathematical proof.

3.2 ADMINISTRATION OF THE MATHEMATICAL THINKING TEST (MTT)[8]

The test of mathematical thinking was administered to 560 students (274 male students and 286 female students), who were in the Year 11 scientific stream in the Irbid governorate. The test took three hours, and there was a break in the middle of the test of 15 minutes. The MTT tested the six aspects of mathematical thinking, which included Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof.

[8] It should be noted that in this test and mathematics achievement test students were not allowed to use calculators in solving the problems.

Each of the six aspects was tested with five questions giving a total of thirty questions. Each item was scored from zero to three points, a maximum possible total of 15 for each aspect and an overall total of 90 (i.e. the range of scores was 0-90). The test was described in detail in a previous section.

3.3 MATHEMATICS ACHIEVEMENT SCALE

Mathematical thinking relates to the processes and specific mathematical abilities such as generalization, understanding of symbols, geometry, and logic that mathematics achievement depends upon. However, for the purposes of this study mathematics achievement will be measured on a single scale incorporating curriculum factors and reflecting school achievement tests and examinations conducted by national examination authorities. The content of the mathematics course for the Year 11 scientific stream in Jordan is closely based on the mathematics textbook published by Ministry of Education. The researcher analyzed the content of the mathematics textbook (first semester) for the Year 11 scientific stream, in Jordan (the textbook is common to all schools used in this study). The Mathematics achievement scale was prepared from the textbook (first semester) for Jordan, covering four topics: real numbers, exponents and logarithms, matrix and determinants, and methods and binomial theorem. The researcher chose items from mathematics books (see Table 3.2 following), and the researcher’s own experience. A copy of the mathematics achievement test can be found in Appendix 2.2 in both the Arabic and English languages.

3.3.1 Content of Mathematics Achievement Test

For real number, the researcher chose question no 1, items 1, 2, 3 from Johnson and Kiokemeister (1977, pp. 9, 6, &7, respectively). For exponents and logarithms, Question no 2, item 2 was chosen from Goodman and Ratti (1979, p.106). However, items no 3 and 5 were chosen from Johnson and Kiokemeister (1977, pp. 269, & 290, respectively). For matrix and determinants, Question no 3, item no 3, was adapted from Goodman and Ratti (1979, p.129). For methods and binominal theorems, Question no 4, item no 1, was adapted from Fouche (1997, p. 228). Other items were devised from the researcher's experience. The researcher sent the items three times to specialists in mathematics such as secondary teachers, supervisors, and specialists in the Ministry of Education for feedback by e-mail. Based on their comments, the researcher continually improved the test until a final version of the test was developed. The feedback received advised the exclusion of items because they were too difficult for students, and would not discriminate well between students, or because the items could easily be misunderstood or misinterpreted by students. As a result, the researcher with the assistance of such specialists changed four of the 17 items. (See Table 3.2).

TABLE 3.2. CONTENT OF MATHEMATICS ACHIEVEMENT

Design and content of mathematics achievement test	
Topic	Item no (source)
Real Number	1 Johnson & Kiokemeister. (1977) 2 Johnson & Kiokemeister. (1977) 3 Johnson & Kiokemeister. (1977) 4 Researcher's experience
Exponents and Logarithms	1 Researcher's experience 2 Goodman & Ratti. (1979) 3 Johnson & Kiokemeister. (1977) 4 Researcher's experience 5 Johnson & Kiokemeister. (1977)
Matrix and Determinates	1 Researcher's experience 2 Researcher's experience 3 Goodman & Ratti. (1979) 4 Researcher's experience
Methods and Binomial Theorems	1 Fouche. (1997) 2 Researcher's experience 3 Researcher's experience 4 Researcher's experience

In relation to what each question tested, topic 1 item 1 required the students to prove a specific theorem. However, the other items required the students to solve inequalities. In topic 2, Items 1 and 2 and topic 4, items 2 and 3 required the students to solve equations using logarithms, exponential laws, factorials and permutations. However, in topic 2, items 3, 4 and 5 required the students to use logarithms, exponential laws, and formula exponential functions to solve these problems without using calculators. Topic 3 item 1 requires the students to find some matrix operations. However, items 2 and 3 require the student to prove or disprove statements in the matrix by example, if the statement is false, or in general, if the statement is true. Finally, item 4 requires the students to solve the system of equations by Cramer's rule. Topic 4, item 1 requires the students to use combinations to find the number of games. However, some students solved the problem by searching for a pattern. Item 4 requires the students to express the specific summation using $\sum$. There are a range of topics used, with some items clearly dependent on mathematical thinking aspects. For example, topic 1, item 1 and topic 3, items 2 and 3, require Mathematical proof. Topic 4, items 1 and 3, require pattern-finding using generalization or induction aspects. Topic 1, items 2, 3 and 4, topic 3, item 4 and topic 4, item 4 require the using of algebra and symbols for the solution. In contrast, other items are less clearly related to the mathematical thinking aspects. However, almost every topic requires Logical thinking.

3.3.2 Administration of the Mathematics Achievement Test (MAT)

The administration of the MAT occurred 15 days after the MTT. In total, 543 students, 268 male students and 275 female students sat this test. In total 527 students attended the two tests (263 male students and 264 female students). The total possible score for MAT was 50.

3.4 POPULATION AND SAMPLE

Jordan has 12 governorates. The sample of schools was selected from the Irbid governorate for the following reasons. It is an important area of Jordan due to high levels of education in this area compared to other governorates. In addition, this location is close to the researcher's home, where the researcher has five years experience as a teacher. Students involved in the study were selected by a two-stage process. First, schools were selected from all government secondary schools in the Irbid governorate of Jordan which included the Year 11 scientific stream. The Year 11 class scientific stream students (17 years old) were selected because they can be assumed to be the most proficient in mathematical thinking. The same stream also requires the students to have a high achievement in scientific subjects, especially in mathematics. Thus, it could be expected that all the students in this stream have a reasonable competence in mathematics. As this study was designed to measure mathematical thinking ability, an investigation of mathematical thinking was best suited to this specific stream.

The Irbid governorate is comprised of six directorates:

1) Irbid First Directorate
2) Irbid Second Directorate
3) Ramtha Directorate
4) Koura Directorate
5) North Jordan Valley Directorate
6) Bani Kenanah Directorate

The second stage of sampling, consisted of selecting students from the selected schools. There were 5185 students in the Year 11 scientific stream in Irbid, 2753 (53%) male students and 2432 (47%) female students. The breakdown by gender and directorate is displayed in Table 3.3.

TABLE 3.3. NUMBERS OF STUDENTS BY GENDER AND DIRECTORAT

Gender / Directorate	Male	Female	Total	Proportion
Irbid First Directorate	1444	1225	2669	51.5%
Irbid Secondary Directorate	444	421	865	16.7%
Ramtha Directorate	233	193	426	8.2%
Koura Directorate	289	240	529	10.2%
North Jordan Valley Directorate	128	138	260	5%
Bani Kenanah Directorate	215	215	430	8.3%
Total	2753	2432	5185	100%

Note: These numbers were provided by Department of Statistics (Jordan) in each directorate in the academic year 2003/2004.

3.5 STUDY SAMPLE

The sample size required for this study was at least 500 students; 250 male students and 250 female students. As schools were not co-educational, selection of the two samples was effectively undertaken separately. The sample size was based on the intended use of linear regression analysis and a lower-bound estimate that a 10 point increase on the Mathematical achievement test (measured over the range zero to 50) could be related to all mathematical thinking factors combined (each measured over the range of zero to 15) in the Mathematical thinking test. This leads to a slope effect in linear regression of 10/15=0.67, rounded to 0.7. Assuming the strongest factor among all six mathematical thinking factors explains 50% of the variance in mathematical achievement, the slope for this factor is 0.35. Applying scenarios using the Power and Sample Size Program (Dupont & Plummer, 1997, 1998, pp. 599-600) with $\sigma=8$ ($1/6^{th}$ of the range of 50) for mathematical achievement and $\sigma_x=3$ ($1/6^{th}$ of the range of 15) shows that a sample size of 500 is the minimum likely to detect such difference, with type one error rate $\alpha=.05$ and power = 0.80.

Although the above estimate of likely relationships between the mathematical thinking factors and mathematics achievement may seem overly pessimistic, the effect on standard errors of measurement of the clustering of students into schools and classes must also be considered. It was also recognised that the magnitude of intra-class correlations may necessitate the use of multilevel regression techniques, but this could not be determined until initial data analyses were done. On balance, dividing the total sample of students approximately equally into male and female students was a prudent course to ensure that at least the major effects of mathematical thinking on mathematics achievement from a gender perspective would be discernible.

In the Irbid governorate there were 121 secondary schools which included the Year 11 scientific stream; 55 schools for male students and 66 schools for female students. These schools contain 90 male classes and 91 female classes, indicating that some schools contain more than one class (12 male schools have more than one class and 13 female schools have more than one class). The researcher numbered the male classes from 1-90 and the female classes from 91-181. The mean number of students per class was approximately 30. Therefore, the required number of schools was at least 20, composed of nine schools to gain a sample of 250 male students and 11 schools to gain a sample of 250 female students. The nine male and 11 female schools were all selected randomly. Table 3.4 shows the number of students in each class (school), because one class was chosen from each school. Five out of six directorates were represented in the sample. All students in secondary public schools are taught by teachers of the same gender throughout Jordan.

Table 3.4 below indicates that 9 out of the 20 schools selected belong to Irbid First Directorate (45%), then Irbid Secondary Directorate had 4 out of the 20 (20%), Bani Kenanah Directorate and Koura Directorate had 3 schools each out of the 20 (15%). However, no school was selected from North Jordan Valley Directorate due to its small size, representing only 5% of the population.

TABLE 3.4. NUMBER OF STUDENTS IN THE CLASSES, TYPE OF SCHOOL, AND DIRECTORATE.

School No	Number of students	Type of school (male or female)	Directorate
1	24	F	Irbid First Directorate
2	31	M	Irbid First Directorate
3	24	F	Koura Directorate
4	36	M	Irbid First Directorate
5	19	F	Irbid Secondary Directorate
6	28	M	Irbid Secondary Directorate
7	32	F	Irbid First Directorate
8	30	M	Koura Directorate
9	24	M	Irbid Secondary Directorate
10	35	F	Irbid First Directorate
11	29	M	Irbid First Directorate
12	32	M	Koura Directorate
13	15	F	Irbid Secondary Directorate
14	40	M	Irbid First Directorate
15	21	F	Ramtha Directorate
16	43	F	Irbid First Directorate
17	23	F	Bani Kenanah Directorate
18	24	M	Bani Kenanah Directorate
19	21	F	Bani Kenanah Directorate
20	29	F	Irbid First Directorate

The researcher defined urban schools as all schools which were situated in the centre of the city, and suburban schools as those 10 km or less from the city, but not in the centre. Otherwise, schools were considered rural schools. Across the 20 selected schools in this study, there were seven urban schools (four female schools and three male schools), four suburban schools (two female schools and two male schools), and nine rural schools (five female schools and four male schools). Most schools that belong to Irbid First Directorate are urban schools (7 out of 9), and the others are suburban schools. Half of the schools in Irbid Second Directorate belong to suburban and the other half to rural, but all schools that belong to other directorates are rural schools. The breakdown of schools by location and gender is displayed in Table 3.5.

TABLE 3.5. SCHOOLS AND STUDENTS BY LOCATION AND GENDER FOR THE MATHEMATICAL THINKING TEST.

Location / Gender	Urban schools (Students)	Suburban schools (Students)	Rural schools (Students)	Total schools (Students)
Male	3 (105)	2 (55)	4 (114)	9 (274)
Female	4 (139)	2 (43)	5 (104)	11 (286)
Total	7 (244)	4 (98)	9 (218)	20 (560)

Schools that are situated in urban areas have a high density population. Schools situated in suburban areas have medium density. In contrast, students in rural schools come from a low density population.

Students who did not wish to participate were given mathematics problems to solve by their teachers during the data collection. Approximately 60 students in total, on average three students per class, took this option. However, there were 16 students absent from schools on the day the mathematical thinking test was administered (approximately one student per class), and 33 students did not wish to participate or were absent from schools the day the mathematics achievement test was administered. However, the achieved sample size was 560 for the test of mathematical thinking and 543 for the test of mathematics achievement.

3.6 INSTRUMENTS AND ADMINISTRATION

This research involved the collection of both quantitative and qualitative data. The first type of data was obtained through the administration of two tests: Mathematical thinking and the test of mathematics achievement as described earlier. The qualitative data was obtained from interviews with teachers and students as described below.

3.6.1 Interviews with Students

These interviews took place after the test of mathematical thinking was administered. It took 30 minutes to interview each group of students who consented to participate in discussing the strategies they used to answer the questions in the test of mathematical thinking, and in describing how they reached their answers. Each interview was comprised of five questions. Three questions were the same for each group, one induction question was discussed, item 3 from the test, one Use of Symbols question, item 5 from the test, and one Mathematical proof question, item 4 from the test. These three items were chosen because the nature of the items meant they were more likely to generate discussion. The remaining two questions utilized in each group interview were derived from a variety of items in the test. They were chosen to expose the interviewees to a greater variety of items in the hope of generating richer data, however, the remaining two interview questions utilized in the first and second groups were the same. The reason for using the same two questions for the first and second groups was to enable a gender comparison because the first group was female and the second group male. There were four different groups from different schools (two male groups from two male schools and two female groups from two female schools). Each group contained five students. The teacher advised the researcher about who should be selected for interview from those students who had previously given their permission, to make sure each group had students with a range of mathematical performance, and were designed to show different ways of thinking in mathematics. The interviews took place in the library, and were audio recorded.

3.6.2 Interviews with Teachers

The teacher interviews took place after the test of mathematical thinking. It took 30 minutes with male teachers, but the female teachers preferred to take the interview questions to answer in their homes after explanation of these questions for them by the researcher. They then returned their answers to the researcher and discussed any issues that had arisen when they returned their responses. The participants were teachers who were prepared and interested to take part in the interview to discuss how they taught mathematical thinking and its aspects in their schools, as well as how many weeks or hours or percentages of time they spent teaching mathematical thinking for the Year 11 scientific stream in Jordan. The male teachers' interviews were recorded in the library after the students' interview. It is recognised that the different data-gathering approaches may affect the nature of responses obtained. A copy of questions of teacher interviews can be found in Appendix 2.3 in both Arabic and English languages.

3.7 PROCEDURES

A number of public secondary schools which taught the Year 11 scientific stream in Irbid region were approached to participate in this study. All schools were single-sex in all directorates. This study was approved by the Ministry of Education, and permission was granted to conduct the study in its schools by the Director of Research Administration and Educational Development. The researcher approached the schools and explained the aim of the study and its procedures to the school principal, teaching staff and students. Each principal was then asked for permission to conduct the study. Consent forms were then disseminated to participants. A copy of the consent forms can be found in Appendix 2.4 in both Arabic and English languages. Participants who gave permission by signing the form were also informed that they were under no obligation to participate and that they could withdraw from the study at any time without giving any reason, as required by the Ethics Committee of the University of Newcastle.

The researcher approached the teachers and students and explained the nature of the teacher interview, that it would take about 30 minutes, and that there were two tests; a test of mathematical thinking and, after a period of time ranging from two weeks to one month, a mathematics achievement test. The mathematical thinking test would take 3 hours; however, the mathematics achievement test would only take 2 hours and was designed from the mathematics curriculum. The mathematical thinking test was followed by a 30 minute recorded group interview with approximately 6 students including 2 students from each mathematical achievement (high, middle, low). The researcher informed the participants that if they had any questions related to this study, they should feel free to ask. Confidentiality of all information was assured, and access to this information was restricted to the researcher and his supervisors. In addition, data were coded so participants could not be identified on answer sheets or on the tape. The researcher emphasized that he could provide school principals with a report of the results as a group, not individual students results. All information letters (principals, teachers, students) can be found in both Arabic and English languages in Appendix 2.5.

There was an instructions page preceding the test of mathematical thinking and test of mathematics achievement to explain to participants how to answer the questions and the time allowed for each test. A copy of the instructions page can be found in Appendix 2.1 and 3.2. The researcher then thanked principals, teachers, and students for their contribution to the success of the study.

Data collection for this study took approximately four months from December 2003 to March 2004. All collection of data was conducted by the researcher to ensure the research proceeded as planned. All tests and interviews were administered during the normal school day.

3.8 STUDY QUESTIONS

This study will attempt to answer the following questions which are reported here.

1) What is the relationship between mathematical thinking and mathematics achievement for the Year 11 scientific stream in Jordan?

2) Do male and female students differ in mathematical thinking and mathematics achievement?

3) Is there an interaction effect of individual schools on mathematical thinking and mathematics achievement?

4) Do urban, suburban, and rural students differ in mathematical thinking and mathematics achievement?

5) Is there an interaction effect of gender and school location on mathematical thinking scales and mathematics achievement?

6) What are the significant effects on mathematics achievement, and what is the relative importance of these effects?

7) What the inconsistencies and consistencies between the teachers' opinions about the importance and difficulty of the aspects of mathematical thinking?

8) Are there differences in mathematical thinking for students of different ability and of different gender? Are the students familiar with solving specific problems (such as rice problem) in scientific ways like searching for patterns rather than more classical methods? Are the students using the fourth step in problem solving according to Polya (1990) (i.e., a checking the answer)?

3.9 SUMMARY

This chapter first focused on describing the development of the instruments for measuring mathematical thinking and mathematics achievement. This was followed by a description of the two tests of mathematical thinking and mathematics achievement. The population and the sample were then described with explanations as to why the researcher chose the specific area and how the sample was chosen from the population. This information was followed by information on the administration of the two tests of mathematical thinking and mathematics achievement and teacher and student interviews, with full explanation of how the study was carried out. The next chapter will outline the descriptive data, tests of reliability for each individual mathematical thinking scale and the overall scale, and the same tests for validity. Some statistical analyses used in the development and testing of the scales is also reported in Chapter four.

CHAPTER FOUR

THE TESTS: SCORING, RELIABILTY, AND VALIDTIY

In this chapter an overview of the instruments used to measure mathematical thinking and mathematics achievement will be provided, including a discussion of test reliability. Indications of construct and content validity of the tests will also be provided.

Scoring, Reliability and Validity of the Instruments

4.1 MATHEMATICAL THINKING TEST

The test comprised 30 items grouped into six subtests, each of five items, assessing different aspects of mathematical thinking. From a total of 576 students involved in at least one aspect of the study, 560 students attempted the mathematical thinking test. These students comprised the sample used in calculating the item statistics and the reliability of this test.

4.1.1 Scoring the Mathematical Thinking Test

During the marking of the test it became evident that not all the students attempted every item in the test, therefore a score of zero was given to each of the items not attempted by students who were present for the test and who responded to other items. The assumption here was that students who did not attempt a particular item, could not do the item at all. In some cases this was particularly obvious because the student then did attempt to answer all subsequent items. In other cases it was not so obvious, particularly for students who were weaker overall and may have failed to respond to several items in the test. This scoring decision, in addition to giving effect to the most likely explanation for missing responses, had the benefit of simplifying the analyses when the mathematical thinking test was broken into six constituent subtests, as the subtests then had the same number of students involved in each analysis.

For all attempted items the scores allocated for each item ranged from 0 to 3. Partially correct answers were recognised according to strict criteria. The criteria are now described, first for the multiple choice items and second for all other items which involved extended responses. A copy of the mathematical thinking answers can be found in Appendix 3.1.

4.1.1.1 Multiple-Choice Items

In addition to selecting the correct response, for each of these items the students were asked to justify their selection, giving reasons why the answer they selected was the correct one. For the precise nature of the task, the reader may refer to the instruction page for the test, shown in Appendix 2.1. This allowed the researcher to allocate a score for each item out of 3 at intervals of 0.5. A copy of the rubric for only one multiple-choice item can be found in Appendix 3.2.

The scoring for the multiple-choice items was as in Table 4.1 below.

TABLE 4.1. ITEM RESPONSE RUBRIC FOR SCORENG THE MATHEMATICAL THINKING TEST (MULTIPLE-CHOICE ITEMS).

Response score	Description
3	Correct choice with correct justification
2.5	Student gave a strong justification that was not totally correct, and then chooses the correct answer.
2	Student made a correct choice with a weak, partly correct justification.
1.5	Correct choice without justification
1	Correct choice with incorrect justification. Or, student justifies why two choices are incorrect, without choosing the correct answer. Or incorrect choice with a plausible justification.
.5	Student just said one or two of the choices are wrong without justification.
0	Completely incorrect, irrelevant, incoherent or no answer.

4.1.1.2 Extended-Response Items

Again the researcher was able to allocate a score for each item in the range 0 to 3, at 0.5 intervals. A copy of the rubric for only one extended response item can be found in Appendix 3.2.

The scoring schedule for these items was as in Table 4.2.

TABLE 4.2. ITEM RESPONSE RUBRIC FOR SCORING THE MATHEMATICAL THINKING TEST (EXTENDED-RESPONSE ITEMS).

Response score	Description
3	Complete and correct. The response demonstrates a thorough understanding of the mathematical concepts and/or procedures embodied in the task. It indicates that the student has completed the task, showing mathematically sound procedures or contains complete clear, explanations and / or adequate work when required.
2.5	Correct, complete and clear explanation, without adequate work when required.
2	Only partially correct and shows only partial understanding of mathematical concepts with some elements of the task correct but may be incomplete.
1.5	Correct solution without showing mathematically sound procedures. Or solve exactly half problem such as solve one unknown from two.
1	Incorrect answer, but with an explanation indicating a correct understanding of some of the mathematical concepts.
0.5	Incorrect solution but applies a mathematically appropriate process.
0	Completely incorrect, irrelevant, incoherent or no answer.

4.1.2 Test Reliability

Given that there was only a maximum of five items in each subtest and each subtest needed to cover a range of skills, individual subtest reliabilities were low, as could be expected. However, the overall test reliability (the alpha coefficient) for the test of the 29 retained items was satisfactory at 0.829 (SPSS, version 12 was used in all statistical analyses throughout the thesis unless another specific program was named). Item 5 in the Logical thinking scale was deleted because it decreased the reliability of the scale considerably. The nature of item 5 in Logical thinking was (if----, then----) to measure Logical thinking in general terms, whereas the other four items were measures of Logical thinking in mathematics. Before item 5 on Logical thinking was deleted the reliability for mathematical thinking test was 0.824. The reliability for each of the mathematical thinking scales, if any item were deleted, can be found below when the results for each subtest are discussed. The reliability for mathematical thinking test as a whole, if any individual item were deleted, can be found in Appendix 3.3.

4.1.3 Distribution of Item and Test Scores

This section will summarize the mean and standard deviation (SD) for each item in each scale. Also included is the reliability of each scale and with the reliability that would eventuate, if any item were deleted from the scales. In addition, this will be followed by a brief description of the score range across items. The maximum possible score for each item in the test was 3. Tables 4.3a-4.3f summarize the mean and SD for each item in each scale together with the score range and the reliability of each scale. All information about facility and discrimination for each individual item for the mathematical thinking test can be found in Appendix 3.4

Generalization:

Items scores for Generalization had means ranging from 1.1 to 2.6 with an overall mean of 8.0 out of 15. Item G1, which required the students to know the relationship between two variables by searching for a pattern which would satisfy all cases, was clearly the easiest item. Three items (G2, G3, and G4) were equally difficult with mean scores slightly more than 1. All items were retained and the scale reliability was 0.564. All information about the items in the Generalization subtest is in Table 4.3a following.

TABLE 4.3a. GENERALIZATION SCALE (G)

Generalization	Mean	SD	Min score awarded	Max score awarded	Scale reliability if item deleted
G1	2.6	0.90	0	3	0.508
G2	1.1	1.42	0	3	0.517
G3	1.2	0.98	0	3	0.515
G4	1.1	1.10	0	3	0.516
G5	1.9	1.31	0	3	0.487
Overall scale	8.0	3.50	0	15	0.564

Induction:

Items scores for Induction had means ranging from 1.2 to 2.2 with an overall mean of 8.5 out of 15. Item I2, which required the students to find the tenth number from a group of numbers using the relationship between the following two numbers, was clearly the easiest item. Three items (I1, I4, and I5) were approximately equally difficult with mean scores slightly more than 1.5. All items were retained and the scale reliability was is 0.607. All information about the items in the Induction subtest is in Table 4.3b.

TABLE 4.3b. INDUCTION SCALE (I)

Induction	Mean	SD	Min score awarded	Max score awarded	Scale reliability if item deleted
I1	1.8	1.34	0	3	0.537
I2	2.2	1.14	0	3	0.541
I3	1.2	1.12	0	3	0.486
I4	1.7	0.88	0	3	0.534
I5	1.6	1.44	0	3	0.513
Overall scale	8.5	3.75	0	15	0.607

Deduction:

Items scores for Deduction had means ranging from 1.2 to 1.9 with an overall mean of 7.3 out of 15. Item D3, which required the students to arrive at a valid conclusion from true premises and was a practical problem, was the easiest item. Three items (D2, D4, and D5) were approximately equally difficult with mean scores slightly less than 1.5. In contrast, item D2 was the most difficult item because it was perceived as easy to understand but involved challenge and careful consideration for most of the students. Item D5 was the next most difficult with only slight difference from item D2. All items were retained and the scale reliability was 0.578. All information about the items in the Deduction subtest is in Table 4.3c.

TABLE 4.3c. DEDUCTION SCALE (D)

Deduction	Mean	SD	Min score awarded	Max score awarded	Scale reliability if item deleted
D1	1.6	1.41	0	3	0.537
D2	1.2	1.43	0	3	0.541
D3	1.9	1.36	0	3	0.486
D4	1.4	1.41	0	3	0.534
D5	1.3	1.44	0	3	0.513
Overall scale	7.3	4.30	0	15	0.578

Use of Symbols:

Items scores For Use of Symbols had means ranging from 1.0 to 2.0 with an overall mean of 7.4 out of 15. Item S3, where the students were required to analyze the shape of the area and the total area, was clearly the easiest item, because it was considered a routine item for the student. Mathematical curricula are rich in such items. In contrast, Item S2 was the most difficult because it required students to use mathematical symbols to solve a geometrical algebraic problem using one variable only to express the circle equation. Three items (S1, S4, and S5) were approximately equally difficult with mean scores approximately 1.5. This subtest was moderately difficult and close to the Deduction subtest in the relation to difficulty level. All items were retained and the scale reliability was 0.644. All information about the items in the Use of Symbols subtest is in Table 4.3d.

TABLE 4.3d. USE OF SYMBOLS SCALE (S)

Use of Symbols	Mean	SD	Min score awarded	Max score awarded	Scale reliability if item deleted
S1	1.5	0.96	0	3	0.566
S2	1.0	1.13	0	3	0.604
S3	2.0	1.23	0	3	0.584
S4	1.3	1.47	0	3	0.651
S5	1.6	1.00	0	3	0.555
Overall scale	7.4	3.76	0	15	0.664

Logical thinking:

Items scores for Logical thinking had means ranging from 0.9 to 2.7 with an overall mean of 7.3 out of 12. This scale was the easiest subtest. Item L3 and L4, which required the students to use their understanding of the truth tables without using these tables, were clearly the easiest items. Two items (L1and L2) were approximately equally difficult with mean scores around 1. These items required the students to use logical understanding to negate statements, and prove symmetry using Venn diagrams.

The Logical thinking test initially had 5 items, but item L5 was removed to increase the test reliability. The reliability for the Logical thinking scale was 0.523 initially and 0.649 after removing item L5 (see pp. 72-73). Finally four items were retained and scale reliability was 0.649. All information about the items in the Logical thinking subtest in Table 4.3e.

TABLE 4.3e. LOGICAL THINKING SCALE (L)

Logical thinking	Mean	SD	Min score awarded	Max score awarded	Scale reliability if item deleted
L1	0.9	1.15	0	3	0.442
L2	1.2	0.95	0	3	0.410
L3	2.7	0.81	0	3	0.412
L4	2.5	1.04	0	3	0.416
L5	1.9	1.40	0	3	0.649
Overall scale	7.3	2.77	0	12	0.649

Mathematical proof (M):

Items scores for mathematical proof had means ranging from 0.2 to 1.5 with an overall mean of 5.0 out of 15. Mathematical proof was a very hard aspect as was generally evident through the students' results, with item M3 the hardest item throughout the mathematical thinking test. This item required the students to use justification, and to use specific theorems and make a connection between them to prove the theorem. In contrast, item M2 was clearly the easiest, in this subtest, although still a difficult item overall. It required the students to use proof by contradiction, representing a routine problem. All items were retained and scale reliability was 0.603. All information about the items in the mathematical proof Induction subtest is in Table 4.3f.

TABLE 4.3f. MATHEMATICAL PROOF SCALE

Mathematical proof	Mean	SD	Min Score Awarded	Max Score Awarded	Scale reliability if item deleted
M1	0.9	1.20	0	3	0.530
M2	1.5	1.19	0	3	0.539
M3	0.2	0.65	0	3	0.569
M4	1.1	1.20	0	3	0.508
M5	1.1	1.20	0	3	0.590
Overall scale	5.0	3.62	0	15	.603

4.1.4 Test Validity

4.1.4. 1 Content validity

To satisfy the requirements of content validity, the researcher chose mathematical thinking items from standardized tests such as the Third International Mathematics and Science Study (TIMSS), tests designed to measure mathematical thinking ability, specialist books in mathematical education, the internet, researcher experience, and from the Shatnawi (1982) scale. In addition, the researcher trialed the mathematical thinking test the year before administering it in the main study with about 30 students comprising two genders from the Year 11 scientific stream in the same area in Jordan. He then omitted some of the items from the mathematical thinking test based on misunderstandings on the part of students, level of item difficulty, item discrimination that was either too low or too high, and teacher advice. Thus, the final form of the test was based on expert opinion and trial testing.

4.1.4. 2 Construct validity

The mathematical thinking test was designed so that each item measured one of the six identified aspects of mathematical thinking. This was tested using principal components factor analysis with a Varimax rotation. In general the items did each load on a single factor indicating good overall construct validity. However, there were four items of the 29 items which cross loaded onto other aspects with a factor loading greater than 0.3.

The first cross-loading item, (item 2) in the Generalization scale also loaded on Use of Symbols. Some aspects of Generalization and Use of Symbols are closely related to each other. In most mathematical problems that require the student to find patterns, the student used mathematical symbols. Item 2 clearly requires the finding of a pattern through Generalization using mathematical symbols (numbers). In addition, in Generalization item 3 there was cross loading with Logical thinking, which requires the students to use logic to arrive at the correct Deduction. Item 3, requires the student to know each even number can be expressed as the summation of two prime odd numbers (this is specific cases), not only as the summation of the odd number. Also, the student is required to know there is at least one expression for each even number. These concepts (‘specific case’, and ‘at least’) are logical concepts and need the Logical thinking.

In Use of Symbols, item 2 cross loaded with Mathematical proof, because this item contains geometrical concepts such as unit circle, straight line, and slope, requiring the students to use symbols to express circle equations. In addition, in Use of Symbols item 3 there was cross loading with Generalization, because item 3 is closely related to Generalization and Use of Symbols aspects. Item 3, requires students to analyse the square shape and find the $(x+2)^2$. Some of students used specific numbers rather than the (x), then put (x) in the place of specific numbers. These aspects clearly refer to Generalization aspects. Table 4.4 shows the loading for each item on each factor if the value was greater than 0.250

However, when each scale is applied individually, each item in each scale was significantly loaded on the appropriate factor, showing that these items correctly measured the intended mathematical thinking scale. A copy of factor analysis for each scale of mathematical thinking can be found in Appendix 3.5.

TABLE 4.4. FACTOR ANALYSIS FOR MATHEMATICAL THINKING TEST

Item No	Component					
	Mathematical proof (M)	Induction (I)	Deduction (D)	Use of Symbols (S)	Logical thinking	Generalization (G)
M4	.617					
M2	.574					.264
M3	.556		.272			
L1	.535				.361	
M5	.523					
M1	.515					
S2	.440		.252			
I1		.660		.267		
I5		.646				
I3		.572		.278		
I4	.338	.551				.318
I2		.469			.268	.277
D3			.651			
D1			.635			
D4			.621			
D2			.472			
D5			.454	.311		
S4				.732		
S1				.548		.370
S5	.337			.526		
G2				.297		
L4					.770	
L3					.751	
L2	.336				.467	
G3				.282	.315	.266
G5						.650
G1						.577
G4	.327					.454
S3	.302			.291		.322

4.2 MATHEMATICS ACHIEVEMENT TEST

4.2.1 Scoring the Test

The mathematics achievement test consisted of 17 items, which covered four areas: real numbers, exponents and logarithms, matrix and determinants, and methods and binomial theorem, with a maximum score range of from 2 to 4 points for each item. The differential scoring reflects how many steps each item needs for solution with a consideration of the approximate time that students would need for each item. A copy of the rubric for scoring the mathematics achievement item can be found in Appendix 3.2. The researcher allocated a score for the items as shown in Table 4.5 below.

TABLE 4.5. ITEM RESPONSE RUBRIC FOR SCORING THE MATHEMATICS ACHIEVEMENT TEST.

Response score	Description
A full point response	Complete and correct. The response demonstrates a thorough understanding of the mathematical concepts and/or procedures embodied in the task. It indicates that the student has completed the task, showing mathematically sound procedures or contains complete clear, explanations and / or adequate work when required.
75 % point response	Correct, complete and clear explanation, without adequate work when required.
66% point response	Only partially correct and shows only partial understanding of mathematical concepts with some elements of the task correct but may be incomplete.
50% point response	Correct solution without showing mathematically sound procedures.
33% point response	Incorrect answer, but with an explanation indicating a correct understanding of some of the mathematical concepts.
25% point response	Incorrect solution but applies a mathematically appropriate process.
0 point response	Completely incorrect, irrelevant, incoherent or no answer.

4.2.2 Distribution of Item and Test Scores

The mathematics achievement test is shown in Appendix 2.1. As described in Chapter 3 section 3.2 this test was closely based on the mathematics curriculum applying in Jordan as exemplified in the text book. A summary of the mean scores and other item statistics is shown in Table 4.6.

TABLE 4.6. DESCRIBUTION MATHEMATICS ACHIVEMENT ITEMS

Item No	Mean score achieved	SD	Min score	Max score possible	Scale reliability if item deleted
Q1a	1.7	1.32	0	3	.815
Q1b1	2.7	0.73	0	3	.817
Q1b2	2.2	0.87	0	3	.813
Q1b3	1.2	1.07	0	3	.812
Q2a1	1.3	0.69	0	2	.810
Q2a2	.3	0.44	0	2	.816
Q2b	2.7	1.41	0	4	.804
Q2c	1.2	0.86	0	3	.807
Q2d	1.5	1.11	0	3	.807
Q3a	3.2	0.88	0	4	.811
Q3b1	.6	0.72	0	2	.812
Q3b2	.4	0.65	0	2	.818
Q3c	1.9	1.11	0	4	.804
Q4a	2.0	1.7	0	4	.824
Q4b1	.6	0.56	0	2	.811
Q4b2	1.1	0.85	0	2	.815
Q4c	2.1	1.32	0	4	.813
Test	27.0	8.79	0	48.5	.822

Mathematics achievement items scores had means ranging from 0.3 to 3.2. For each item, the mean performance ranged from 15% to 90% correct. The most difficult item was Q2a2 which required the student to solve an equation using exponential laws. This item is a non-routine item, and requires metacognitive processes such as the student's ability to solve a complex exponential equation, and then change it to quadratic equation to facilitate this problem. Next in level of difficulty was item Q3b2 which required students to prove or disprove a statement by example, if the statement is false, or in general, if the statement is true. This item is considered a puzzle problem, because most students believe this statement is correct due to the fact that this statement is true numerically, but it is not necessarily correct in matrix operations. Item Q1b1, which required the students to solve inequalities in the first degree, and all the problems in the first degree of equations and inequalities were the easiest problems. However, this specific type of item was often a standard class exercise and was clearly the easiest item, followed by item Q3a, which required students to find out some algorithm operations for a simple matrix, and is considered a routine item. That means the students had also faced many problems like this in their classes. Eight items (Q1a, Q1b3, Q2c, Q2d, Q3c, Q4a, Q4b2, and 4c) were about moderate in difficulty with performance ranging from 40% to 50% success. According to Cronbach's Alpha the reliability was .822 for mathematics achievement. Almost all items contributed to this high reliability coefficient, the exception being item 4a. All information about facility and discrimination for each individual item for the mathematics achievement test can be found in Appendix 3.6

4.2.3 Test Validity

4.2.3.1 Content validity

To satisfy the requirement of content validity, the researcher sent the items to specialists in mathematics such as secondary teachers, supervisors, and specialists in the Ministry of Education for feedback. Some of their comments related to unclear language or need for specific changes. In addition, some items were too difficult for the students or needed information from later classes. Based on their comments, the researcher reworded some items, omitted other items, and continually improved his test until a final version of the test was developed. This test was designed to measure general abilities in mathematics consistent with the Year 11 scientific stream syllabus in Jordan, that is, it was different to the mathematical thinking test that was designed to measure specific abilities not limited to the Year 11.

4.2.3.2 Construct validity

Each item of the mathematics achievement test has a significant factor loading on a single scale which indicates all items belong to one scale (mathematics achievement). Only one item out of 17 items had a weaker loading, being less than 0.4 (0.379), and it was considered acceptable. Table 4.7 shows in descending order the factor loading for each item in the mathematics achievement test.

TABLE 4.7. FACTOR LOADINGS FOR MATHEMATICS ACHIEVEMENT TEST

Item	Loading
Q3d	.635
Q4b	.634
Q2c	.625
Q2d	.622
Q2a	.593
Q2e	.591
Q3b	.547
Q3a	.533
Q1d	.522
Q4d	.518
Q2b	.518
Q1c	.499
Q1a	.496
Q4a	.467
Q4c	.460
Q1b	.413
Q3c	.379

4.3 STUDY QUESTIONS

This study will attempt to answer the following questions:

1) What is the relationship between mathematical thinking and mathematics achievement for the Year 11 scientific stream in Jordan?

The relationships between the aspects of mathematical thinking will be examined using Pearson product-moment correlation coefficients. The relationships of the aspects of mathematical thinking and total mathematical thinking score and mathematics achievement will also be examined using the same analyses.

2) Do male and female students differ in mathematical thinking and mathematics achievement?

As all schools involved in this study were single-sex, analyses will be considered separately. The male and female students' mean scores on each of the six aspects of mathematical thinking and the total scores of mathematical thinking and mathematics achievement will be compared using t-tests.

3) Is there an interaction effect of individual schools on mathematical thinking and mathematics achievement?

An ANOVA will be conducted in turn with each of the six aspects of mathematical thinking as the dependent variable and school as the independent variable. For completeness, the same analysis will be performed for each of the aspects and the total score for mathematical thinking, and for mathematics achievement.

4) Do urban, suburban, and rural students differ in mathematical thinking and mathematics achievement?

The mean scale scores of mathematical thinking and mathematics achievement for each of the three locations (urban, suburban, and rural) will be compared. A series of one-way analyses of variance will be used with a Scheffe test to assist in distinguishing individual locations that differed from each other. Several ANOVAs will be conducted to determine any significant differences between locations and the different aspects of Mathematical thinking, and Mathematical thinking (total). Location difference for mathematics achievement will also be examined.

5) Is there an interaction effect of gender and school location on mathematical thinking scales and mathematics achievement?

Two-way analyses of variance will be undertaken with the mathematical thinking scales (Generalization, Induction, Deduction, Use of Symbols, Logical thinking, Mathematical proof, and mathematical thinking (total)) and mathematics achievement as dependent variables with gender and location as independent variables.

In order to test the strengths of the relationships of each aspect of mathematical thinking with mathematics achievement, all six aspects of mathematical thinking will be entered into a regression equation with mathematics achievement as the dependent variable. The relative importance of each aspect and the overall importance of mathematical thinking for mathematics achievement will be estimated.

6) What are the significant effects on mathematics achievement, and what is the relative importance of these effects?

These questions involve testing the complete model. Gender and school location will be included in a model with the six aspects of mathematical thinking as independent variables, with mathematics achievement as the dependent variable. Figure 4.1 following shows the complete model that linked sequentially gender and location as background variables, the six aspects of mathematical thinking, and mathematics achievement.

7) What the inconsistencies and consistencies between the teachers' opinions about the importance and difficulty of the aspects of mathematical thinking?

The researcher will examine whether there are any inconsistencies or consistencies between the teachers' opinions of the aspects of mathematical thinking in relation to level of importance and level of difficulty, and the results derived from the quantitative analyses of the students' responses to the mathematical thinking test.

8) Are there differences in mathematical thinking for students of different ability and of different gender? Are the students familiar with solving specific problems (such as rice problem) in scientific ways like searching for patterns rather than

more classical methods? Are the students using in the fourth step in problem solving (i.e., a checking the answer)?

The researcher will answer the questions using student interviews.

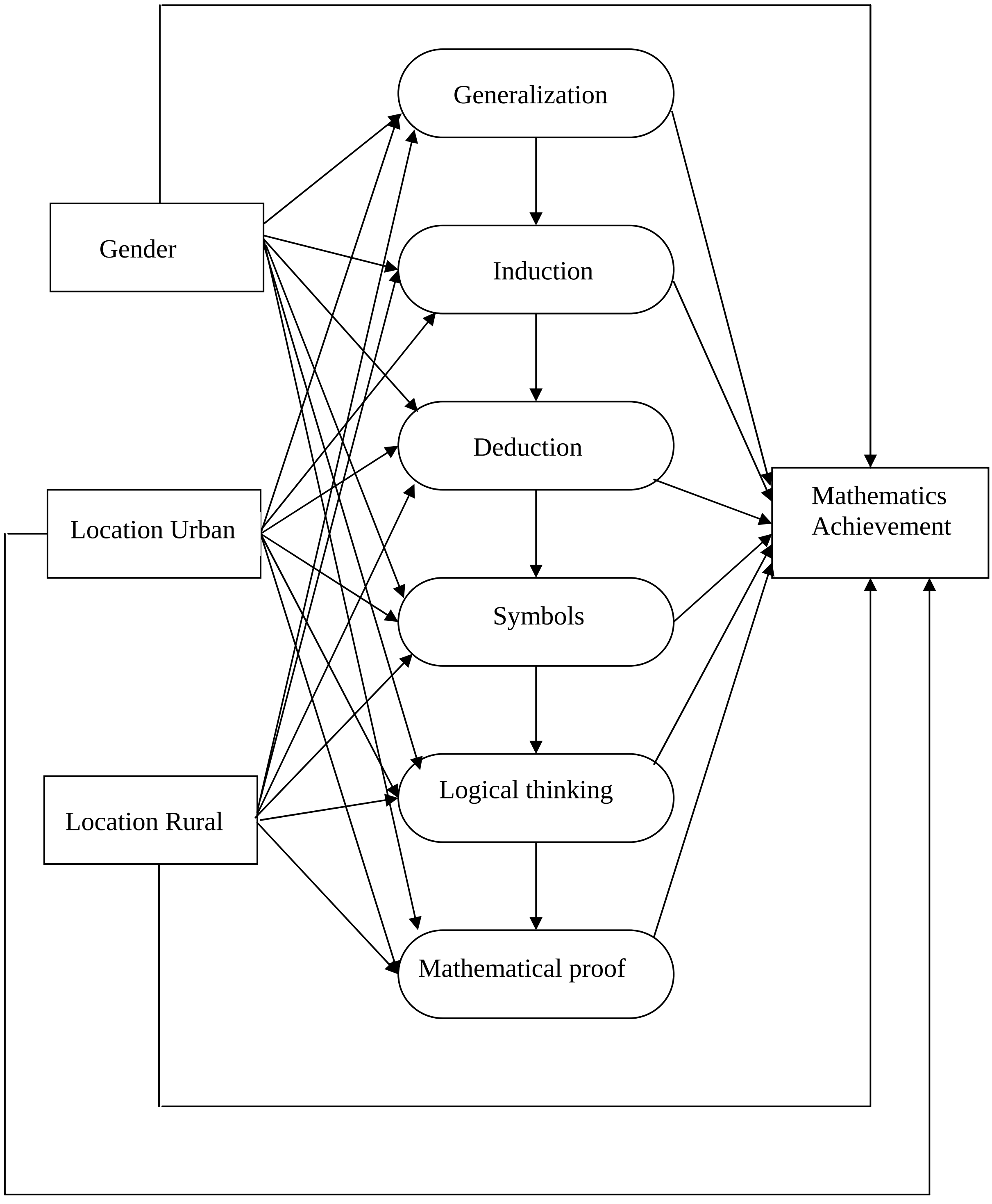

FIGURE 4.1. THE COMPLETE MODEL LINKING BACKGROUND VARIABLES WITH MATHEMATICAL THINKING AND MATHEMATICS ACHIEVEMENT

4.4 SUMMARY

This chapter focused on presenting the descriptive test data, tests of reliability for each individual scale and the overall scale, and the same tests for validity. Some analyses were also carried out in this chapter to determine test reliability and validity. The next chapter will provide an analysis of the quantitative data, in relation to the test of mathematics achievement and the test of mathematical thinking, including the six aspects of mathematical thinking (Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof). Student performances on these aspects of mathematical thinking are compared on a range of variables including gender, school, and school location and subsequently, the simple descriptive analyses and multivariate analyses are described.

CHAPTER FIVE

MATHEMATICAL THINKING AND ACHIEVEMENT

This chapter will provide an analysis of the quantitative data, in relation to the test of mathematical thinking and the test of mathematics achievement, including the six aspects of mathematical thinking: Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof. Student performances on these six aspects of mathematical thinking will be compared on a gender, school, and school location. Following the simple descriptive analyses, multivariate analyses are described. These analyses are used to test the model shown in Figure 5.2 (p.100).

A range of statistical tests will be used in this chapter. In each case, unless otherwise specified, where appropriate, a two-tailed significance test using the 0.05 probability level will determine statistical significance.

5.1 RELATIONSHIPS BETWEEN SCALES

As an initial step, the relationships between the six aspects of mathematical thinking were examined using Pearson product-moment correlation coefficients. The relationships of each of the six aspects and the total score for mathematical thinking and mathematics achievement were also examined. The coefficients are shown in Table 5.1.

Although all coefficients are statistically significant, there is a notable difference between the levels of the correlations within the six individual aspects, which range from 0.17 to 0.45, with a mean of 0.33. The level of correlation within the six aspects indicates a clear relationship, but also a clear indication that each of these scales is measuring something largely independent of the other five scales. Even the two scales with the highest correlation (Generalization and Use of Symbols – correlation 0.45) are largely independent with only 20 per cent of their variance in common. The correlations of the six aspects with the total scores for mathematical thinking and mathematics achievement range from 0.45 to 0.74, with a mean of 0.61.

TABLE 5.1. CORRELATION MATRIX FOR MATHEMATICAL THINKING SCALES, MATHEMATICAL THINKING (TOTAL) AND MATHEMATICS ACHIEVEMENT (MA).

	General-ization	Induction	Deduction	Use of Symbols	Logical thinking	Mathematical proof	MT	MA
Generalization	1							
Induction	.336	1						
Deduction	.301	.167	1					
Use of Symbols	.452	.345	.364	1				
Logical thinking	.397	.327	.251	.376	1			
Mathematical proof	.374	.298	.249	.397	.372	1		
Mathematical thinking total (MT)	.707	.619	.620	.740	.639	.668	1	
Mathematics achievement (MA)	.633	.464	.454	.602	.549	.614	.820	1

Notes to the table:
1. All coefficients different from zero at $p<.01$.
2. The number of students overall was 560.
3. The number of students who completed the two tests was 527.
4. MT (total) denotes mathematical thinking and MA denotes mathematics achievement.

The higher level of correlation between the six scales and the total for mathematical thinking is to be expected, given that the score for each aspect is included in the total score. However, the almost equally-high correlations between the six scale scores and mathematics achievement are of greater interest, because the assessments made were entirely independent with separate measuring instruments being used. However, it was hypothesised that the different aspects of mathematical thinking would be related to mathematics achievement to varying extents. In summary, the correlations indicate that, when considered in isolation from each other, all six aspects of mathematical thinking were strongly related to mathematics achievement, with Generalization the strongest relationship and with Deduction and Induction the weakest. The relationships with mathematics achievement of the six scales considered jointly will be taken up subsequently.

5.2 GENDER DIFFERENCES

All 20 schools involved in this study were single-sex, as is the case throughout Jordan for all public schools at this senior secondary level 9 being schools for male students and 11 being schools for female students. Consequently any differences in mathematical thinking and achievement that might exist between the genders would also be related to school differences, at least for the two groups of schools. However, at this point gender and school membership will be considered separately with subsequent analyses including interactions between these two variables.

The male and female student mean scores on each of the six aspects of mathematical thinking and the total scores for mathematical thinking and mathematics achievement were compared using t-tests. There was a significant gender difference for two of the scales and both total scores. Female students had significantly higher scores than male students for Logical thinking, Mathematical proof, and for total mathematical thinking and mathematics achievement. None of the other differences between gender groups was significant. The mean scores by gender are shown in Table 5.2.

TABLE 5.2. RESULTS FOR GENDER DIFFERENCES IN MATHEMATICAL THINKING AND MATHEMATICS ACHIEVEMENT.

Scale	Mean (M)	Mean (F)	SD (M)	SD (F)	T-value	Prob
Generalization	7.9	8.2	3.30	3.67	1.112	.267
Induction	8.7	8.3	3.70	3.79	-1.095	.274
Deduction	7.3	7.3	4.44	4.16	.036	.972
Use of Symbols	7.5	7.4	3.89	3.64	-.236	.814
Logical thinking	6.7	8.0	2.94	2.44	5.648	.000
Mathematical proof	4.4	5.6	3.61	3.53	3.959	.000
Mathematical thinking(total)	42.4	44.8	14.68	14.10	1.980	.048
Mathematics achievement	25.5	28.5	8.90	8.4	4.048	.000

5.3 SCHOOL DIFFERENCES

It was of interest to determine whether there were overall differences between schools in each aspect of mathematical thinking and, where an overall difference was found, which individual schools differed. To do this, an ANOVA was conducted in turn with each of the six aspects of mathematical thinking as dependent variable and school as the independent variable. For completeness, the same analysis was performed for each of the aspects and the total score for mathematical thinking, and for mathematics achievement.

It was found that five of the six aspects of mathematical thinking differed overall by school, and both the total scores also differed (see Table 5.3). The exception was Deduction for which there was no significant inter-school difference. The mean scores for individual schools are shown in Appendix 4.1.

TABLE 5.3. RANGE FOR SCHOOL MEANS AND SIGNIFICANCE OF SCALE SCORE DIFFERENCES BETWEEN SCHOOLS

Scale	Overall mean (SD)	School range	F-value	Prob
Generalization	8.0 (3.50)	5.4 - 10.7	3.95	.000
Induction	8.5 (3.75)	6.1 - 11.6	5.92	.000
Deduction	7.3 (4.30)	4.9 - 9.0	1.31	.171
Use of Symbols	7.4 (3.76)	4.8 - 9.8	4.13	.000
Logical thinking	7.3 (2.77)	4.8 - 9.5	6.90	.000
Mathematical proof	5.0 (3.62)	3.1 - 7.9	3.60	.000
Mathematical thinking (total)	43.6 (14.42)	35.4 - 54.7	4.51	.000
Mathematics achievement	27.0 (8.79)	19.3 - 33.4	5.56	.000

Students at school 1 most consistently had among the highest scores for five aspects of mathematical thinking (Generalization, Induction, Use of Symbols, Logical thinking and Mathematical proof) and for mathematics achievement which differed significantly from some of the other schools. Schools 5, 13, 19 and 9 were also higher for Logical thinking, and schools 13, 20, and 5 were higher for mathematics achievement. There was less consistency for schools with significantly lower scores, school 11 being lowest for Generalization and mathematics achievement, school 10 was lowest for Induction, and school 2 was the lowest for Logical thinking.

5.4 SCHOOL LOCATION DIFFERENCES

The mean scale scores of mathematical thinking and mathematics achievement for each of the three locations (urban, suburban, and rural) were compared. A series of one-way analyses of variance was used with a Scheffe test to assist in distinguishing individual locations that differed from each other. The overall results for each scale are shown in Table 5.4. Individual location means are shown in Appendix 4.2.

TABLE 5.4. RANGE FOR LOCATION MEANS AND SIGNIFICANCE OF SCALE SCORE DIFFERENCES BETWEEN LOCATIONS.

Scale	Overall mean (SD)	Location range	F-value	Prob
Generalization	8.0 (3.50)	7.7 - 8.7	3.16	.043
Induction	8.5 (3.75)	7.5 - 9.8	16.88	.000
Deduction	7.3 (4.30)	6.9 - 7.8	1.96	.142
Use of Symbols	7.4 (3.76)	7.1 - 8.6	6.08	.002
Logical thinking	7.3 (2.77)	6.9 - 7.7	5.17	.006
Mathematical proof	5.0 (3.26)	4.7 - 5.5	1.73	.178
Mathematical thinking (total)	43.6 (14.42)	41.5 - 48.2	7.77	.000
Mathematics achievement	27.0 (8.79)	26.0 - 30.0	7.26	.001

Several ANOVAs were conducted to determine any significant differences between locations and the aspects of Mathematical thinking (Generalization, Induction, Deduction, Use of Symbols, Logical thinking, Mathematical proof, and Total Mathematical Thinking). Location difference for mathematics achievement was also examined. There were significant differences in four of the scales: Generalization, Induction, Use of Symbols, Logical thinking, as well as Mathematical thinking (total), and Mathematics achievement. For Generalization and Logical thinking, suburban students had higher mean scores than urban students (for Generalization means 8.7, and 7.7 respectively and for Logical thinking means 7.7 and 6.9 respectively). For Induction, suburban and rural students both had higher mean scores than urban students (Means 9.7, 9.0, and 7.5 respectively), For the Use of Symbols, suburban students had higher mean scores than both urban and rural students (Means =8.6, 7.3, and 7.1 respectively). For Mathematical thinking (total) and mathematics achievement suburban students had higher mean scores than both rural and urban students (Means 48.2, 43.9, and 41.5 respectively for Mathematical thinking (total) and Means 33.0, 26.8, and 26.0 respectively for Mathematics achievement). In general, suburban students had the highest scores and urban students the lowest.

5.5 GENDER AND SCHOOL LOCATION DIFFERENCES

The independent variables considered here were gender and school location with the Mathematical thinking scales, mathematical thinking (total), and Mathematics achievement as dependent variables. The numbers of students, by category, for these variables are shown below in Table 5.5. Two-way analyses of variance were undertaken for the six mathematical thinking scales and for Mathematical thinking (total), and Mathematics achievement. A summary of the results of the analyses of variance is shown in Table 5.6, and the mean scores for the Mathematical thinking test and Mathematics achievement test are shown in Table 5.7.

TABLE 5.5. NUMBERS OF MALE AND FEMALE STUDENTS BY MATHEMATICAL THINKING AND MATHEMATICS ACHIEVEMENT

School location								
Mathematical thinking					Mathematics achievement			
Gender	Urban	Suburban	Rural	Total	Urban	Suburban	Rural	Total
Male	105	55	114	274	103	53	112	268
Female	139	43	104	286	126	42	107	275
Total	244	98	218	560	229	95	219	543

School location was the variable more consistently related to the dependent variable, being related to scores on four of the six scales, Mathematical thinking and Mathematics achievement. While the level of achievement in rural and urban schools was often similar, there were larger and more consistent differences between achievement of the suburban students and other students in rural and urban locations. Only for two scales (Deduction and Mathematical proof) were the scores of the suburban students similar to the scores of the other students.

When location was taken into consideration, gender was significant only on three of the six scales (Generalization, Logical thinking, Mathematical proof), Mathematical thinking (total), Mathematics achievement, in which the females had a higher achievement than the males, whereas on the other three scales (Induction, Deduction, Use of Symbols), the males had the same or slightly higher (Induction) but not statistically significant scores than the females.

As also shown in Table 5.6, there were some significant interactions between gender and location. Female students in suburban schools performed better for four of Mathematical thinking scales (Generalization, Induction, Logical thinking, and Mathematical proof), and Mathematical thinking (total). The means for these five scores simultaneously broken down by gender and school location re shown in Table 5.8

TABLE 5.6. SUMMARY OF SIGNIFICANT LEVELS FOR ANOVA FOR MATHEMATICAL THINKING SCALES AND MATHEMATICS ACHIEVEMENT BY GENDER AND SCHOOL LOCATION.

Outcome	Gender	Location	Interaction	F (Interaction)
Generalization	<.05	<.05	<.01	5.11
Induction	n/s	<.01	<.01	8.48
Deduction	n/s	n/s	n/s	0.73
Use of Symbols	n/s	<.01	n/s	1.06
Logical thinking	<.01	<.01	<.01	6.73
Mathematical proof	<.01	n/s	<.05	3.48
Mathematical thinking (total)	<.01	<.01	<.01	6.73
Mathematics achievement	<.01	<.01	n/s	2.84

In almost every case where the interaction between gender and school location was significant, the pattern of interaction was the same. For suburban schools, the females had higher scores for these five scores (Generalization, Induction, Logical thinking, and Mathematical proof), and Mathematical thinking (total), but were different for rural schools where the females had the higher scores on Logical thinking and Mathematical proof. The differences for the males were small but not statistically significant. Differences in achievement found between gender and school location for the total group of students were almost entirely due to the females. For Induction, however, the males had a significantly higher mean score than the females for urban schools.

TABLE 5.7. MEANS SCORES FOR MATHEMATICAL THINKING SCALES, MATHEMATICAL THINKING (TOTAL), AND MATHEMATICS ACHIEVEMENT.

Mean scores					
Outcome	Gender		Location		
Range	Male	Female	Urban	Suburban	Rural
Generalization (0-15)	7.9	8.2	7.7	8.7	8.1
Induction (0-15)	8.7	8.3	7.5	9.8	9.0
Deduction (0-15)	8.3	7.4	7.5	7.8	6.9
Use of Symbols (0-15)	7.5	7.4	7.1	8.6	7.3
Logical thinking (0-15)	6.7	8.0	6.9	7.7	7.6
Mathematical proof (0-15)	4.4	5.6	4.7	5.5	5.0
Mathematical thinking (total) (0-87)	42.4	44.4	41.5	48.2	43.9
Mathematics achievement (0-50)	25.5	28.5	26.0	30.0	26.8

TABLE 5.8. MEANS AND STANDARD DEVIATIONS FOR MATHEMATICAL THINKING SCALES AND MATHEMATICAL THINKING (TOTAL) WHERE INTERACTION WAS SIGNIFICANT BETWEEN GENDER AND SCHOOL LOCATION.

School location						
	Urban		Suburban		Rural	
Outcome / Gender	Mean	SD	Mean	SD	Mean	SD
Generalization						
Male	7.7	3.10	7.6	3.46	8.1	3.42
Female	7.7	3.64	10.1	3.85	8.1	3.41
Induction						
Male	8.4	3.68	9.1	3.56	8.7	3.80
Female	6.9	3.43	10.6	3.65	9.4	3.51
Logical thinking						
Male	6.6	2.90	6.6	3.33	6.8	2.80
Female	7.1	2.21	9.2	2.35	8.5	2.39
Mathematical proof						
Male	4.4	3.62	4.3	4.00	4.4	3.46
Female	5.0	3.26	7.1	3.86	5.7	3.57
Mathematical thinking (total)						
Male	42.1	13.49	43.3	16.57	42.2	14.87
Female	41.0	12.55	54.4	14.87	45.9	13.77

Note: The results for gender shown here are different from those in Table 5.2 once location is taken into consideration.

5.6 RELATIONSHIP OF MATHEMATICAL THINKING WITH MATHEMATICS ACHIEVEMENT

In order to test the strengths of the relationships between each aspect of mathematical thinking and mathematics achievement, all six aspects of mathematical thinking were entered into a regression equation with mathematics achievement as the dependent variable. All aspects were significantly related to mathematics achievement. When the standardized regression coefficients are compared, Mathematical proof was found to be the most important scale for mathematics achievement followed by Generalization. Use of Symbols and Logical thinking were next in importance and finally Deduction and Induction. Overall, the six aspects accounted for 69.1% of the variance in mathematics achievement. The results are shown in Table 5.9 where both the unstandardized (metric) coefficients and the standardized coefficients are shown. Comparison of the standardized coefficients is appropriate in comparing the relative strengths of variables in the same regression equation. The results are also shown in the model illustrated in Figure 5.1 where a section of the full model is displayed.

TABLE 5.9. STANDARDIZED AND UNSTANDARDIZED COEFFICIENTS, WITH T-VALUE AND SIGNIFICANCE LEVEL SHOWN FOR THE MATHEMATICAL THINKING ASPECTS AS INDEPENDENT VARIABLES AND MATHEMATICS ACHIEVEMENT AS DEPENDENT VARIABLE.

Model	Unstandardized coefficients		Standardized coefficients		T-value	Prob
	B	SD	Beta	Std. Error		
(Constant)	6.046	.729			8.30	.000
Generalization	.667	.074	.268	.030	9.03	.000
Induction	.307	.064	.131	.027	4.82	.000
Deduction	.316	.054	.156	.027	5.81	.000
Use of Symbols	.445	.070	.191	.030	6.34	.000
Logical thinking	.546	.088	.176	.029	6.17	.000
Mathematical proof	.692	.068	.289	.028	10.22	.000

5.7 THE COMPLETE MODEL TO BE TESTED

Finally gender and location were included in a two stage model with the six aspects of mathematical thinking as independent variables, with mathematics achievement as the dependent variable.

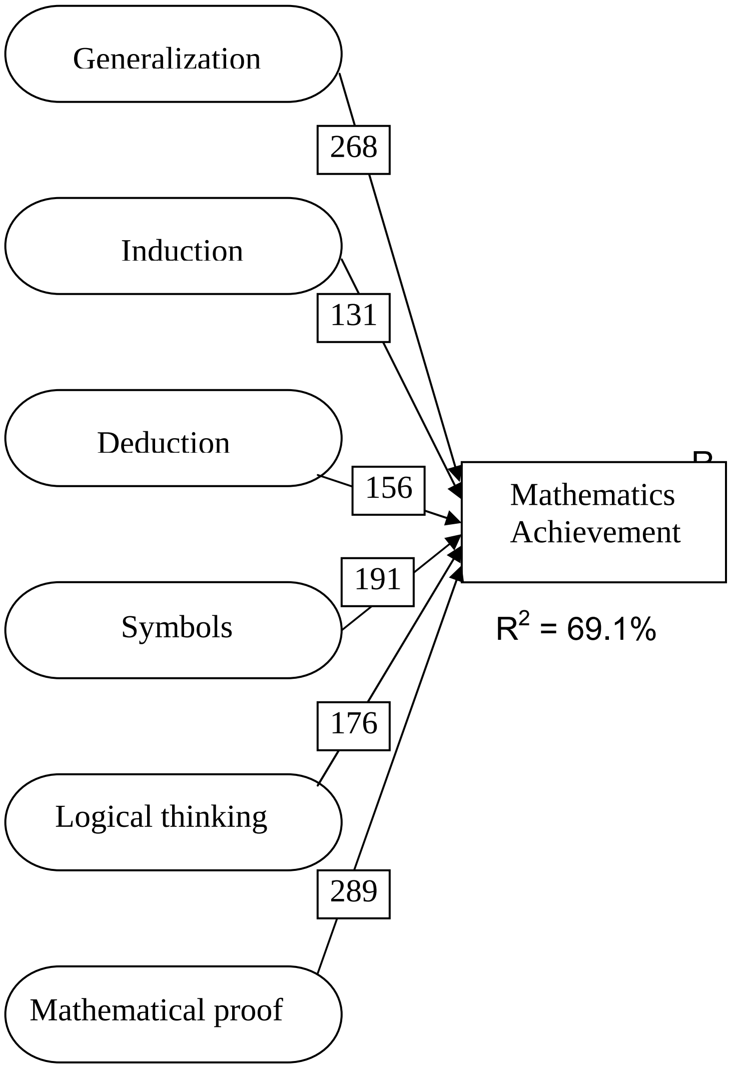

FIGURE 5.1. MULTILEVEL PATH MODEL EXPLAINING VARIATION IN MATHEMATICS ACHIEVEMENT BASED ON ASPECTS OF MATHEMATICAL THINKING (ONLY SIGNIFICANT STANDARDIZED PATHS ×1000 SHOWN)

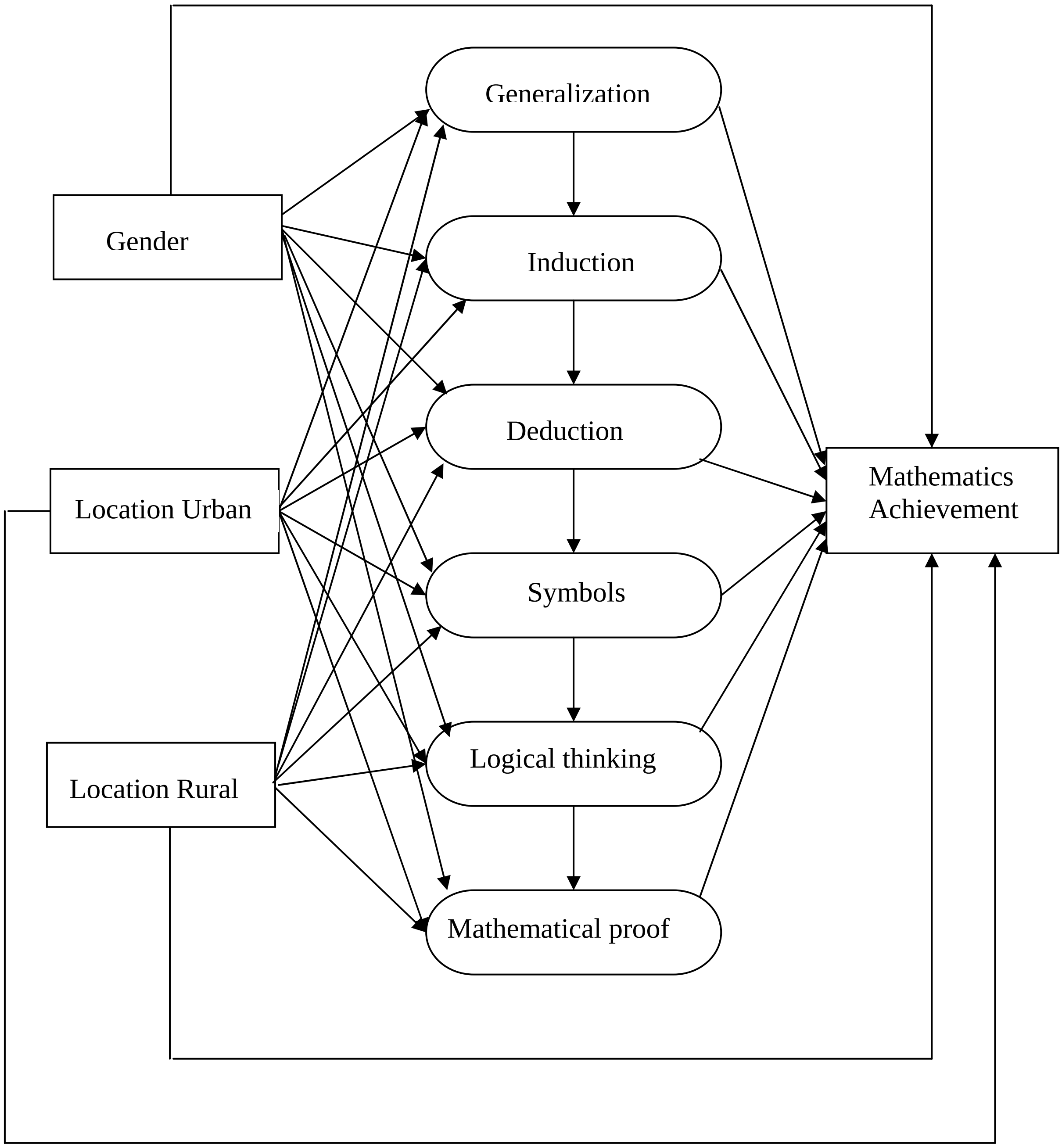

FIGURE 5.2. SCHEMATIC DIAGRAM OF THE COMPLETE MODEL TO BE TESTED

Note: For location, suburban schools are the omitted category and the other two categories are compared with it.

The model hypothesised that student gender and school location were causally related to the six aspects of mathematical thinking and together with them were related to mathematics achievement. For convenience, a schematic representation of the full model to be tested is re-presented in Figure 5.2 above.

The results of the two-stage model developed to examine the relationships of student gender, school location and the six aspects of mathematical thinking with mathematics achievement are shown in table 5.11. For the first stage of the model, the relationships of the background variables, gender and location, with mathematical thinking are examined. For the second stage, the relationships of the background variables and the six aspects of mathematical thinking measures with mathematics achievement are examined. School location has three categories urban, suburban, and rural. The omitted category is suburban and the other two categories are compared with it.

When each of the six aspects of mathematical thinking was regressed on student gender and school location, the results indicated that there were significant differences in Generalization for urban schools, in Induction and Use of Symbols for urban and rural schools, in Deduction for rural schools, and finally in Logical thinking and Mathematical proof for gender and urban schools. All results for this stage of the analysis are shown in Table 5.10 regression coefficients and standard errors.

Once gender and school location were included in analyses with mathematical thinking scales as the dependent variable, gender was not significant for four of these scales (Generalization, Induction, Deduction, Use of Symbols), and significant only in the last two scales (Logical thinking and Mathematical proof) in favour of females. School location was significant in these scales in different ways. An urban location was significantly lower than the omitted category (suburban schools) in four of these aspects (Induction, Use of Symbols, Logical thinking, and Mathematical proof). However, rural location was lower than suburban schools in three of these aspects (Induction, Deduction, Use of Symbols, Logical thinking). No statistically significant difference was evident for the other scales.

TABLE 5.10. MATHEMATICAL THINKING SCALES AS DEPENDENT VARIABLES WITH GENDER AND SCHOOL LOCATION AS INDEPENDENT VARIABLES.

Independent variables	All standardized regression coefficients (SE)	Sig. coefficients (SE) only included
Generalization		
Gender	.058 (.042)	
Location urban	-.155 (.059)	-.085 (.042)
Location rural	-.090 (.059)	
R^2	1.5%	0.7%
Induction		
Gender	-.021 (.041)	
Location urban	-.294 (.058)	-.297 (.058)
Location rural	-.096 (.058)	-.097 (.058)
R^2	5.8%	5.7%
Deduction		
Gender	-.001 (.043)	
Location urban	-.029 (.060)	
Location rural	-.101 (.059)	-.081 (.042)
R^2	0.7%	0.7%
Use of Symbol		
Gender	.001 (.042)	
Location urban	-.200 (.059)	-.199 (.059)
Location rural	-.174 (.059)	-.174 (.059)
R^2	2.1%	2.1%
Logical thinking		
Gender	.251 (.041)	.250 (.041)
Location urban	-.182 (.057)	-.160 (.041)
Location rural	-.032 (.057)	
R^2	8.0%	8.0%
Mathematical proof		
Gender	.175 (.042)	.174 (.042)
Location urban	-.132 (.059)	-.082 (.042)
Location rural	-.071 (.058)	
R^2	3.6%	3.4%

Mathematics achievement was then regressed on the scores for all six aspects of mathematical thinking plus gender and school location. Again a backwards elimination process was used to remove any independent variables with non-significant relationships with the dependent variable. The results indicated all aspects were significantly related to mathematics achievement, and all background variables were significantly related to mathematics achievement. All results for the second stage of the analysis are shown in Figure 5.3.

TABLE 5.11. MATHEMATICS ACHIEVEMENT SCALE AS DEPENDENT VARIABLE WITH MATHEMATICAL THINKING SCALES, GENDER AND SCHOOL LOCATION AS INDEPENDENT VARIABLES.

Independent variables	Unstandardized coefficients (SE)	All standardized regression coefficients (SE)	Sig. coefficients (SE) only included
Mathematics achievement			
Constant	3.887 (.928)	-	-
Generalization	.665 (.073)	.267 (.029)	.268 (.029)
Induction	.335 (.063)	.138 (.028)	.143 (.027)
Deduction	.323 (.054)	.160 (.027)	.159 (.027)
Use of Symbols	.473 (.070)	.200 (.030)	.203 (.030)
Logical thinking	.466 (.090)	.149 (.029)	.150 (.029)
Mathematical proof	.654 (.068)	.273 (.028)	.273 (.028)
Gender	1.634 (.443)	.096 (.025)	.094 (.025)
Location urban		-.060 (.035)	
Location rural		-.056 (.034)	
Variance explained		R^2= 70.1%	R^2 = 69.9%

In most cases the standardised regression coefficients (or path coefficients) indicating the relative effects of the six aspects of mathematical thinking on mathematics achievement are relatively stable whether or not the background variables of gender and school location are included in the regression equation. Mathematical proof has the strongest relationship with mathematics achievement with Generalization almost as strong. The Use of Symbols and Deduction are the next most important for mathematics achievement, followed in order by Logical thinking and Induction.

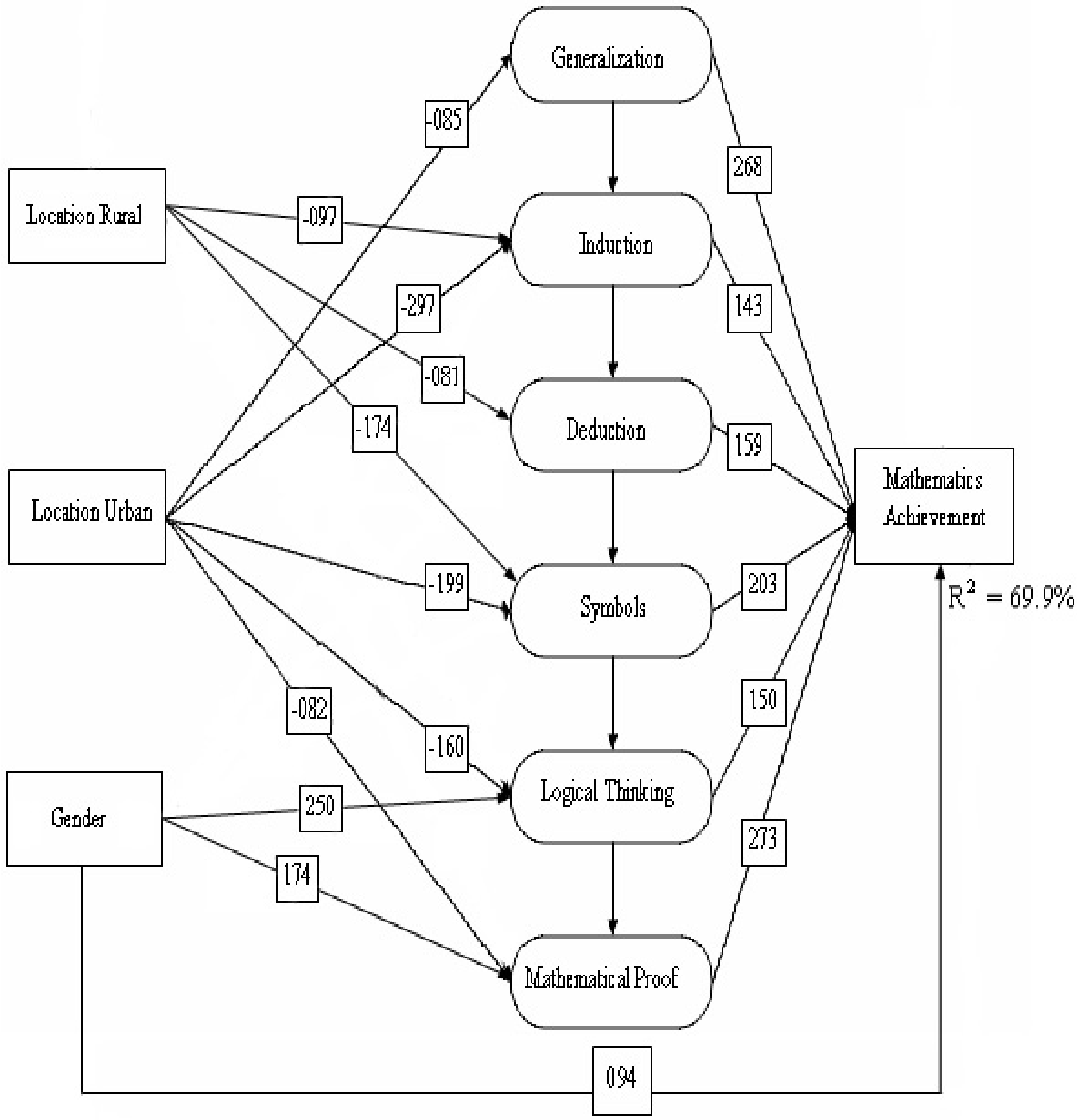

FIGURE 5.3. FULL PATH MODEL EXPLAINING VARIATION IN MATHEMATICS ACHIEVEMENT WITH MATHEMATICAL THINKING, GENDER, AND SCHOOL LOCATION (ONLY SIGIFICANT STANDARDIZED PATHS ×1000 SHOWN).

Gender has both a direct effect and indirect effects through aspects of mathematical thinking on mathematics achievement. School location does not have direct effects, but it has several indirect effects. The direct effect of gender on mathematics achievement is of the same magnitude as being in a rural school, with female students having higher achievement. Overall, 69.9 per cent (the R^2 value for the full model) of the variation in mathematics achievement can be explained by the nine independent variables.

In order to establish the most important effects overall on mathematics achievement, the direct and indirect effects were added. The indirect effects from school location and gender to mathematics achievement through each of the aspects of mathematical thinking are calculated as the sum of the products of the path coefficient from the background variable to the aspects of mathematical thinking multiplied by the coefficient from mathematical thinking to mathematics achievement (see Penhazur, 1982. pp. 600-605.). The direct, indirect and total path coefficients from independent variables to mathematics achievement as the dependent variable are shown in Table 5.12.

TABLE 5.12. DIRECT, INDIRECT, AND TOTAL EFFECTS OF THE INDEPENDENT VARIABLES ON MATHEMATICS ACHIEVEMENT.

Independent variables	Direct effects	Indirect effects	Total effects
Gender	0.094	.038+.048 = 0.086	0.180
School loc: Urban		-.024+-.042+-.040+-.024+-.022= -.152	-0.152
School loc: Rural		-.014+-.013+-.035 = - .062	-0.062
Generalization	0.268		0.268
Induction	0.143		0.143
Deduction	0.159		0.159
Use of Symbols	0.203		0.203
Logical thinking	0.150		0.150
Mathematical proof	0.273		0.273

In terms of total effects Mathematical proof has the strongest effect on individual student achievement with Generalization almost as strong. The Use of Symbols and Gender are the next most important variables for mathematics achievement, followed in order by Deduction and Urban school locations (negative effect), then Logical thinking, Induction, and Rural location schools (negative effect).

5.8 MULTILEVEL ANALYSES OF EFFECTS ON MATHEMATICS ACHIEVEMENT

The sample of students involved in this study was structured with students clustered in classes within schools, and schools clustered in three types of locations (urban, suburban and rural). The clustering gives rise to intra-class correlations for each of the measures which would affect the standard errors obtained. Consequently the levels of statistical significance of the regression coefficients calculated when regressing mathematics achievement on the independent (explanatory) variables would be inflated (Goldstein, 1995). It was therefore considered desirable to check the robustness of the model developed and tested at the student level of analysis (described above). The relative independent power of the model at each of the three levels, namely individual students (N = 527 when cases with any missing data had been removed), schools (N = 20) and locations (N = 3), was also of interest. For these purposes the multilevel analysis program Ml wiN (Rasbash et al, 2000) was used.

The second stage of the model, previously described in Figure 5.2, with mathematics achievement as dependent (or response) variable and with gender, location and each of the mathematical thinking sub-tests as independent variables was analysed using a multilevel regression analysis. The results obtained from these analyses indicated that gender and each of the mathematical thinking sub-tests were significantly related to mathematics achievement, while school location was not. These results, shown in Table 5.13 are almost consistent with those obtained from the simpler, single-level regression analyses reported above, with a change in the order of importance for the lowest two aspects (Logical thinking and Induction).

TABLE 5.13. SIGNIFICANT STANDARDIZED AND UNSTANADARIZED REGRESSION COEFFICIENTS IN THE MODEL

Independent variables	Unstandardized coefficients	Standard errors	Standardized coefficients
Constant	4.501	1.184	-
Generalization	0.634	0.071	0.255
Induction	0.355	0.062	0.152
Deduction	0.316	0.052	0.156
Use of Symbols	0.465	0.067	0.200
Logical thinking	0.430	0.087	0.139
Mathematical proof	0.668	0.065	0.279
Gender	1.493	0.643	0.086

As was found previously in the single-level model, Mathematical proof and Generalization were the most important aspects of mathematical thinking for mathematics achievement, Use of Symbols was next followed by Deduction and Logical thinking, with Induction the least important, although clearly still statistically significant. Gender was less important than all aspects of mathematical thinking, although again statistically significant.

The variance unexplained at each level is shown in Table 5.14. Initially (in the null model) it can be seen that most of the variance in mathematics achievement was between individual students (82.5%), with 16.2% between schools and only 1.3% between locations. Unexplained variances at all levels in the final model were considerably reduced.

Of perhaps greater interest is the proportion of variance in mathematics achievement explained at each level. The model was approximately equally effective at levels 1 and 2, explaining almost 70% of the variance. The unexplained variance at level 3 (that is between locations), small in the null model, was too small to be calculated in the final model. This suggests that all the initial variance of mathematics achievement between locations results from differences in mathematical thinking and gender, and that there are no additional effects of being at urban, suburban or rural schools. Overall, 69.2% of the variance was explained by the model including only the statistically significant paths shown in Table 5.13.

TABLE 5.14. UNEXPLAINED VARIANCE IN TWO MODELS AND PERCENTAGE OF EXPLAINED VARIANCE IN THE INDEPENDENT MODEL

Unexplained variance	Null model		Independent model		Percent variance explained
	Variance	%	Variance	%	
Between locations (Lev 3)	1.00	1.3	0	0	100
Between schools (Lev 2)	12.13	16.2	4.04	17.5	66.7
Between students (Lev 1)	67.71	82.5	19.06	82.5	69.1
Total	74.84		23.10		69.2

5.9 SUMMARY

1) When mathematics achievement was regressed on the background variables (gender, school location) and on all aspects of mathematical thinking identified for this study, almost 70 per cent of the variance in mathematics achievement was explained.

2) All aspects of mathematical thinking were important for mathematics achievement, particularly, Mathematical proof and Generalization, followed by Use of Symbols, Deduction, Logical thinking, and Induction.

3) Although all aspects of mathematical thinking were significantly correlated with each other, each of the six aspects was also independently important for mathematics achievement, with Mathematical proof and Generalization being the most important, and Induction the least important. Gender and school location were also important. Gender had almost equal direct and indirect effects on mathematics achievement, the indirect effects being through Logical thinking and Mathematical proof. School location had only indirect effects on mathematics achievement, differentially through all aspects of mathematical thinking.

4) Gender and school location were important to some extent. Gender was important in the last two scales Logical thinking and Mathematical proof, and in mathematics achievement. However, for school location, most commonly,

suburban schools had the highest levels of mathematical thinking and mathematics achievement and urban schools had the lowest levels of achievement.

5) The simple model developed and tested was reasonably powerful in determining mathematical achievement, and suggests that particular emphases on Mathematical proof and Generalization in mathematics teaching for these students would be most beneficial in improving mathematics achievement for students of both genders and for students in different school locations.

6) In the single level analyses, all mathematical thinking aspects were important for mathematics achievement, particularly, Mathematical proof and Generalization, followed by Use of Symbols, then Deduction, Logical thinking, and Induction. Again, in the multilevel analyses, all aspects of mathematical thinking were important for mathematics achievement, particularly, Mathematical proof then Generalization, followed by Use of Symbols, then Deduction, Induction, and Logical thinking.

This chapter focused on the quantitative data analyses, in relation to the two tests, the test of mathematical thinking with six aspects (Generalization, Induction, Deduction, Use of Symbols, and Logical thinking, and Mathematical proof) and the test of mathematics achievement. Student performances on mathematical thinking and mathematics achievement were compared with regard to the background variables of gender, individual schools, and school location, followed by descriptive analyses, correlations between the aspects, t-tests, ANOVA, and multivariate analyses. In contrast, the next chapter will provide an analysis of the qualitative data, in relation to the teacher and student interviews. Teacher interviews focused on their perceptions of mathematical thinking, the strategies they use to teach it, and how they encourage students learning of mathematical thinking. Student interviews were conducted as a group interview between the researcher and the students and focussed on explaining how the students arrive at their answers to specific mathematical thinking problems.

CHAPTER SIX

THE TEACHER AND STUDENT INTERVIEWS

6.1 INTRODUCTION

In this chapter a description and analysis of the qualitative data (teacher and student interviews) will be provided. The first section will summarise and discuss the teacher interviews, where the researcher interviewed 13 teachers individually, each from a different school. The researcher initially asked all 20 teachers from the 20 schools involved in the research project if they were willing to participate in an interview and 13 consented. The researcher interviewed these teachers with regard to how they teach mathematical thinking in their classes, how they help their students to learn mathematical thinking, what they believe mathematical thinking is, what the different aspects of mathematical thinking are, how they rank the aspects of mathematical thinking according to the level of importance and the level of difficulty, and what they consider to be the most helpful strategies to use in teaching mathematical thinking.

There were seven male teachers and six female teachers included. All male teachers were interviewed at their schools by the researcher. In contrast, all female teachers preferred to take the interview questions home and then return their answers to the researcher on the day of the second test session. The researcher explained the questions to the female teachers before they took these interviews home, and discussed with them any issues that had arisen when they returned their responses.

The second section in this chapter will summarise and discuss the data collected from focus group discussions with students. The researcher interviewed four groups of which two were all male groups and two were all female groups, each comprising five participants. The researcher chose five questions from the test of mathematical thinking; three of the questions were the same for each group, because the nature of these questions was more likely to generate discussion. The remaining two questions were different for each group except for the first and second groups, one female group and one male group, for whom the remaining two questions were the same in order to enable gender comparison of the responses. The use of different questions for the last two discussion groups was to expose the interviewees to a variety of questions with the intention of generating richer data. The interviews were designed to elicit information about different ways of thinking in mathematics. The researcher discussed with the students the strategies they use in answering these questions, and how they reached their answer.

The purpose of these two sets of interviews was to examine any inconsistencies and consistencies between the teachers' opinions about aspects of mathematical thinking, such as level of importance, level of difficulty, and meaning of mathematical thinking and the results derived from the quantitative analyses of the student responses to the mathematical thinking test. However, the purpose of the student interviews was to identify the popular strategies they used to arrive at solutions, to differentiate between the thinking skills they used and to ascertain their attitude toward checking their solutions.

6.2 THE TEACHER INTERVIEWS

This section first gives the actual questions asked of teachers, before describing the responses from each teacher.

1) In your opinion, what does mathematical thinking mean? Do you think mathematical thinking is restricted to the domain of mathematical computation and formula (e.g. it is restricted to the use of numbers and formulas to find answers to specific problems) or can it be used like a game to explore mathematical processes? Do you think mathematical thinking is "effective thinking" or the basis of mathematics, and contributes to the development of the student through the study of mathematics, in particular, and other sciences in general?

2) What are the aspects of mathematical thinking? For example Generalization is one of the aspects; do you know what the others may be?

The researcher then gave the teachers his list of six aspects of mathematical thinking to consider them in answering questions three to seven.

3) How important is each of the aspects in teaching mathematics? Rank these aspects according to level of importance for mathematics achievement.

4) Why do you consider the -----aspect the most important aspect? And how useful is it for the students to improve their progress in mathematical thinking?

5) What is the most difficult aspect for the students, and what is the easiest, and why? Rank these aspects according to their level of difficulty.

6) How many weeks or hours (lessons) do you spend to teach each aspect?

7) What are most effective strategies you use when you teach mathematical thinking?

Each of the teacher interviews is now summarised using the school number as identification. Following a summary of responses to the interview questions for each teacher, comparisons are made between the teacher opinions of importance and difficulty of the six aspects of mathematical thinking with test results for students at that school. The relative importance for student achievement is assessed by the magnitudes of the standardized regression coefficient linking the aspects of mathematical thinking with mathematics achievement. The relative difficulty for students was taken as the mean scores for the subtests assessing the six aspects of mathematical thinking. In particular, consistencies and inconsistencies between the teacher opinions and student results are noted. Participants in the interviews comprised seven male teachers and six female teachers, located in five urban schools, five rural schools and three suburban schools. A copy of the transcripts of the teacher interviews can be found in Appendix 5.1 for each individual interview and a summary of each is given in the following section.

6.2.1 School 1

This school was a moderately sized school located in a suburban area with only two streams, humanities and scientific streams, with 24 students from the scientific stream participating in the study. The teacher interviewed was female. When asked for her opinion of what mathematical thinking was, she answered: “ability to make inferences, connections and solve problems. It is not restricted to the use of numbers and formulas to find answers to specific problems, and looks like a game. It is development by the student through his study of mathematics in particular and other science in general”.

When asked to identify the aspects of mathematical thinking, she answered:

- Generalization.
- Induction.
- Deduction.
- Use of Symbols.
- Logic.
- Plane Geometry.

When asked which the most important aspect was, she answered: The Use of Symbols aspect is considered the most important aspect because it contains the basis for the other aspects, when the students are able to use of symbols as the first step, then the other aspects become easier such as Generalization, Induction, and Mathematical proof.

When the teacher was asked to name the most effective strategies she used when teaching mathematical thinking, she answered: Search for pattern and try a simple problem.

The interviewee agreed with the researcher’s description of the aspects of mathematical thinking with a slight difference in the sixth aspect, which the interviewee considered to be plane geometry rather than Mathematical proof. Most proofs in mathematical thinking test covered geometry so it was a case of the teacher focussing on the topic rather than the process. With regard to order of importance, the interviewee considered Use of Symbols the most important aspect, because it is the first step in most problem solving, and other following steps will be easier. The other aspects were ordered as follows: Deduction, Generalization, Logical thinking, Mathematical proof, and Induction from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Deduction, Induction, Generalization, Use of Symbols, and Logical thinking. The interviewee spent 15% of the time teaching Use of Symbols and Mathematical proof, then Generalization and Induction (10% of each), however, she considered the curriculum lacked coverage of Deduction, and Logical thinking. Finally, she used searching for patterns to develop students' generalizations and elicitation skills and trying a simple problem when she taught difficult problems to facilitate responses to the problems.

TABLE 6.1a. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.1

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Generalization	Use of Symbols	Mathematical proof	Mathematical proof
Use of Symbols	Deduction	Deduction	Deduction
Mathematical proof	Generalization	Use of Symbols	Induction
Induction	Logical thinking	Generalization	Generalization
Logical thinking	Mathematical proof	Logical thinking	Use of Symbols
Deduction	Induction	Induction	Logical thinking

In respect to importance level, the consistencies and inconsistencies between the teacher opinion and results extracted from student responses indicate some patterns. Use of Symbols and Generalization were considered among the more important aspects in terms of both student responses and teacher opinion. However, the results were inconsistent for two other aspects, Deduction and Mathematical proof. In particular, Deduction was the least important aspect from the test results but was the second most important aspect in teacher opinion.

In relation to level of difficulty, consistency between student performance and teacher opinion was much higher. Only the difficulties of Use of Symbols and

Induction were inconsistent. Induction was the least difficult aspect from the results that derived from students and a moderately difficult aspect in teacher opinion. Use of Symbols was found to be moderately difficult according to student results and the second least difficult according to teacher opinion. It was not possible to compare student responses on the test and the interviews, because it was required by the University's ethics committee that the names of students interviewed were not obtained.

This school had second highest mean score for the mathematical thinking test and the highest mean score in mathematics achievement. For this high performing school, there was a high level of consistency between students and the teacher for the relative difficulty of the aspects of mathematical thinking. It would seem that the teacher had a clear perception of what was difficult for her students. However, even for this school, the teacher and students did not agree on the relative importance of different aspects of mathematical thinking.

6.2.2 School 2

This school was a large, secondary, comprehensive school located in a suburban area, with 31 students participating in the study. This interviewee was male. He considered the aspects of mathematical thinking to be Generalization and its applications, Specialization and Problem solving. The interviewee considered Generalization the most important aspect, because arriving at Generalization and finding the patterns from specific cases requires high levels of thinking, so if the student has high levels of ability in terms of arriving at Generalizations, then he will achieve highly in mathematics. He then ordered other aspects with regard to importance as follows: Logical thinking, Mathematical proof, Induction, Use of Symbols, and Deduction. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Logical thinking, Generalization, Induction, Deduction, and Use of Symbols.

The interviewee spent 20% of the time teaching Logical thinking and Mathematical proof, followed by Generalization and Induction (15% of each), then, Deduction, and Use of Symbols (10% for each). The interviewee emphasised that Logical thinking is integral to most of mathematical areas although it is omitted from the curriculum. Finally, the interviewee mostly used ‘looking for patterns’ which is suitable for Generalization, ‘using a model’, and ‘drawing a picture’ in teaching mathematical thinking.

TABLE 6.1b. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.2

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Use of Symbols	Generalization	Mathematical proof	Mathematical proof
Induction	Logical thinking	Logical thinking	Logical thinking
Mathematical proof	Mathematical proof	Deduction	Generalization
Logical thinking	Induction	Use of Symbols	Induction
Generalization	Use of Symbols	Generalization	Deduction
Deduction	Deduction	Induction	Use of Symbols

With regard to level of importance, there was consistency only for Mathematical proof and Deduction. Mathematical proof was considered a moderately important aspect, while Deduction was considered the least important. However, the results were inconsistent, in particular, for two other aspects, Generalization and Use of Symbols. Generalization was the second least important aspect from the test results but the most important aspect in teacher opinion. In contrast, Use of Symbols was found the most important aspect from the test results but the second least important aspect in teacher opinion.

In relation to difficulty level, Mathematical proof and Logical thinking were among the more difficult aspects in terms of test results and teacher opinion. However, the results were inconsistent for other aspects; Generalization and Induction were found to be the least difficult aspects in test results but moderately difficult aspects in teacher opinion. In contrast, Deduction and Use of Symbols were found to be moderately difficult according to test results and the least difficult aspects in teacher opinion.

6.2.3 School 3

This school was a relatively moderately sized school with only humanities and scientific streams located in a rural area, with 24 students participating in the study. This interviewee was female. She agreed with the researcher about the aspects of mathematical thinking but believed that logic should be expressed as logic with mathematical cognition. The interviewee considered Logical thinking the most important aspect, because it is fundamental to any area of mathematics, and it is the first step for all other aspects of mathematical thinking. Next, Generalization, Mathematical proof, Use of Symbols, Induction, and Deduction were ordered from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most to least difficult as follows: Mathematical proof, Induction, Deduction, Use of Symbols, Logical thinking, and Generalization.

Although the interviewee considered Deduction the least important, she spent 25% of time teaching Deduction, next Mathematical proof (20%), followed by Logical thinking and Induction (15% for each), then Generalization and Use of Symbols (10% for each). Finally, the interviewee mostly used ‘looking for patterns’ (Generalization), ‘writing an equation’, ‘trying a simple problem’, and ‘more challenging problems’ when teaching mathematical thinking.

TABLE 6.1c. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.3

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Mathematical proof	Logical thinking	Mathematical proof	Mathematical proof
Generalization	Generalization	Deduction	Induction
Logical thinking	Mathematical proof	Generalization	Deduction
Use of Symbols	Use of Symbols	Use of Symbols	Use of Symbols
Induction	Induction	Induction	Logical thinking
Deduction	Deduction	Logical thinking	Generalization

In terms of level of importance, consistency between the student test results and teacher opinion was high for four of the aspects. Only the importance of

Mathematical proof and Logical thinking were inconsistent. Mathematical thinking was the most important aspect for the student test and a moderately important aspect in teacher opinion. Logical thinking was found to be moderately important aspect to student results and the most important according to teacher opinion.

In relation to level of difficulty, the consistency between student test and teacher opinion was much higher. Only the difficulties of Generalization and Induction were inconsistent. Generalization found to be moderately difficult according to test results and the least difficult according to teacher opinion. Induction was considered the second least difficult aspect in test results and the second most difficult aspect according to teacher opinion.

6.2.4 School 6

This school was a relatively moderately sized school with only humanities and scientific streams located in a rural area, with 28 students participating in the study. This interviewee was male. He agreed with the researcher about the aspects of mathematical thinking with a slight difference with regard to the Use of Symbols aspect because he considered Use of Symbols problems as translations from words to equations. The interviewee considered Mathematical proof the most important aspect, because it requires high ability in thinking, connections, justification, and understanding the mathematical concepts. Generalization, Logical thinking, Use of Symbols, Induction, and Deduction were ranked from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most to least difficult as follows: Mathematical proof, Induction, Deduction, Logical thinking, Use of Symbols, and Generalization. Although the interviewee considered Deduction the least important, he spent 25% of the time teaching Deduction, next Mathematical proof (20%), followed by Logical thinking and Induction (15% for each), then Generalization and Use of Symbols (10% for each). Finally, when teaching Mathematical proof, the interviewee mostly used ‘try then adjust’ and retry particularly in Generalization and through this strategy can check whether his answers are correct or not, and ‘draw a picture’.

TABLE 6.1d. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.6

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Generalization	Mathematical proof	Mathematical proof	Mathematical proof
Deduction	Generalization	Deduction	Induction
Mathematical proof	Logical thinking	Logical thinking	Deduction
Use of Symbols	Use of Symbols	Use of Symbols	Logical thinking
Induction	Induction	Induction	Use of Symbols
Logical thinking	Deduction	Generalization	Generalization

In relation to level of importance, consistency between student performance and teacher opinion was very high. Generalization and Mathematical proof were among the more important aspects in terms of test results and teacher opinion. Use of Symbols and Induction also were identical according to test results and teacher opinion. Only Logical thinking and Deduction were found to be inconsistent. Deduction was the second most important aspect in test results and the least important in teacher opinion. Logical thinking was found to be moderately important aspect in teacher opinion but the least important in test results.

In respect of level of difficulty, consistency between student performance and teacher opinion was very high. Only the difficulty of Induction differed, with induction being the second least difficult aspect in student results but the second most difficult in teacher opinion.

6.2.5 School 7

This school was a relatively large, secondary, comprehensive school located in an urban area, with 32 students participating in the study. This interviewee was female. The interviewee agreed in general with the researcher when identifying the aspects of mathematical thinking but excluded Logical thinking, replacing this with Specialization. The interviewee considered Use of Symbols the most important aspect, because it has a fundamental role in many mathematical areas such as Generalization, Logical thinking, algebra, geometry. Generalization, Mathematical proof, Logical thinking, Deduction, and Induction were ranked from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most to least difficult as follows: Mathematical proof, Generalization, Use of Symbols, Logical thinking, Induction, and Deduction.

The interviewee spent 20% of her teaching of mathematical thinking on both Generalization, and Mathematical proof, next Induction (15%), then, Use of Symbols and Logical thinking (10% for each), and only 5% for Deduction. Finally, she used 'search for patterns' and 'more challenging problems' in teaching mathematical thinking.

TABLE 6.1e. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.7

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Deduction	Use of Symbols	Mathematical proof	Mathematical proof
Mathematical proof	Generalization	Induction	Generalization
Induction	Mathematical proof	Use of Symbols	Use of Symbols
Generalization	Logical thinking	Generalization	Logical thinking
Use of Symbols	Deduction	Deduction	Induction
Logical thinking	Induction	Logical thinking	Deduction

With respect to level of importance, inconsistency between student performance and teacher opinion was much higher in this case. Only the importance of Mathematical proof and Logical thinking were consistent. Mathematical proof was among the more important aspects in terms of test results and teacher opinion, while Logical thinking was considered among the less important aspects according to test results and teacher opinion.

In relation to level of difficulty, consistency between student performance and teacher opinion was much higher. Only the difficulties of Induction and Generalization were considered inconsistent. Induction was considered the second least difficult aspect in teacher opinion but was the second most difficult according to the test results. Generalization found to be a moderately difficult aspect in test results and the second most difficult in teacher opinion.

6.2.6 School 8

This school was a moderately sized school with only humanities and scientific streams located in a rural area, with 30 students participating in the study. This interviewee was male. He agreed with the researcher for four of the six aspects of mathematical thinking but excluded Use of Symbols and Logical thinking, replacing these two aspects with Challenges. He considered both Use of Symbols and Logical thinking aspects to be represented under the heading Challenges, to express that these aspects are more suited to the more adept mathematics students. The interviewee considered Mathematical proof the most important aspect, because it plays a fundamental role in geometry. This is necessary for the students in discovering their environment and their world, because the student can understand the environment through their understanding of geometry. He then ordered other aspects with regard to importance as follows: Induction, Generalization, Deduction, Logical thinking, and Use of Symbols. In relation to difficulty level, the interviewee ranked these aspects from most to least difficult as follows: Mathematical proof, Logical thinking, Use of Symbols, Induction, Generalization and Deduction.

The interviewee spent 33% of his time in teaching mathematical thinking on Mathematical proof, followed by Generalization, Induction and Logical thinking (17% of each), then, Deduction, and Use of Symbols (8% for each). Finally, the strategies the interviewee mostly used in teaching mathematical thinking were ‘analysis of figures’, ‘translations from words to Use of Symbols’, and ‘making a sketch’.

TABLE 6.1f. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.8

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Logical thinking	Mathematical proof	Mathematical proof	Mathematical proof
Generalization	Induction	Deduction	Logical thinking
Mathematical proof	Generalization	Use of Symbols	Use of Symbols
Use of Symbols	Deduction	Induction	Induction
Deduction	Logical thinking	Logical thinking	Generalization
Induction	Use of Symbols	Generalization	Deduction

In respect to level of importance, inconsistency between test results and teacher opinion was much higher in this case. In particular, Logical thinking was the most important aspect according to test results and the second least important aspect in teacher opinion. However, Induction was found to be the least important in terms of test results and the second most important aspect according to teacher opinion.

In relation to difficulty level, consistency between student test results and teacher opinion was much higher. Only the difficulties of Deduction and Logical thinking were inconsistent. Deduction was considered the least difficult aspect in teacher opinion and was the second most difficult according to test results. Logical thinking was found to be the second most difficult aspect according teacher opinion but was the second least difficult according to test results.

6.2.7 School 9

This school was a relatively large, comprehensive school located in a suburban area, with 24 students participating in the study. The teacher interviewed was male. The interviewee agreed with the researcher with regard to the aspects of mathematical thinking as Logical thinking, Generalization, Induction, proof, but excluded Deduction and Use of Symbols. With regard to order of importance, the interviewee considered Use of Symbols the most important aspect, because it is the first step in problem solving, and after that the student is required to use his mathematical knowledge to arrive at solutions. Use of Symbols is the basis of mathematical thinking. The other aspects were ordered as follows: Logical thinking, Mathematical proof, Deduction, Induction, and Generalization from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Induction, Deduction, Use of Symbols, Logical thinking and Generalization.

The ratio of time that the interviewee spent in Generalization, Induction, Deduction, Use of Symbols was as follows: 1:2:1:2. However, there are no specific lessons in Logical thinking, because this aspect is involved in each mathematical area. There are limited Mathematical proof lessons in the curriculum and each theorem needs one lesson. Finally, he used ‘searching for patterns’, ‘using figures’ and ‘Logical analysis’ when he taught mathematical thinking.

TABLE 6.1g. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.9

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Generalization	Use of Symbols	Mathematical proof	Mathematical proof
Use of Symbols	Logical thinking	Generalization	Induction
Mathematical proof	Mathematical proof	Deduction	Deduction
Deduction	Deduction	Use of Symbols	Use of Symbols
Logical thinking	Induction	Logical thinking	Logical thinking
Induction	Generalization	Induction	Generalization

In relation to level of importance, consistency between test results and teacher opinion was high. Only Generalization and Logical thinking were inconsistent. Generalization was the most important aspect according to test results and the least important aspect according to teacher opinion. Logical thinking was the second least important to test results and the second important in teacher opinion.

In relation to level of difficulty, four of six aspects of mathematical thinking were totally consistent between test results and teacher opinion. Only the difficulties of Generalization and Induction were inconsistent. Generalization was the second most difficult aspect from test results and the least difficult according to the teacher. Induction was found to be the least difficult from test results and the second most difficult according to teacher opinion.

6.2.8 School 11

This school was moderately sized school with only humanities and scientific streams located in an urban area, with 29 students participating in the study. This interviewee was male. The interviewee considered the aspects of mathematical thinking to be Induction, Investigation, Proof, and Find the optimal solution. The interviewee considered Induction the most important aspect, because through this method the student can arrive at Generalizations via specific cases, then find a pattern from the observed, and hypothesise that the pattern will be true in other similar cases (National Council of Teachers of Mathematics, NCTM, 1971, p.53). He then ordered other aspects with regard to importance as follows: Use of Symbols, Generalization, Deduction, Logical thinking, and Mathematical proof. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Use of Symbols, Logical thinking, Deduction, Induction, and Generalization.

The interviewee spent 40% of the class teaching Deduction, followed by Induction 20%, then Use of Symbols 15%, Logical thinking and Generalization 10% for each, and Mathematical proof 5 %. Finally, the interviewee mostly used 'Induction' and 'Deduction' in teaching mathematical thinking.

TABLE 6.1h. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.11

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Mathematical proof	Induction	Mathematical proof	Mathematical proof
Generalization	Use of Symbols	Generalization	Use of Symbols
Deduction	Generalization	Induction	Logical thinking
Use of Symbols	Deduction	Use of Symbols	Deduction
Induction	Logical thinking	Logical thinking	Induction
Logical thinking	Mathematical proof	Deduction	Generalization

The consistencies and inconsistencies between the teacher opinion and test results indicate some patterns. There is consistency in the level of importance for Deduction and Logical thinking. Deduction was moderately important in respect to test results and teacher opinion. Logical thinking was found to be among the least important. However, Mathematical proof was found to be the most important aspect to test results but the least important in teacher opinion. Induction was the second least important aspect in the test results and the most important according to teacher opinion.

In regard to difficulty level, inconsistency between student performance and teacher opinion was much higher. For example, Logical thinking and Deduction found to be the least difficult aspects according to student performance and moderately difficult in teacher opinion. Generalization was the second difficult aspect to student performance and the least difficult in teacher opinion. This school had the lowest results on the mathematics achievement test and the second lowest results on the mathematical thinking test. Perhaps for this reason, most of aspects of mathematical thinking in respect to level of importance and difficulty were inconsistent between student results and teacher opinion.

6.2.9 School 13

This school was a relatively small school with only humanities and scientific streams located in a rural area, with 15 students participating in the study. The teacher interviewed was female. The interviewee considered different aspects of mathematical thinking to be: reasoning, ability to apply and check answers, ability to analyse and discuss, translating word problems to equations, and agreed with the researcher about the last aspect Mathematical proof. With regard to order of importance, the interviewee considered Mathematical proof the most important aspect, because it makes a connection with other mathematical areas, and if the student is skilled at proofs that means the student will be skilled in mathematics. The other aspects were ordered as follows: Use of Symbols, Logical thinking, Generalization, Induction, and

Deduction from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Induction, Use of Symbols, Deduction, Logical thinking, and Generalization.

The interviewee spent 15% of the time teaching each of Generalization, Induction, Use of Symbols, and Mathematical proof, followed by Logical thinking 10%, then Deduction 5%. Finally, she used ‘writing an equation’, ‘connection among mathematical ideas’, and ‘making an organizing list’ in teaching mathematical thinking.

TABLE 6.1i. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.13

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Induction	Mathematical proof	Deduction	Mathematical proof
Use of Symbols	Use of Symbols	Mathematical proof	Induction
Mathematical proof	Logical thinking	Induction	Use of Symbols
Generalization	Generalization	Use of Symbols	Deduction
Logical thinking	Induction	Generalization	Logical thinking
Deduction	Deduction	Logical thinking	Generalization

In regard to level of importance, consistency between student tests and teacher opinion was high. Only the importance of Induction was inconsistent, with Induction being the most important aspect in the test results and the second least important according to teacher opinion.

In relation to difficulty level, consistency between test results and teacher opinion was high. Only the difficultly of Deduction was inconsistent, with Deduction being the most difficult aspect in the test results and moderately difficult in teacher opinion. According to test results and teacher opinion this is the only school among all participating schools where Mathematical proof was found to be the second most difficult aspect not the most difficult aspect according to the student test results.

6.2.10 School 14

This school was a very large school with only humanities and scientific streams located in an urban area, with 40 students participating in the study. This interviewee was male. The interviewee generally agreed with the researcher about the aspects of mathematical thinking but excluded Induction and included specializing. The interviewee considered Deduction the most important aspect, because it is the first step in the transformation to abstract thinking to arrive at valid conclusions. He then ordered other aspects with regard to importance as follows: Use of Symbols, Generalization, Induction, Logical thinking, and Mathematical proof. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Deduction, Use of Symbols, Logical thinking, Induction, and Generalization.

The interviewee spent 20% of the time teaching Use of Symbols and Mathematical proof, followed by Generalization and Induction (15% for each), then, Deduction 10%. However, Logical thinking was omitted from the mathematics curricula. Finally, the strategies the interviewee mostly used when teaching mathematical thinking were ‘looking for patterns’, ‘making inferences from premises’, and ‘optimization’.

TABLE 6.1j. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.14

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Generalization	Deduction	Mathematical proof	Mathematical proof
Logical thinking	Use of Symbols	Use of Symbols	Deduction
Induction	Generalization	Deduction	Use of Symbols
Mathematical proof	Induction	Logical thinking	Logical thinking
Deduction	Logical thinking	Induction	Induction
Use of Symbols	Mathematical proof	Generalization	Generalization

In relation to level of importance, inconsistency between student performance and teacher opinion was much higher than normal. In particular, Deduction and

Use of Symbols were found to be the least important aspects according to test results but the most important aspects in the teacher's opinion.

In relation to difficulty level, consistency between student tests and teacher opinion was much higher. Mathematical proof, Use of Symbols and Deduction were the most difficult aspects in terms of both the student tests and teacher opinion. Similarly, the other three aspects were considered the least difficult aspects.

6.2.11 School 16

This school was a relatively large, secondary, comprehensive school located in an urban area, with 43 students participating in the study. The teacher interviewed was female. The interviewee considered the aspects of mathematical thinking to be the same six aspects as the researcher, without any difference. With regard to order of importance, the interviewee considered Generalization the most important aspect, because it leads students to arrive at general laws and formulas through searching for patterns. As mathematics is rich in patterns this will assist the students to achieve the highly in mathematics. The other aspects were ordered as follows: Mathematical proof, Logical thinking, Induction, Deduction, and Use of Symbols from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Induction, Use of Symbols, Logical thinking, Deduction, and Generalization.

The interviewee spent more time in teaching Generalization and Logical thinking (20% for each), followed by Induction, Deduction, Use of Symbols, and Mathematical proof (15% for each). Finally, strategies she used in teaching mathematical thinking were 'discussing results', 'sketching pictures and figures', and 'looking for a pattern'.

TABLE 6.1k. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.16

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Generalization	Generalization	Mathematical proof	Mathematical proof
Mathematical proof	Mathematical proof	Use of Symbols	Induction
Deduction	Logical thinking	Deduction	Use of Symbols
Logical thinking	Induction	Induction	Logical thinking
Use of Symbols	Deduction	Generalization	Deduction
Induction	Use of Symbols	Logical thinking	Generalization

In relation to level of importance, consistency between student tests and teacher opinion was high. Only the importance of Induction and Deduction were inconsistent. Induction was the least important aspect from the test results and moderately importance aspect in teacher opinion. Deduction was found to be a moderately important according to student results and the second least important according to teacher opinion.

In relation to difficulty level, the consistency between test results and teacher opinion was also high. In particular, Mathematical proof was the most difficult aspect in terms of test result and teacher opinion. Generalization was found to be among the least difficult aspects.

6.2.12 School 18

This school was a relatively moderately sized school with only humanities and scientific streams located in a rural area, with 24 students participating in the study. This interviewee was male. The interviewee agreed that the aspects of mathematical thinking were generally the same as the researcher but with the inclusion of mathematical inferences. These are similar to Deduction items in that they require inferences from general statements, and the ability to interpret to accept or reject the solution. The interviewee considered Use of Symbols the most important aspect, because it is the basic step in many practical problems such as area, volume, and applications on maximum and minimum values. It is considered the most difficult step in solving these problems, and then the following steps are easier. He then ordered other aspects with regard to importance as follows: Mathematical proof, Logical thinking, Induction, Generalization and Deduction. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Use of Symbols, Logical thinking, Deduction, Induction, and Generalization.

The interviewee spent the more time in teaching Logical thinking and Mathematical proof (25% and 20% respectively), followed by Generalization, Induction, and Deduction (15% for each) , then, Use of Symbols 10%. Although the interviewee considered the most important aspect to be Use of Symbols, he spent the minimum in teaching it, due to little focus on it in the curriculum. Finally, the strategies the interviewee mostly used were 'discussing the results', 'sketching pictures and figures', and 'Generalization' in teaching mathematical thinking.

TABLE 6.1I. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.18

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Induction	Use of Symbols	Mathematical proof	Mathematical proof
Mathematical proof	Mathematical proof	Use of Symbols	Use of Symbols
Deduction	Logical thinking	Generalization	Logical thinking
Use of Symbols	Induction	Deduction	Deduction
Logical thinking	Generalization	Induction	Induction
Generalization	Deduction	Logical thinking	Generalization

In relation to level of importance, inconsistency between student performance and teacher opinion was high. Only the importance of Mathematical proof and Generalization were consistent. Mathematical proof was the second most important aspect in terms of both the student tests and teacher opinion. Generalization was among the least important aspects on both measures.

In relation to level of difficulty, consistency between student performance and teacher opinion was much higher. Only the difficulties of Logical thinking and Generalization were inconsistent. Logical thinking was found to be moderately difficult in teacher opinion and the least difficult aspect according to test results. Generalization was moderately difficult in the test results and the least difficult aspect according to teacher opinion.

There was an absolute consistency with respect to teacher opinions between school No.11 and school No.18 in relation to the order of mathematical thinking aspects according to the level of difficulty, but there was a large difference between these schools in test results at this level. There was a large consistency in school No.18 between test results and teacher opinion and an inconsistency for school No.11. School No.11 also had the lowest performance in mathematical thinking and mathematics achievement tests whereas school No.18 had a moderate school performance on the mathematical thinking test and was one of the best 25% of schools in the mathematics achievement test. Perhaps the teacher in school No.11 was not aware of the relative difficulty of the aspects of mathematical thinking, which could explain the relatively poor performance of the students in his class.

6.2.13 School 20

This school was a large school with only humanities and scientific streams located in an urban area, with 29 students participating in the study. The teacher interviewed was female. The interviewee agreed generally with the researcher with regard to the aspects of mathematical thinking with the inclusion of specialization. With regard to order of importance, the interviewee considered Generalization the most important aspect, because it is important in arriving at a Generalization, and because it develops inductive thinking skills. The other aspects were ordered as follows: Mathematical proof, Use of Symbols, Induction, Deduction, and Logical thinking from most to least important. In relation to difficulty level, the interviewee ranked these aspects from most difficult to least difficult as follows: Mathematical proof, Logical thinking, Induction, Deduction, Use of Symbols, and Generalization.

The interviewee spent 20% of class time in teaching Use of Symbols, followed by Generalization, Induction, and Mathematical proof 15%, then Deduction and Logical thinking (10% for each). Finally, the strategies she used in teaching mathematical thinking were ‘Logical analysis’, ‘proving the results’, and ‘looking for a pattern’.

TABLE 6.1m. THE COMPARISON BETWEEN THE ORDER OF ASPECTS OF MATHEMATICAL THINKING IN RELATION TO LEVEL OF IMPORTANCE AND DIFFICULTY ACCORDING TO STUDENTS AND THE TEACHER IN SCHOOL NO.20

Level of importance		Level of difficulty	
Student order (test results)	**Teacher order**	**Student order (test results)**	**Teacher order**
Mathematical proof	Generalization	Mathematical proof	Mathematical proof
Use of Symbols	Mathematical proof	Induction	Logical thinking
Logical thinking	Use of Symbols	Generalization	Induction
Deduction	Induction	Use of Symbols	Deduction
Generalization	Deduction	Deduction	Use of Symbols
Induction	Logical thinking	Logical thinking	Generalization

In relation to level of importance, consistency between student performance and teacher opinion was high. Mathematical proof and Use of Symbols were among the important aspects, whereas Induction and Deduction were among the least important aspects. Only the importance of Generalization and Logical thinking were inconsistent. Generalization was the second least important aspect in the test results but the most important in teacher opinion. Logical thinking was found to be moderately important aspect in the student test and the least important aspect according to teacher opinion.

In relation to level of difficulty, consistency between student performance and teacher opinion was high. Only the difficulties of Generalization and Logical thinking were inconsistent. Generalization was a moderately difficult aspect from the results that derived from students and the least difficult aspect in teacher opinion. Logical thinking was found to be the least difficult according to student results and the second most difficult in teacher opinion.

6.3 GENERAL SUMMARY

This section described and discussed the results of the teacher interviews with 13 individual teachers, six of whom were females and the remaining seven were males. The researcher asked the interviewees seven questions. The first two questions were open questions; Question 1 asked for their opinion on what mathematical thinking means and the second asked them to identify what they believe to be the most important aspects of mathematical thinking. The conclusion with regard to what constitutes mathematical thinking in the teachers' opinions (not in any order) is effective thinking, the basic power of mathematics, analytical thinking, anticipatory thinking, Generalizations and theorems, thinking which depends on Mathematical proof, Logical thinking, the ability to make inferences, connections, and proof. The teachers felt that mathematical thinking is developed through practice and reflection and through the study of all sciences in general, and mathematics in particular. Some mathematical thinking problems are like games and challenges, and this is not restricted to the domain of mathematical computation and formula.

We now move to the second open-ended question (Question 2) which asked interviewees to identify the aspects of mathematical thinking. According to data collected from teacher interviews, the most frequently mentioned aspects of mathematical thinking were Generalization, and Mathematical proof (approximately 85% of the teachers considered these two aspects to be part of mathematical thinking). Next in importance were Induction, Deduction, using Symbols and mathematical expression (approximately 62% of the teachers considered each of these three aspects to be part of mathematical thinking). Logical thinking or reasoning followed (approximately 54% of the teachers considered this aspect to be part of mathematical thinking). Approximately 8% of teachers considered other mathematical thinking aspects such as problem solving, application of Generalization, challenges, using patterns, investigation, finding the optimal solution, inferences, and the ability to sketch pictures and figures.

Questions, 3, 5 and 6 addressed the level of importance of the six aspects of mathematical thinking, level of difficulty, and time spent in teaching each aspect, respectively[9]. In these questions, the researcher provided a list of aspects of mathematical thinking: Generalization, Induction, Deduction, Use of Symbols, Logical thinking, and Mathematical proof. A Likert scale was employed to differentiate level of importance and level of difficulty, ranging from 6 for most important / most difficult and 1 for least important / least difficult. In relation to the level of importance the analysis of the teachers' interviews for each aspect would have possible mean range of 6 to 1, 6 representing very important and 1 least important.

TABLE 6.2. LEVEL OF IMPORTANCE ACCORDING TO TEACHERS' OPINIONS.

Teacher No	Generalization	Induction	Deduction	Use of Symbols	Logical thinking	Mathematical proof
1	4	1	5	6	3	2
2	6	3	1	2	5	4
3	5	2	1	3	6	4
6	5	2	1	3	4	6
7	5	1	2	6	3	4
8	4	5	3	1	2	6
9	1	2	3	6	5	4
11	4	6	3	5	2	1
13	3	2	1	5	4	6
14	4	3	6	5	2	1
16	6	3	2	1	4	5
18	2	3	1	6	4	5
20	6	3	2	4	1	5
Mean	4.2	2.8	2.4	4.1	3.5	4.1

Generalization was the most important with mean of (4.2), followed by Mathematical proof and Use of Symbols (4.1 for each). Then Logical thinking (3.5), and finally, Induction and Deduction (2.8 and 2.4, respectively). However, in relation to the level of difficulty, the analysis of teachers' interviews for each

[9] Question 4 asked interviewees for reasons that they chose the most important from the six aspects. Their reasons are reported in each individual summary.

aspect could range from 6 to 1, 6 representing the most difficult aspect and 1 the least difficult aspect. Mathematical proof was the most difficult aspect (6), followed by Induction (3.5), then Use of Symbols, Logical thinking, and Deduction (3.3, 3.2, and 3.1 respectively), and finally Generalization (1.8) was the least difficult aspect. All results about teacher interviews in relation to level of importance and level of difficulty are shown in Tables 6.2 and 6.3.

Although one teacher rated Generalization as the least important aspect, six teachers rated it as either the most important or second in importance. In contrast, Deduction was the least important aspect, with five teachers rating it such, two teachers rated it as either the most important or second most important. Use of Symbols, was the second most important aspect (as was with Mathematical proof) with seven teachers rating it as either the most important or the second most important aspect, with two teachers rating it as least important. For Mathematical proof, six teachers rated it as the most important aspect, with two teachers rating it as either the least important or second most important. Induction, was the fifth in importance, nine teachers rated it as either the fourth or fifth in importance. Finally, Logical thinking, was considered of moderate importance in comparison to the other aspects with almost half the teachers rating it such.

All teachers rated Mathematical proof as the most difficult of the aspects. Although Generalization was considered the easiest aspect overall, one teacher rated it as the second most difficult. Although Induction was the second aspect in the level of difficulty, four teachers rated it as the second easiest aspect. However, Deduction, Logical thinking, and Use of Symbols were considered as almost the same level of difficulty (moderate level); almost half of the teachers rated them as moderate level of difficulty.

TABLE 6.3. LEVEL OF DIFFICULTY ACCORDING TO TEACHERS' OPINIONS

Teacher No	Generalization	Induction	Deduction	Use of Symbols	Logical thinking	Mathematical proof
1	3	4	5	2	1	6
2	4	3	2	1	5	6
3	1	5	4	3	2	6
6	1	5	4	2	3	6
7	5	2	1	4	3	6
8	3	2	1	4	5	6
9	1	5	4	3	2	6
11	1	2	3	5	4	6
13	1	5	3	4	2	6
14	1	2	5	4	3	6
16	1	5	2	4	3	6
18	1	2	3	5	4	6
20	1	4	3	2	5	6
Mean	1.8	3.5	3.1	3.3	3.2	6.0

Level of importance is sometimes reflected in the time spent in teaching each aspect. Mathematical proof was the second most important aspect recording the highest time spent in teaching. In contrast, Induction recorded the second highest time spent in teaching and was the fifth aspect in terms of level of importance. In addition, Logical thinking was considered to be of moderate level of importance and the least time was spent in teaching this aspect. However, Generalization was considered the most important aspect with recoding the third highest time spent teaching this aspect, along with Use of Symbols which was considered to be the second most important aspect.

The relationship between level of importance and level of difficulty reflects the fundamental role that some aspects of mathematical thinking play. For example, Generalization was the most important aspect with an average ranking of (4.2) and the least difficulty aspect (1.8). In contrast, Mathematical proof is of similar importance (4.1) but unanimously regarded as the most difficult aspect of mathematical thinking (6.0) by the teachers. Use of Symbols were of the same importance (4.1) as Mathematical proof but nearly of mid level of difficulty (3.3). Aspects such as Logical thinking fall midway in range of importance and difficulty (3.5 and 3.2 respectively). However, Induction was of less importance (2.8) but the second in level of difficultly, with mid level of difficulty (3.5). Finally, Deduction was of least importance (2.4) and nearly was of the middle level of difficulty (3.1).

According to responses in the teacher interviews, we calculated the average time spent in teaching each aspect. Overall, the teachers spent the greatest time teaching Mathematical proof (17.6%), followed by Induction (16%), then Use of Symbols and Generalization (14.2% for each aspect), Deduction (14%), and finally, Logical thinking (11.7%). The low result for Logical thinking was expected because it was omitted from curricula about 20 years ago. The percentages calculated are low because the six aspects addressed in the interviews were not the only aspects of mathematics that are taught by these teachers, however, the total of mathematical thinking aspects was about 88%. All results about teacher interviews in relation to time spent teaching aspects are shown in Table 6.4, expressed as percentages.

The final question (Question 7) in the teacher interviews was, what are the most effective strategies they use when they teach mathematical thinking. The most important strategies that the teachers were more likely to use in their teaching were, looking (searching) for a pattern, for example, searching for patterns, helping the students to find general formulae that assist in solving many problems in Generalization and Induction. Drawing a picture was used to help students think about the relationships in a problem. The strategy of trying a simple problem, for example, was used to encourage their students when they faced complex problems. This strategy involves trying the same problem with smaller numbers or by dropping some conditions. The strategy of writing an equation was used with some algebraic problems which require writing in equations or inequalities to solve. Other strategies employed were Logical analysis, using a model, discussion of results, and optimization.

TABLE 6.4. PERCENTAGE OF TIME SPENT TEACHING THE ASPECTS ACCORDING TO TEACHERS' OPINIONS.

Teacher No	Generalization (%)	Induction (%)	Deduction (%)	Use of Symbols (%)	Logical thinking (%)	Mathematical proof (%)
1	10	10	---	15	---	15
2	15	15	10	10	20	20
3	10	15	25	10	15	20
6	10	15	25	10	15	20
7	20	15	5	10	10	20
8	16.5	16.5	8.5	8.5	16.5	33.5
9	13.5	26.5	13.5	26.5	---	10
11	10	20	40	15	10	5
13	15	15	5	15	10	15
14	15	15	10	20	---	20
16	20	15	15	15	20	15
18	15	15	15	10	25	20
20	15	15	10	20	10	15
Mean	14.2	16.0	14.0	14.2	11.7	17.6

Comparing the six aspects of mathematical thinking in relation to level of importance, level of difficulty, and time spent of teaching each aspect. Mathematical proof was considered the most difficult aspect, and the most time was spent in teaching this aspect, even though it was considered to be the second in importance. Deduction was considered the least important aspect, and the fifth in relation to level of difficulty and time spent in teaching. Generalization was considered the easiest aspect, and the most important aspect, but a moderate amount of time was spent in teaching it. Induction was considered the fifth aspect in importance, and of moderate level of difficulty, but it was considered the second aspect in relation to most time spent in teaching it. Logical thinking was given the least time in teaching, and considered of moderate importance and difficulty. Use of Symbols was considered to be moderate in relation to level of importance, level of difficulty, and time spent in teaching it. Table 6.5 shows level of importance, level of difficulty, and time spent by all teachers.

TABLE 6.5. LEVEL OF IMPORTANCE, LEVEL OF DIFFICULTY, AND TIME SPENT BY ALL TEACHERS.

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time (%)
Generalization	4.2	1.8	14.2
Induction	2.8	3.5	16
Deduction	2.4	3.1	14
Use of Symbols	4.1	3.3	14.2
Logical thinking	3.5	3.2	11.7
Mathematical proof	4.1	6	17.6

The teachers' opinions about aspects of mathematical thinking, in terms of level of importance, and level of difficulty are now discussed in relation to the students test results. Consistencies and inconsistencies are noted. Teachers' opinions of importance were almost the same as importance of the mathematical thinking aspects for mathematics achievement in the student tests. There was some change in the order of the first two aspects and the last two aspects. The order for Generalization and Mathematical proof (the two most important aspects) and Induction and Deduction (the two least important aspects) were reversed in each case.

Teacher opinions of difficulty differed more from the student tests results than was the case of importance, although Mathematical proof was consistently the most difficult aspect, and Use of Symbols was the third. Logical thinking was moderately difficult according to the teachers and the easiest aspect according to student tests. Further, Generalization was the least difficult aspect according to teachers but moderately difficult for the student tests. Other two aspects almost considered moderate in relation to level of difficulty in both teachers and student tests.

6.4 THE STUDENT INTERVIEWS

Each of the student interviews is now summarised using the school number as identification. Participants in the interviews comprised two groups of male students, five students in each school, and two groups of female students, five students in each school. The groups were located in three suburban schools and one rural school. The interviewer discussed with the interviewees the strategies that they used to answer five questions in the test of mathematical thinking, what are the different ways in thinking that they used, and how they reached their answers. A copy of each of the student interviews can be found in Appendix 5.2.

6.4.1 School 1

This school was located in a suburban area, with 24 students participating in the study. The interviewees were female students. There were five interview participants, two students with high achievement in mathematics, two moderate, and one student with low achievement. The interview was comprised of five questions: item 3 Generalization, item 3 Induction, item 1 Deduction, item 5 Use of Symbols, and item 4 Mathematical proof.

Item 3 Generalization: the initial idea about the expression was ‘the summation for each two odd numbers is even number’, but after the researcher gave them some hints (eg, If so, why we did not write 12 = 3+ 9, although 3 and 9 are odd numbers, (see pp. A-99-A-100 for the complete interview), two of the students were able to find the correct answer.

Item 3 Induction: Some students were able to find a pattern to the number of rice grains in each square; another student answered the question using ‘double the amount each time’. In relation to the total number of grains, after the interviewer’s hint (eg, could you please rewrite the number of total grains of rice up to the specific square using 2 to the power of the number of the square, (for the complete solution see pp. A-100-A-101), one student was able to find the pattern for the total. However, other students used the classical way to find the total. No student was able to find the total amount, but they agreed it was a huge number, and one student said perhaps it might represent the world production of rice for ten years.

Item 1 Deduction: The item was clearly non-routine from the responses interviewees gave to the researcher's question (have you seen it before?). All interviewees knew that y must be positive, because positive times positive gives a positive number. Some of the students were able to find the correct answer that required the students to know the converse relationship that indicates if x increases, then y must decrease. In contrast, the highest achieving student answered: x is less than 1, and y is less than 1. For example, x = .2, and y = .5, then xy = 1. However, after recalculating her answer, she found her mistake.

Item 5 Use of Symbols: Most students saw the same problem in a slightly different form, which required finding the whole areas for the shapes. Finding the known areas was routine, and finding the shape area in general, was non-routine. Most students were able to express the solution in specific terms such as 11^2, 12^2, 13^2, but only one student was able generalize and correctly find the expression for n^2, and she checked her answer correctly.

Item 4 Mathematical proof: The item was to prove a specific theorem. This was considered a routine problem, because most students faced problems with the same idea in the mathematics curriculum. The interviewer began by asking the students to prove that the Δ BSC is an isosceles triangle, then used other theorems that related to opposite angles - altitude angles, to prove the Δ ABC is an isosceles triangle. However, one student proved that Δ ABC is an isosceles triangle in a different way after proving that the Δ BSC is isosceles, she used the two following theorems for proof: the altitudes make 90° with the intersection lines, and the total of the angles for any triangle is 180°. The students knew the relationship between equilateral and isosceles triangles is: "equilateral is a specific case of an isosceles triangle".

6.4.1.1 Conclusion

The findings from this interview revealed that some students were unable to think mathematically, particularly for items that required them to find the Generalization or patterns. These students attempted to answer the items using classical methods until they received some explanations prompting them how to think. In addition, some of them believed that checking whether the solution is correct is necessary, but they did not activate this step. The researcher found there were some weaknesses in writing proof in mathematics to show each step in the process of answering the item such as item 4 in Mathematical proof.

6.4.2 School 2

This school was located in a suburban area, with 31 students participating in the study. The interviewees were male students. There were five interview participants, two students with high mathematics achievement, two moderate, and one student with low achievement. The interview was comprised of five questions, item 3 Generalization, item 3 Induction, item 1 Deduction, item 5 Use of Symbols, and item 4 Mathematical proof.

Item 3 Generalization: Student responses were that the summation of two odd numbers is an even number. One student (the highest achieving student) was able to find the correct answer. Some students (male students) responded with no answer to the interviewer's questions until he gave them some explanations. In contrast, in school 1 (female students) there were some responses without prompting or explanation to the interviewer's questions.

Item 3 Induction: Low and middle achieving students solved this item using classical methods. One student was able to find the total rice grains in the whole chessboard using an unusual method which found the relationship between the total rice grains up to a specific square and the number of rice grains on the following square such that the total of rice grains up to n square = the number of rice grains in the (n + 1) -1 after the observation of specific cases. No other students were able to find the correct answer without classical methods until the interviewer gave them several suggestions (eg, the researcher began by asking what is the total of the rice grains up to specific squares, could you please rewrite these numbers as 2 to the power number, etc, (see pp. A-105-A-107 for the complete interview) to enable find them to the other relation that indicated the number of rice = $2^{\text{square number}} - 1$. Students with low and middle achievement believed that the item answer was a relatively small amount of rice, particularly after only a first glance at the item.

Item 1 Deduction: Students agreed that y will be negative to give a positive number, because x>0. However, one student answered that to get 1 x will be less than 1, and y will be less than 1, for example, x = ½ and y = ½. Then, the interviewer asked the same student (Does ½ X ½ = 1), and after recalculating he realised his mistake. Two students were able to find the correct answer, and one student knew the name of the relationship between x and y.

Item 5 Use of Symbols: Most of the students were able to find an approximately correct expression of analyses of specific numbers. In contrast, most of them were unable to find a correct expression for n^2. One student was able to find the correct expression for n^2 after the interviewer hinted (eg, what are the constants and variables in each specific case, see pp. A-110-A-111 for the complete solution) and he was able to check whether his answer was correct or not by substituting specific numbers.

Item 4 Mathematical proof: The interviewer started by asking the students, for example, What is meant by altitude? and most students did not know the meaning of altitude until given some explanations. Then the researcher asked them about the most related theorems needed to prove the theorem. One student was able to prove the theorem, and other students were unable to prove the theorem using either the same or different methods.

6.4.2.1 Conclusion

The researcher chose the same five items for the first and the second groups (one female group and one male group), so as to ascertain the differences between females and males. The researcher found female students more active than male students in their responses to these items. More males than females gave initial answers such as ‘I have no idea’, ‘I do not know’, or gave no answer .Some students believed that checking whether their answer is correct is necessary but unfortunately, they did not check their answers, particularly when doing tests.

6.4.3 School 5

When the interviews were conducted, this class teacher was on extended leave, so there was no interview with teacher in school No.5 to compare the test results and teacher opinion. However, the order of importance and difficulty levels according to test results are shown in Table 6.6.

TABLE 6.6. THE ORDER OF IMPORTANCE AND DIFFICULTY LEVELS ACCORDING TO STUDENT RESULTS IN SCHOOL NO.5.

Student order (test results)	
Level of importance	Level of difficulty
Use of Symbols	Mathematical proof
Generalization	Deduction
Mathematical proof	Generalization
Induction	Induction
Deduction	Use of Symbols
Logical thinking	Logical thinking

This school was located in a suburban area, with 19 students participating in the study. The interviewees were female students. There were five interview participants, one student with high mathematics achievement, two moderate, and two students with low achievement. The interview was comprised of five questions, item 4 Generalization, item 2 Induction, item 3 Induction, item 5 Use of Symbols, and item 4 Mathematical proof.

Item 4 Generalization: Some of students had previously faced the same problem with a slight difference that required finding the number of diagonals for specific polygons. Some of students found the number of diagonals for the specific polygons by drawing the polygons. Other students attempted to use a quadratic relation, but only two students were able to find the correct answer for specific and general polygons using quadratic relation. One student was able to check whether the answer was correct or not using the number of diagonals for the pentagon polygon.

Item 2 Induction: Students were able to distinguish that this sequence was not arithmetical or geometrical. The interviewer asked the students whether the integer number was an arithmetical or geometrical sequence and the students knew it was an arithmetical sequence “odd number”, but that the fraction was not an arithmetical or geometrical sequence. Students also knew the term n in general after receiving some hints (eg, the researcher asked the interviewees to check their answers to find the correct general solution, see pp.A-113-A114 for the interview questions). Most students answered the question using classical methods until the tenth term in the sequence. Some students mistakenly found the next missing term in the sequence rather than the tenth term.

Item 3 Induction: One student said we put one grain of rice in the 1^{st} square, 2 in the second, and 3 in the third square and so on, but other students said “no this is wrong answer”. Only one student had previously faced a similar problem, and although the previous chessboard had contained wheat rather than rice, she knew the answer would be a large number. Students knew the number of rice grains in each square is double the number in the previous square and they were able to rewrite these numbers as a number to the power 2. The interviewer simplified the problem by asking the students to consider it contained 4 squares rather than 64 squares to find the total number of rice grains and the pattern. The students were able to find the pattern with a little explanation. However, the students thought the solution to this problem required only a few kilograms of rice.

Item 5 Use of Symbols: Students had faced the same problem in a slightly different form in previous classes. Students were able to express specific numbers without using exponential and addition as the same terms. The interviewer gave them a small hint (eg, observe the constants and variables in each expression to generalise, for the complete interview see pp.A-116-A-117), and they were then able to correctly express the specific numbers and generalise.

Item 4 Mathematical proof: Students had faced a slightly different problem in previous classes. Interviewer and students proved the theorem by considering Δ

BSC to be an isosceles triangle, and then proved Δ ABC to be an isosceles triangle (common proof). However, other students proved this theorem using a different method such that $\angle$ BSC = 140° and as it is considered outside Δ MSB and Δ NSC and equal to the total of two opposite interior angles, then completed the proof.

6.4.3.1 Conclusion

Most interview questions involved finding patterns and the researcher found most students answered these questions using classical methods. For example, the first question was answered using drawing the shape, the second question was answered finding all terms until the tenth term. However, classical methods are sometimes not practical. It is not the best method if we want to find the general solution, and it takes a long time. The last item was proved in two different ways, suggesting that these students were able to adopt a variety of approaches to the problem.

6.4.4 School 6

This school was located in a rural area, with 28 students participating in the study. The interviewees were male students. There were five interview participants: two students with mathematics achievement, two moderate, and one student with low achievement. The interview was comprised of five questions, item 2 Generalization, item 3 Induction, item 4 Use of Symbols, item 5 Use of Symbols, and item 4 Mathematical proof.

Item 2 Generalization: This item was a difficult item and non routine. However, some students had previously faced the same problem in a slightly different form that required them to find the summation for the first n numbers. One student observed the difference between the first outcome and the second is 3, and between the second and the third is 5, then 7, and so on, but he was unable to find the summation of the last statement, because the summation for the previous statement was unknown. The interviewer asked students to rewrite the last term in each statement as 2n-1 to link n in each term with the total.

Other students answered this problem when asked if they had another way to solve this problem linking the number of terms in each statement and the total to indicate that the total = the square of the number of terms.

Item 3 Induction: Some students knew that there is double the number of grains of rice in the following square, and only one student had previously faced such a problem but had not achieved any solution. The students were able to rewrite number of rice grains in each square using the exponential method without any difficulty. However, the students were unable to find the total of the grains of rice on the whole chessboard until prompted by the interviewer's explanations and hints (the researcher began to simplify the original question so that chessboard contained of 9 squares rather that 64 squares and asked them to rewrite the total numbers as an exponential expression, for the complete interview see pp.A-120-A-121). They believe that mathematics comes with correct answers, but sometimes these answers are not practical.

Item 4 Use of Symbols: Students had faced such a problem in year 10 which required them to determine if the shape was circular or not. The initial answer was the first shape was circular, because the total of the opposite angles was 180°, but the second shape was not, because the known was the neighbouring angles not the opposite angles. This emphasis that students did not know if neighbours' angles = 180° perhaps the opposite angles will be 180° as well. The interviewer asked some questions to explain the idea that if two neighbouring angles = 180° then, perhaps the two opposite angles = 180° as well.

Item 5 Use of Symbols: Students expressed 11^2 correctly, but they did not express 13^2 and n^2 correctly until the interviewer gave them further information (see pp.A-122-A-124) and asked them to check their answers by calculation to find the correct answers.

Item 4 Mathematical proof: Some students mentioned that "if we drop an altitude from the vertex triangle it will divide the base into equal parts, and divide the vertex angle into equal angles". Then the researcher explained that we can not use this theorem until we prove that Δ ABC is isosceles. The interviewer then used almost the same procedure to prove the theorem. The students from this school who were able to prove this theorem, all used the same method.

6.4.4.1 Conclusion

The researcher learned from the interview which items were clearly non-routine items, and the different methods students used to solve these problems. In addition, for item 4 Use of Symbols, the researcher wanted to know whether students knew that the total of each opposite two angles is equal to 180°, then the shape is circular, and it is not necessary that the two neighbouring angles equal 180° for the shape to be not circular. However, students were able to express specific square numbers with few hints, but they needed more hints to express general n^2.

6.5 GENERAL SUMMARY

This section described and discussed the results of student interviews with 4 groups of students, two male groups, and two female groups. Each interview involved discussion of 5 problems; three problems were the same for each group, and two problems were different among the groups, but the two different problems were the same for the first and second groups.

Most of interview problems according to interviewer expectation were non-routine mathematical problems so as to be better able to assess the students' mathematical thinking ability. The first common problem ”chessboard problem”, was a non routine problem; only one student (the highest achieving student) among each group had heard about this problem, with wheat or rice from different books. We found from these interviews that some of the students solved this problem using traditional approaches, without looking for a pattern. For example, they doubled the number in each square to find the number of rice in each square and to find the total of rice grains in the chessboard they used the normal summation from the first square to the final square. Students were able to find the pattern for the number of rice grains in each square sometimes after a little explanation. However, most students were unable to find the pattern for the total number of rice grains up to specific square and on the whole board.

The second common problem, the “three shapes” problem, was a non routine problem, because none of the students had seen similar problems which required them to find the shape area in general. In addition, most of the students had previously faced some problems which required them to find the area for the whole shape through addition of the areas that represented the shapes.

The third common problem, the “isosceles triangle” problem, was familiar to most of the students, because our mathematical curricula are rich in problems like this, albeit with slightly different forms. However, in general the other two problems were non routine problems with the exception of problem two in school number five which was a routine problem.

The fourth step in mathematical problem solving according to Polya (1990) is checking answers. In most of the student interviews students agreed that checking whether the answers are correct or not is an important step in any problem solving. The researcher found, however, that most students did not check their answers, even though they agreed it is an important step. The main reasons cited were there is not enough time, particularly in tests, nervousness, and that their teacher did not encourage them to check answers and told them the most important thing was to find the correct answer rather than to check their answers. In addition, rarely did their teachers check their answers during problem solving. Some students considered checking their own answers to be unimportant, particularly, if they answered the problems carefully and reviewed the method they had used.

The researcher found females more likely to answer the interview questions than males. They were more responsive and the interaction during interviews between females themselves was much better than that between males. The researcher was required to prompt and provide further information to elicit information more often from males than from females. In some cases, the researcher received no response from the male students until he gave them a hint or more explanation. However, the number of hints necessary was fewer for females than males.

Item 4, in Mathematical proof was approximately routine problem, because the students had faced such a problem in year 9. Female students proved the theorem in this item using different methods, but male students proved it using the same method. For example, some female students proved that the Δ BSC is an isosceles triangle first, then they proved the Δ ABC is an isosceles using the relation between opposite angles and altitude angles. Other female students used a different method to prove it using different theorems after they proved Δ BSC is an isosceles triangle. For example, altitudes of triangles are drawn from the apex to intersect the opposite side at 90° and the summation of the angles for any triangle is 180°. A third group of female students proved Δ CSB is an isosceles triangle, and then they used the ∠ BSC = 140° the exterior angle is equal to the two opposite interior angles. However, all successful male students proved the theorem using the first method described above. This suggested that females were more likely to have a general understanding and were more capable of thinking logically, reasoning and making connections between different related theorems to prove the theorem. A copy of transcripts of student interviews can be found in of Appendix 5.2.

6.6 CONCLUSION

The previous chapter provided some of results derived from the quantitative data in regard to the effect of gender differences, location differences, interaction between gender and location, and linking mathematical thinking and mathematics achievement. However, this chapter described and discussed the 13 teacher interviews and four student interviews, two female groups and two male groups. The first section presented a summary of each individual teacher interview, followed by the general summary for all teacher interviews. The general summary presented the overall meaning of mathematical thinking, the most important aspects that comprise mathematical thinking, level of importance and level of difficulty, and time spent in teaching the six aspects of mathematical thinking. In addition, this chapter presented the relationship between level of importance, level of difficulty, and time spent in teaching. Teacher opinions and results derived from student responses in relation to level of importance, level of difficulty, and time spent in teaching the six aspects of mathematical thinking were also presented in this section.

The second section presented the individual student interviews, followed by a general summary that summarised all interviews. Following this chapter, a discussion of results is presented. Chapter 7 discusses the results that were reviewed in Chapter 5 and Chapter 6. Chapter 8 presents the conclusions, recommendations, limitations, and suggestions for future research.

CHAPTER SEVEN

Discussion of Results

This chapter comprises a discussion of the results of the mathematical thinking test and the mathematics achievement test presented in chapter five and the results of teacher and student interviews presented in chapter six.

7.1 STUDY AIMS

This study aimed to investigate the relationships between aspects of mathematical thinking and mathematics achievement in the Year 11 scientific stream in Jordan. This stream includes the high achieving students with respect to mathematics. In addition, this study also investigates the gender, school, and school location differences in mathematical thinking and mathematics achievement. This study also investigates the interaction between gender and school location in relation to mathematical thinking and achievement.

Further, this study examined any consistencies and inconsistencies between the teachers' opinions about the aspects of mathematical thinking in terms of level of importance, level of difficulty and the results derived from the quantitative analyses of the student answers to the mathematical thinking test. In relation to the student interviews, the study aims to identify popular strategies used by students to arrive at solutions, and to differentiate between the thinking skills they used and to ascertain their attitude toward checking their solutions.

7.2 RELATIONSHIP BETWEEN SCALES

The individual relationships between the six aspects of mathematical thinking were all positive and statistically significant at $p < .01$ using Pearson product-moment correlation coefficients. The highest correlation was between Generalization and Use of Symbols, and the lowest correlation was between Induction and Deduction. The correlations of the six aspects of mathematical thinking with the total score for mathematical thinking and mathematics achievement were also statistically significant, ranging from 0.45 to 0.74, with the highest level of correlation between the six scales and mathematics achievement in Generalization. The second highest level was Mathematical proof.

The statistically significant relationships between the six aspects of mathematical thinking were expected, because collectively they comprise the power of mathematics. In relation to the highest correlation, relationship between Generalization and Use of Symbols, this was also an expected result. However, even the highest correlation indicates that only 20% of the variance was shared. This indicates a high degree of independence between these two aspects. Algebraic thinking describes generalizations succinctly by being concerned with the structure of a mathematical statement (MacGregor, 1993). In contrast, the lowest relationship was between Induction and Deduction; we found this result because Induction and Deduction are opposite approaches to mathematical problem solving. Induction requires arriving at general laws from specific cases (specific to general). In contrast, Deduction requires arriving at a specific conclusion from valid premises (general to specific).

The relationships between the individual aspects of mathematical thinking and the total aspects of mathematical thinking demonstrated strong correlation coefficients, which ranged from 0.62 to 0.71. In part these results were found because the total score for mathematical thinking includes each of the six specific abilities. However, the relationships between the six aspects of mathematical thinking with mathematics achievement also had strong correlation coefficients, which ranged from 0.45 to 0.63, with a mean of 0.55. The mathematics achievement score is a single scale that measures students' achievement based on the school curriculum and reflects school achievement tests and examinations. The highest correlation for among the six aspects of mathematical thinking with mathematics achievement was Generalization, then

Mathematical proof, perhaps because generalizations are considered the life blood of mathematics (Mason et al, 1991, p.8), and because many results have been discovered through lucky generalizations in mathematics (Polya, 1990, p.108). This leads us to accept the prime importance of generalization in mathematics achievement. However, Mathematical proof in geometry is considered one of the NCTM standards (Hynes, 1995, 1996). It also plays a critical role in teaching mathematics (Schoenfeld, 1994, p. 274). Proofs and geometry are also considered an important part in any mathematics curriculum.

7.3 GENDER DIFFERENCES

When mean scores for the male and female students on each of the six aspects of mathematical thinking and the total scores of mathematical thinking were compared, there was a significant gender difference for two aspects and for the total score of mathematical thinking. Females had significantly higher scores than males for Logical thinking, Mathematical proof, and for total mathematical thinking. For both Logical thinking and Mathematical proof, the results contrast to other findings which indicated that males outperformed females or that there were no statistical differences between them (Al-Hassan, 2001; Battista, 1990; Bitner-Corvin, 1987; Cox, 2000; Hanna, 1986; Huntley, 1990; Ma, 1995; Mills, Ablard, & Stumph, 1993; Senk & Usiskin, 1983; TIMSS, 2003), However, that females outperformed males in Logical thinking is consistent with Cox (2000). In Mathematical proof, these are interesting findings that are consistent with Senk and Usiskin (1983) and TIMSS (2003). In TIMSS (2003) Jordanian females had a significantly higher average score than males consistent with seven other countries in geometry.

The year 8 students in TIMSS (2003), for both Islamic and non Islamic countries, results indicated that for a large proportion of the countries there was no gender difference in mathematics achievement. In more than 20% of the countries males outperformed females, and in less than 20% of the countries females outperformed males. Perhaps the results showing that females outperformed males on two aspects of mathematical thinking, and no statistically significance was the evident for the other four aspects, indicate that in Jordan, as with any Arabic and Islamic country, females tend to spend a lot more of their time at home than male students due to the strictures of religion (Alkhateeb, 2001). It is possible that in non Islamic countries where females may not spend as much time at home that differences in achievement between males and females may not be as pronounced. This argument depends on the implication that because the female students are at home, they are likely to spend more time in studying. It is perhaps also relevant that when a female teacher teaches female students as is the case throughout Jordan public schools, it is possible that the female students feel more comfortable asking questions of their teachers than do male students of their male teachers. Finally, In Jordan female students seem to have a greater incentive than males to complete their education. Males are more likely to be able to get a job without graduating such as in the military, industries and private businesses. However, as it is difficult for females to get a job without completing their education, they perhaps have a greater incentive to do well at school in all subjects, including mathematics. This contention is supported by the fact that only 70% of males continue into upper secondary school (year 11 and 12), whereas 75% of females continue their secondary education (Ministry of Education, 2002).

In contrast, there were no gender differences in mathematical thinking for the other four aspects. For two of the aspects, Generalization and Use of Symbols, this study found there was no gender difference. This result is supported by other research that investigated gender difference in algebra (symbols) (Armstrong, 1981; Hanna, 1986; Low & Over, 1993; Stites, Kennison, & Horton, 2004; TIMSS, 2003). In TIMSS (2003), in almost half of participating countries there were no gender differences in the algebra subtest. For the other two aspects, Induction and Deduction, there were also no gender differences evident.

It should be recalled that the reliabilities of the mathematical thinking scales used in this investigation were relatively low. This raises the possibility that more reliable tests might have better identified any differences that did exist. As there were no previous studies specifically focusing on this area, it is not prudent to speculate further on this possibility.

In the mathematics achievement test, females outperformed males. This result is an interesting finding, because it is inconsistent with other research that found males generally outperformed females in mathematics achievement (Baya'a, 1990; Cox, 2000; El Hassan 2001; Hanna, 1986; Leder, 1990b, cited in Leder & Frogasz, 1992; Low & Over, 1993; Ma, 1995; Randhawa & Hunt, 1987; Randhawa, 1988; Uekawa & Lange, 1998; Young, 1994), or research where no relationship between mathematics achievement and gender was found (Ai, 2002; El Hassan 2001; Hanna, 1986; Lauzon, 1999; Lantz & Smith, 1981; Low & Over, 19983: Uekawa & Lange, 1998; Young, 1994). This result, however, is consistent with some research findings (Ai, 2002; Alkhateeb, 2001; Cook, 2000; Cox, 2000; Dennis, 1993; Ma, 1995; Randhawa & Hunt, 1987). In addition, this result is consistent with results from the Ministry of Education (2001) test for the 9th grade throughout Jordan where females achieved significantly higher scores than males. In the Irbid governorate females with a mean score of (50%) outperformed males (36%) in the national test (ibid, 2001). These results are similar to the results of this study where females also (57%) outperformed males (51%)[10]. For TIMSS (2003) in Jordan, as with some of other Arabic countries participating in TIMSS (2003) such as Bahrain, females also outperformed males in mathematics achievement.

Due to the strong relationship that was found between mathematical thinking and mathematics achievement, most of possible reasons why females had higher mean scores than males for mathematical thinking are also relevant for mathematics achievement. Further possible reasons include that most small and rural schools contain only the humanities stream, and relatively moderate and large schools include humanities and scientific streams. Most of the females prefer to study in their villages rather than go to other schools. For this reason, they are more likely to study in the humanities stream rather than the scientific stream except for those who have high ability in sciences and mathematics who are more likely to travel to schools. The males generally tend

[10] The 9th grade includes all students with differing mathematical ability; however, participants in this study were derived from the Year 11 scientific stream which includes only high achieving students in mathematics.

to be more prepared to travel to attend another school. However, the Islamic religion treats both genders equally, and encourages families to teach their daughters in the same way as their sons. The nature of females in Jordanian culture tends to be sensitive and shy, so when a female teacher teaches female students (as is the case in gender specific schools), the students feel free to ask their female teacher any question. When female students are educated and tested alone, their performance tends to be better; and this is the case throughout Jordan. The evidence here supported that of Cook (2000) who found that, even for students taught in co educational classes, when she tested male and female students separately, females outperformed males by 12% more than when they were tested jointly with males.

7.4 LOCATION DIFFERENCES

All schools in different locations in Jordan receive the same curriculum as set by Ministry of Education but school provision would vary to some extent. In relation to location, there were significant performance differences for four of the six scales: Generalization, Induction, Use of Symbols, Logical thinking, as well as mathematical thinking (total). In Generalization and Logical thinking, suburban students outperformed urban students. For Induction, suburban and rural students outperformed urban students. For Use of Symbols, and for mathematical thinking (total) suburban students outperformed urban and rural students. In general, the order of performance from highest to lowest was suburban, rural, and finally, urban.

In relation to mathematics achievement by students in different school locations, suburban students outperformed rural and urban students. This result is inconsistent with other researchers who found urban school students outperformed their counterparts in other locations (Cox, 2000, Kleinfeld, et al, 1985, cited in Young, 1994, 1998; Uekawa and Lange, 1998; Young, 1994; 1998) or other researchers who found there was no relationship between mathematics achievement and locations (Fan & Chen, 1999; Haller, Monk, & Tien, 1993; Howley, 2003; Lee & McIntire, 2000; Monk & Haller, 1986). However, this result is consistent with the results of other research (Clarke, et al, 1980, cited in Randhawa, 1988; Cox, 2000; Lee and McIntire, 2000; Randhawa, 1988; Uekawa & Lange, 1998).

A partial explanation of the lower mathematics performance in urban schools may be that, being central, they have more science equipment than other schools. That equipment may encourage the school to increase time allocated to laboratory work. Although this may improve achievement in subjects in related to sciences and computing, it also decreases the time available to be spent in mathematics, this potentially impacting negatively on mathematics achievement.

The researcher's definition of these three school locations may also have influenced the results obtained. The researcher's definitions for the three locations were that all schools located in the city centre were defined as urban, all schools that were distant from the city centre by 10 km or less, but not in the centre were defined as suburban schools. Otherwise, the schools were defined as rural schools. These definitions did not take account of the population size, and socio economic background of the city. The, suburban students are likely to have a higher socio economic status than students in other locations. In addition, in urban areas there are many places for students to be distracted such as internet cafes, and other places of entertainment, so for this reason the students, particularly male students, may spend less time in studying. In relation to rural students achieving better than urban students it may be the case that rural students study more than urban students in order to enter university and to find a good job after high school, because most rural areas in Jordan depend on agriculture which has much lower earning potential. In contrast, the urban areas depend on commerce, government jobs and industry, so urban students may find it easier to obtain a job, because there are more opportunities than in rural areas.

Our interpretation of this interesting result is that in Jordan, all schools in different locations receive the same curricula as set by the Ministry of Education. However, in urban locations there are many distractions like internet cafes, place to play for games, and many other entertainment options. In addition, male students from urban locations tend to be more independent and more likely than students in suburban and rural areas to go to these places. This result is also inconsistent with other research which may be attributed to the differences in definition provided by individual researchers.

In terms of the difference in achievement between rural and suburban students, the higher levels achieved by suburban students could be explained by the lower value accorded educational or academic achievement in rural areas. While people in rural areas are becoming increasingly educated, it is possible a difference still exists between suburban and rural areas. Finally the socio economic status of the suburban areas may be higher than the urban and rural areas and this may affect on achievement.

7.5 THE INTERACTION BETWEEN GENDER AND SCHOOL LOCATION

When gender and location are combined as independent variables with the six aspects of mathematical thinking and mathematics achievement as dependent variables, females outperformed males in three of the mathematical thinking aspects rather than two as shown in a simple t-test when gender was considered alone. Location was significant for four aspects of mathematical thinking in the one-way ANOVA and the two-way analyses, where suburban students outperformed their peers in Generalization, Induction, Use of Symbols, and Logical thinking. However, mathematical thinking (total) and mathematics achievement were consistent in both the one-way and two-way analyses in relation to gender and location differences. Females outperformed males in mathematical thinking (total) and mathematics achievement, and suburban students outperformed their rural and urban peers in mathematical thinking (total) and mathematics achievement.

The possible interaction between gender and school location, suggested above were investigated, and there were some significant interactions between gender and location. Females in suburban schools were different from others in four of the mathematical thinking aspects: Generalization, Induction, Logical thinking, Mathematical proof and mathematical thinking (total). Females had higher mean scores for these scales. However, the differences that were found between locations were mostly due to the female students. Males had almost the same mean scores in all three locations for the four aspects. In contrast, for the induction aspect, males had significantly higher mean scores than females for urban schools.

The possible reasons for males having approximately the same mean scores in urban, suburban and rural locations are that males in all schools spend more time outside their homes with friends and play familiar games. Moreover, males in urban location spend some time also in internet cafes, or other places of entertainment, particularly those in urban and suburban areas, rather than studying. In contrast, females in suburban schools had higher mean scores than those in rural and urban schools. Although less than males, perhaps females in urban areas are more likely to go out to places of entertainment or visit their female friends than females in suburban and rural locations. In addition, females in urban areas are more likely to find a job in a private company than their peers in the other locations. In relation to gender differences in urban schools, the exception was that for induction males had higher mean scores than females.

7.6 DISCUSSION OF INTERVIEW RESULTS

This section will describe and discuss the similarities and differences between interview data and test results of mathematical thinking, in terms of level of importance, and level of difficulty and the results derived from student responses. This will be followed by a discussion of the time spent in teaching mathematical thinking, and interpretations of meaning of mathematical thinking, based on the opinions of teacher interviewees.

7.6.1 Importance

The consistencies and inconsistencies between the teachers' opinions about aspects of mathematical thinking, with regard to level of importance, and the results derived from student responses were discussed. In respect of the relative levels of importance of the six aspects of mathematical thinking, the results for teachers' opinions and student responses were almost the same with some change in the order for the first two aspects (Mathematical thinking and Generalization) and the last two aspects (Induction and Deduction). There was generally consistency between teachers' opinions and test results that Mathematical proof and Generalization were most significant related to mathematics achievement. These consistencies between teacher opinions of importance and test results indicate that those teachers who participated in this study were generally accurate about what were the most significant aspects of mathematical thinking that lead to high mathematics achievement. These results were expected, because generally teacher opinions reflect student performance across the six aspects of mathematical thinking. Moreover, the teachers in schools where students had high performance in both mathematical thinking and mathematics achievement were more accurate in their opinions of importance and difficulty levels than other teachers. This result also was expected due to the teachers' opinions reflecting student achievement. In the earlier study conducted in 1998 by the researcher, his colleagues and peers were asked to order the different aspects according to importance level. The order they gave was: Generalization, Induction, Deduction, Use of Symbols, Logical thinking and Mathematical thinking (see p.48). However, the order of importance of these aspects according to teacher opinions in the present study was: Generalization, Mathematical proof and Use of Symbols, Logical thinking, Induction, and Deduction.

7.6.2 Difficulty

With regard to level of difficulty, all teachers agreed that Mathematical proof was the most difficult aspect among the mathematical thinking aspects which is consistent with the test data collected. This result was expected, because of the nature of proof which is needed to understand concepts and procedures, and justification of each procedure and which also requires high ability in thinking. This indicates that many students faced a difficulty in constructing Mathematical proof (Baker & Campbell, 2004). This result is also consistent with Senk (1985) who claimed that writing proofs is one of the most difficult processes for students to achieve. In contrast, the least difficult aspect was Generalization according to the opinions of teachers, although this result was inconsistent with the test results which indicated that Generalization was of moderate difficulty, and Logical thinking was the easiest aspect. The teachers believed that Generalization is the easiest aspect, because this aspect is the most common aspect in mathematics and the student develops Generalization skills in mathematics and other subjects as well. However, the test results indicated that Logical thinking was the easiest aspect, possibly because the nature of items that measure this aspect focused on the meaning of some of the logical relations concepts such as intersection, union, negation of the statements, and the meaning of symmetry, which are also concepts familiar in other contexts.

7.6.3 Time Spent in Teaching Aspects of Mathematical Thinking

In relation to time spent in teaching the different aspects of mathematical thinking, Mathematical proof received the greatest time allocation. This result was expected, because the mathematics curriculum for each class contains one chapter of geometry, due to the importance of geometry in understanding the environment and the world. Mathematical proof was one of NCTM standards (Hynes, 1995, 1996). Induction received the second largest time allocation, perhaps because the teachers believed that Induction has a more general application in the curriculum than Generalization. Logical thinking received the least time. This result was expected as well, because the Ministry of Education in Jordan recently omitted specific reference to this aspect from the curriculum. Other aspects received approximately the same moderate percentage of class time.

7.6.4 Teacher Understandings of Mathematical Thinking

In respect of what is mathematical thinking, teachers reported that mathematical thinking is developed through practice through all sciences and particularly in mathematics. Effective thinking, the basic power of mathematics, analytic thinking (Bruner, 1960), the ability to make inferences and logical analysis, these two meanings being consistent with Schielack et al. (2000) that they considered these two meanings as two of aspect of mathematical thinking. These findings were expected, because collectively they comprise the concept of mathematics and real meaning of mathematics.

7.7 SUMMARY AND CONCLUSION

This chapter presented further discussion of the results presented in Chapters 5 and 6. The researcher presented the aims of the study, reviewed the results with regard to the mathematical thinking test, mathematics achievement test and teacher interviews. The researcher also presented possible reasons for his findings.

The key findings for the current study are that, in relation to level of difficulty Mathematical proof was the most difficult aspect and Logical thinking was the least difficult. The other four aspects were moderately difficult. In general, females had higher mean scores on some of the mathematical thinking scales, mathematical thinking (total), and mathematics achievement. Also, suburban students outperformed their counterparts in other regions in four of the aspects of mathematical thinking, mathematical thinking (total) and mathematics achievement. There were some consistencies and inconsistencies between interview (teacher opinions) data and test results. The consistency in respect of level of importance was that Mathematical proof and Generalization were the most important aspects in interview data and test results, and Induction and Deduction were the least important aspects in the interview data and test results. In respect of level of difficulty, Mathematical proof was consistently the most difficult aspect, whereas Generalization was considered the least difficult aspect in terms of teacher opinion but was moderately difficult in test results. Logical thinking was the least difficult aspect in test results but a moderately difficult in interview data. These results are similar to some previous research (Alkhateeb, 2001; Baker & Campbell, 2004; Cook, 2000; Ministry of Education (Jordan), 2001; Randhawa, 1988; Senk, 1985; TIMSS, 2003), but dissimilar to other research (El Hassan, 2001; Cox, 2000; Howley, 2003; Uekawa & Lange, 1998).

Chapter 8 will describe the conclusions of the study according to single level and multilevel analyses. Recommendations with regard to the most significant aspects that are related to mathematics achievement and level of difficulty will be discussed. In addition, limitations in respect to the researcher's definitions about location, sample size for the three location categories, and the possible restriction on results as a consequence of having only six aspects of mathematical thinking will be also discussed. This will be followed by some suggestions for future research.

CHAPTER EIGHT

CONCLUSIONS AND RECOMMENDATIONS

In this chapter conclusions and recommendations arising from this study will be provided. This will include conclusions in relation to the major study findings for both the quantitative and qualitative components of the study, followed by study recommendations and limitations, and suggestions for future research.

8.1 OVERVIEW OF CONCLUSIONS

An overview of conclusions based on the results of the current study is presented in this section. First, the importance of all six aspects of mathematical thinking for student mathematics achievement should be emphasised. Although the six aspects were significantly inter-correlated, each was also independently related significantly with mathematics achievement. Further than this, a regression analysis indicated that each of the six aspects of mathematical thinking was important for mathematics achievement, in the presence of all the other aspects and with gender and school location included in the model tested. Mathematical proof and Generalization were the most important, followed by Use of symbols, Deduction, Logical thinking, and finally Induction.

Given the known effects on significance testing of student clustering in schools and classes in studies of this type, the results of the single-level regression analysis predicting mathematics achievement, reported above, were checked in a multi-level regression analysis. It was found that the results were robust, with only one small difference, being that the order of importance of Logical thinking and Induction were reversed in the multi-level model. The total variance in mathematics achievement explained in the model was approximately 69.2%, with most of the explanation coming from the mathematical thinking scales.

Gender was also an important focus in this study. Female students outperformed male students for two of the mathematical thinking scales (Logical thinking and Mathematical proof), for the overall measure of mathematical thinking and for mathematics achievement. In no instance did the male students outperform the female students in this study.

There were several differences in both mathematical thinking and mathematics achievement based on school location. In general, students attending suburban schools had higher performance than students attending either urban or rural schools. There were also some interactions between gender and school location with the differences almost entirely due to the female students. Whereas the male students generally had similar performances across suburban, urban and rural school locations, the female students differed with those at suburban schools generally having higher achievement than students in the other two locations.

The teacher opinions of relative importance of the aspects of mathematical thinking and the student test results for mathematics achievement were almost the same. There were, however, changes in the order for the two most important aspects (Mathematical proof and Generalization) and for the two least important (Induction and Deduction).

In contrast, the teacher opinions of difficulty of mathematical thinking and student test results were more inconsistent. Although mathematical proof was consistently the most difficult aspect for teachers and students and Use of Symbols the third most difficult, other aspects differed between the two groups.

8.2 RECOMMENDATIONS

Given the results of the research into mathematical thinking and mathematics achievement in relation to gender and school location, the following recommendations are put forward:

In relation to mathematics achievement, the most significant relationships were evident with Mathematical proof and Generalization. It is therefore suggested that, if mathematics achievement of the Year 11 scientific stream in Jordan is to be maximised, the mathematics curriculum include a sound and carefully structured joint emphasis on these two aspects of mathematical thinking.

This suggestion is reinforced by the finding that the most difficult aspect of mathematical thinking for the students was Mathematical proof. Therefore, it is suggested that the teaching and learning strategies be modified in order to promote better understanding of the concept of proof. The other aspect of mathematical thinking most closely related to mathematics achievement was Generalization, which had a moderate level of difficulty.

Perhaps the other aspects of mathematical thinking were also important for mathematics achievement, so teachers should be assisted in their application to lessons.

Further, in relation to mathematics achievement, a significant relationship was evident between mathematics achievement and Logical thinking. It is therefore suggested that, for the Year 11 scientific stream in Jordan, the mathematical curricula focus on this aspect rather than it be omitted from the curriculum, as recommended by the Ministry of Education.

At a more detailed level, future researchers could include two items included in this research in Use of Symbols in other aspects of mathematical thinking. For example, item 2 should be included in Generalization, and item 3 included in Mathematical proof. Similarly, two items included in Generalization in this research should be included in other aspects. For example, item 2 should be included in Use of Symbols, and item 3 in Logical thinking.

It is also recommended that teachers encourage the use of the fourth step in mathematical problem solving, that is, checking the answer and to encourage their students to check their answers. It is an important part of mathematical problem solving.

From the student responses and interviews it was found that most students use classical methods in solving some of their problems, which often take a longer time, and are not always practical, particularly if a general solution is needed. The recommendation, therefore, is for teachers to encourage their students to use the strategy of looking for patterns, because this strategy is closely integrated with mathematical thinking processes.

Each of the six identified aspects of mathematical thinking was found to be significantly related to mathematics achievement. However, the relatively low reliabilities of the individual scales assessing these six aspects of mathematical thinking would have had the effect of reducing the strengths of their relationships with mathematics achievement. It is recommended that further research in this area focus on the refinement of the mathematical thinking scales to ensure higher reliabilities. This could be done by working on the content of individual items used in the scales and by increasing the number of items in each scale. Both approaches are recommended.

8.3 LIMITATIONS

The generalizability and limitations and of the results of the current study will now be discussed. Given the way in which the samples of schools and students were selected, it is suggested that these results could be generalized to the Year 11 scientific stream students in the Irbid governorate in Jordan. The possibility of further generalization to similar areas to the Irbid governorate in terms of socio economic status, population size, and urban, suburban, rural breakdown, etc should also be considered.

The generalizability of the results in relation to school location is limited as the definition of school location could differ from one researcher to another. This could also lead to the results differing. The researcher’s definitions in this study were that urban schools referred to schools located in the centre of the city, suburban referred to all schools located 10 km or less from the city, but not in the urban, and outside these locations, schools were defined as rural schools. However, we must also be aware that there were inequities in the urban, rural, suburban breakdown of the sample with suburban schools representing less than 20% of the whole sample.

The investigation carried out was limited to consideration of only six aspects of mathematical thinking and their relationships with mathematics achievement. It is possible that the results would differ if a wider range of aspects of mathematical thinking were included. Additional aspects that could also be tested in any future study include: specialization, searching for patterns, find the optimal solution, and reasoning.

The qualitative data were based on the teacher and student interviews, and the sample size of 13 teachers and about 20 students clearly limits the generalizability of the related findings. It should also be noted that those teachers and students were not chosen randomly, but chosen because they were interested and willing to participate in the research.

Another limitation is the relatively low reliabilities of the sub-scales assessing the six aspects of mathematical thinking. Despite this limitation, however, the six aspects were still found to have strong relationships with mathematics achievement. The trend that female students performed better may have reached statistical significance more generally if the sub-scales had been more reliable measures of the six aspects of mathematical thinking.

8.4 SUGGESTIONS FOR FUTURE STUDIES

Given the importance of mathematical thinking for mathematics achievement found in this study, it would be of interest to conduct research similar to that conducted with the secondary scientific stream into relationships between mathematical thinking and mathematics achievement in primary education settings and general secondary education settings.

The researcher also suggests that future studies replicate this study in primary schools with classes at different levels to investigate any difference in the relationships between mathematical thinking and mathematics achievement. Clearly the tests used would need to be developed to be appropriate for their mathematics capability at this level.

The researcher also suggests that future studies include different aspects and combinations of aspects of mathematical thinking in relation to mathematics achievement that correspond to the various curricula and for classes at different levels around the world.

If these relationships were known at an earlier phase of the students' education, it would be more possible to work with the students to improve their performance before they reached the senior secondary schools. Such a study would also have possible benefits for all students, not only those in the Year 11 scientific stream.

For example, due to the relatively strong relationship between the two aspects of mathematical thinking, Generalization and Use of Symbols, it is suggested that test items be developed to jointly assess those two aspects.

Mathematics is a particularly important subject for the secondary scientific stream, because these students are required to attain high levels of achievement in science, particularly in mathematics. We suggest that future researchers also continue the study of other aspects of mathematical thinking that have a high correlation with mathematics achievement worldwide, so as to enable teaching and learning of mathematical thinking that will improve levels of achievement of the most able students. In addition, future researchers should also focus on those aspects of mathematical thinking which have a high correlation with mathematical achievement.

8.5 SIGNIFICANCE OF THE STUDY

Mathematics is important for the learning of science in all countries. This is particularly the case in Jordan where scientific and technological development is so necessary. Application of mathematics is fundamental to the study of all science subjects such as physics, chemistry and biology, and even medicine. In addition, computer science is dependent on logic considered to be the foundation of mathematics. For example, connective words (and, or) as considered a part of Boolean logic is considered the basis for computer programming and for the development of computer games (Huetinck & Munshin 2004). This study was intended to provide a basis for improving learning in mathematics for the most able students.

Mathematics is considered to be a difficult subject internationally and any developments that can improve the teaching of mathematics more generally are important. This study found significant relationships of the different aspects of mathematical thinking and also mathematical thinking in general for mathematics achievement. Consequently, it is suggested that a greater focus on mathematical thinking in primary schools would be likely to result in an increase in mathematics achievement for all students. A consequence of an overall improvement in mathematics achievement could be that more students become capable and interested in pursuing careers in science and technology areas. In turn this would be likely to lead to personal satisfaction and to be of ultimate benefit to the nation.

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Appendix One

This Appendix will be included in the Future Projects and National Educational Processes Projects in order to develop the Educational system in Jordan, and to focus on the development of quality education and scientific thinking.

1.1 Future Renewal Projects[1]

In order to continue to develop the educational system in Jordan, the Ministry of Education will focus on the following five renewal educational projects.

1) Construction of King Abdullah II schools for gifted and talented: These schools are for students with high ability in thinking and creativity for all ages. These schools will be established nationally across all kingdom governorates over three years.

2) Queen Rani Project to use computers in learning: This project aims to be implemented in all schools over three years. This project was designed to promote use of the educational software in learning different subjects such as: Arabic language, English language, Mathematics, and Science, and to use the internet in learning and research, and many other applications.

3) Generalization of compulsory English language teaching from 1st grade to 12th grade.

4) Generalization of the improved school model: including integration of educational supervisors and school administration, and a focus on improving school processes such as school and class administration, connection, class interaction, openness to community, and development of a transparent democratic process.

5) Inclusion of kindergarten in compulsory schooling: Government kindergarten schooling will be implemented over a period of time beginning with remote and country schools, then suburban and urban schools. This will provide compulsory free schooling for five year olds.

[1] *Year Book*, 1999-2000, Ministry of Education, Jordan.pp.60-62.

1.2 National Educational Processes Project

In order to continue the development of the educational system in Jordan, the Ministry of Education has instituted a national process comprised of the following steps:

1) Improve and develop curricula: Continuous improvement and development of curricula, with development of flexibility in curricula and textbooks.

2) Teaching development of some subjects such as Arabic language, English language, science, and mathematics, including

a) Focus on Arabic language teaching.
b) Focus on English language skills as a basic foreign language for improvement and development.
c) Integration of science and mathematics.

3) Improvement in scientific research skills and critical thinking.

a) Encourage students to transfer skills toward research, investigation, and critical thinking.
b) Qualify students and train them to acquire scientific research and critical thinking skills.

Appendix Two

This appendix will relate to Chapter 3 “The Instruments and Sample”
(i) Appendix 2.1: Test of Mathematical Thinking in both English and Arabic Languages

(ii) Appendix 2.2: Test of Mathematics Achievement in both English and Arabic Languages

(iii) Appendix 2.3: The Teacher Interviews in both English and Arabic Languages

(iv) Appendix 2.4: Consent Forms

(v) Appendix 2.5: Information Letters

Appendix 2.1

Test of Mathematical Thinking Test in Both of English and Arabic Languages

Test Instructions of Mathematical Thinking.

Dear student,

This test is designed to measure six aspects of mathematical thinking. The test consists of 30 questions with five questions for each aspect, each designed to measure different capabilities. It should take three hours.

Please read each question carefully and accurately and answer every question objectively.

- Use the information given to answer the question.
- Multiple-choice questions have only one correct answer. Please write down a justification for your answer when you have answered.
- For each question, you are asked to explain your answer. That is to write down the way you thought and found the answer.
- Please don't write anything on the question sheet and use the answer sheet only to write down the answers.
- Please write down the code number in place of your name on both the answer and the questions, to study the relationship between mathematical thinking and mathematics achievement of 1^{st} secondary class scientific stream.
- You are kindly required to take this test seriously since it will greatly affect the result of the study conducted.
- Kindly return both answers and questions sheet once you have finished the test.
- If any student wishes to know the result of their tests, they can contact the researcher and will be provided their results.
- Finally, the results of this test will be treated with complete confidentiality and will not affect school assessment for students. They will be merely used for the purposes of study.

Part one: Generalization (G).

G1) If n→n is a function as n= [1, 2, 3---), and 1→1, 2→8, 3→27, 4→64, then X→--- (As X∈n).

G2) Complete the last statement.

1=1
1+3=4
1+3+5=9
1+3+5+7=16
1+3+5+7+---+ (2n-1) =---------.

G3) Notice the two numbers on the right of the equals mark and their totals to its left in the following, and then discuss any deductions that can be made.

6=3+3   8=5+3   10=5+5   12=5+7   14=7+7   14=3+11
16=11+5   16=13+3.

G4) Complete the table:

Number of sides of the polygon	3	4	5	7	-----	n
Number of diagonals	0	2	5		------	

(Hint: This involves a square relation).

G5) Contemplate each pair of the following equal numbers and notice the relation between the numbers to the left of equal mark and the two numbers to the right of the equal mark.

a) 6×6=36   b) 7×7=49   c) 8×8=64
3×9=27   4×10=40   5×11=55.

If X×X=289, then (x-3) × (x+3) =----------.

Part second: Induction (I).

I1) The number of bacteria in a colony was growing exponentially. At 1 pm yesterday the number of bacteria was 1000, and at 3 pm yesterday it was 4000. How many bacteria were there in colony at 6 pm yesterday?

I2) A group of the numbers appeared classified as follows:

3 ½, 5 1/3, 7 ¼, 9 1/5,---
What is the tenth number?

I3) A long time ago, a mathematician invented the game of chess and presented it to the king. The king was so pleased with the game that he asked the mathematician to name a reward. The mathematician looked at the chess board, consisting of 64 squares, and asked for the amount of rice according to this rule:
One grain of rice on the first square of chessboard, two grains on the second square, four grains on the third square, and so on until the last square.
How many grains of rice are there on 64^{th} square? And how many grains of rice did the mathematician ask for in total? Explain the pattern you are using.

I4) Contemplate the following algebra statement and write an analysis of the last statement.
$(x-1)^2=x^2-2x+1.$

$(x-1)^3= x^3 -3x^2 + 3x -1.$

$(x-1)^4 = x^4 -4x^3 +6x^2 -4x +1.$

$(x-1)^n=$ ------------------------------.

I5) The following three cards to the left are written according to a certain rule form "If-----, Then ------, whereas the fourth card is written in a form that does not correspond to that rule.

The cards that correspond to the rule   The card that does not correspond

A	B	C	D
○	14	M	A
4	□	15	Δ

The rule is:
a) If a shape appears in the upper half of the cards, a number appears its lower half.

b) If a number appears in the upper half of the card, a shape appears in its lower half.
c) If a letter appears in the upper half of the card, a number appears in its lower half.
d) If a letter appears in the upper half of the card, a shape appears in its lower half.

Part three: Deduction (D).

D1) If (xy) =1, and x is greater than 0, which of the following statements is true?

a) When x is greater then 1, y is negative.
b) When x is greater than 1, y is greater than 1.
c) When x is less than 1, y less than 1.
d) As x increases, y increases.
e) As x increases, y decreases.

D2) All of the numbers in group A are divisible by 5, number (20) is divisible by 5, and belongs to group B, we infer from that:

a) Group A is equal to group B.
b) A is a sub-group of B.
c) B is a sub-group of A.
d) Nothing from what is mentioned above.

D3) Read both of the following hypotheses.

1) All engineering students in (J.S.T.U) are intelligent.
2) All science students in Y.U are intelligent

What is the correct deduction from the following?

a) All engineering students in both universities are intelligent.
b) All science students in both universities are intelligent.
c) All science and engineering students in both universities are intelligent.
d) We can't induce anything from what has been mentioned above.

D4) The vertices of the triangle PQR are the point p (1, 2), q (4, 6), r (-4, 12), which one of the following statements above triangle PQR is true?

a) PQR is a right triangle with the right angle$\angle$P.
b) PQR is a right triangle with the right angle$\angle$Q.
c) PQR is a right triangle with the right angle$\angle$R.
d) PQR is not a right triangle.

D5) Some of the isosceles triangles are right triangles.
The medians of all triangles intersect at one point.
ABC is a right triangle.
What do you induce from triangle ABC?

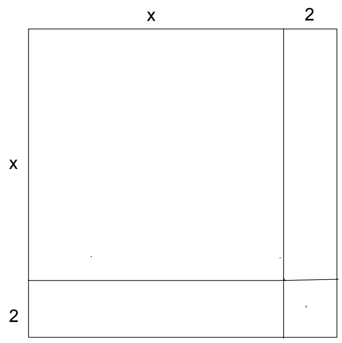
a) ABC is a triangle whose medians are of equal length.
b) ABC is an isosceles triangle whose medians intersect at one point.
c) ABC is a triangle whose medians intersect at one point.
d) ABC is a triangle whose medians intersect at one point, but is not an isosceles triangle.

Part four: Use of Symbols (S).

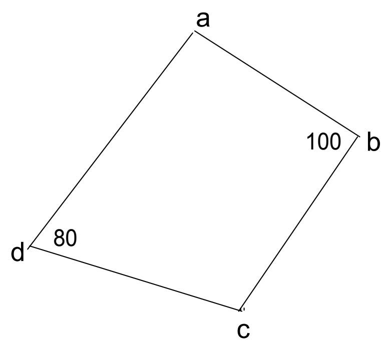
S1) There are two classes, A and B, in a school. The number of students in class A is ten more than in B. If five students move from class B to A, then the number of students in A becomes triple the number in B. Express the above in equations.

S2) Unit circle, its centre (0, 0), a line L through (0, 1), with rational slope (m). Write an expression for the equation of circle in terms of X.

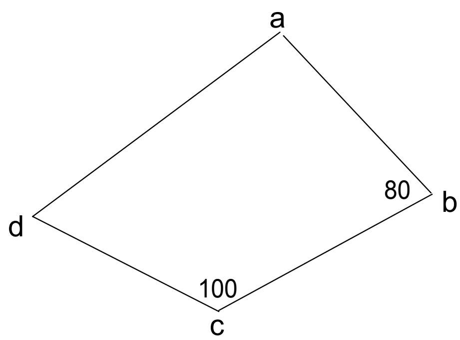
S3) Find $(x+2)^2$ using the following shape.



S4) If the quadrilateral shape is cyclic, then the total of each opposite angle in it is equal to its two right angles (180)°and vice versa.



Shape 1

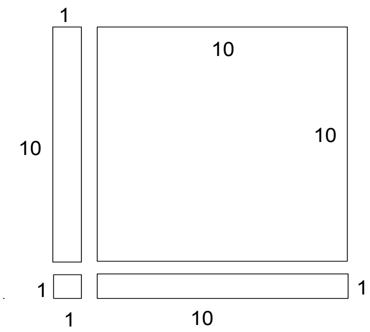


Shape 2

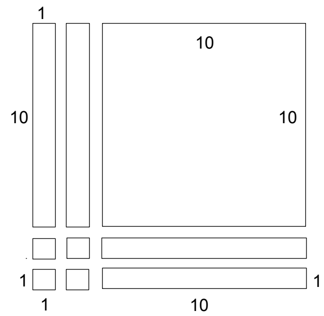
What can we induce in regards of the two shapes “1” and “2”

a. The shape “1” and “2” are cyclic.
b. Shape “1” is cyclic, “2” is not cyclic.
c. Shape “2” is cyclic, “1” is not cyclic.
d. Shape “1” is cyclic, “2” we don’t know.

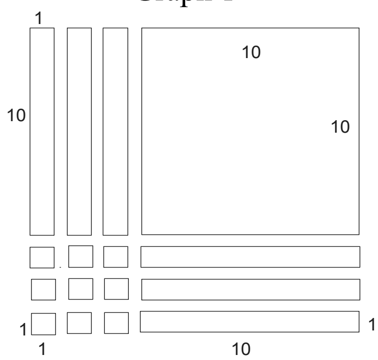
S5) Express the following shapes by symbols.



Graph 1



Graph 2



Graph3

Consequently, n^2=-----------.

Part five: Logical thinking (L):

L1) The symmetric difference of two sets A and B is defined to be.
$A \Delta B = (A-B) \cup (B-A)$.

a) Draw a Venn diagram to illustrate $A \Delta B$.
b) Prove that $(A-B) \cup (B-A) = (A \cup B)-(A \cap B)$.

L2) Negate the following statements in such a way your resulting sentence does not use the word “not”.

a) There is a real number whose square is negative.
b) There exists $X \in \Re$ such that $f(x) > 100$.
c) For all $\delta > 0$, there exists $n \in N$ such that $1/n < \in$.

(**) In the items L3 and L4 that follow an explicit rule is written, and you are requested to choose the card that corresponds to the rule, from the following four cards written under the rule.

L3) A number or a shape doesn’t appear on the card.

D	○	M	L
Δ	A	L	4
A	B	C	D

L4) A letter and a number appear on the card.

5	L	○	M
Δ	4	15	□
A	B	C	D

L5) Mary's sister made these statements. If Vera told the truth, who else must have told the truth?
Lucy” If the rug is in the car, and then it is not in the garage”.
Sally:” If the rug is not in the car, then it is in the garage”.
Vera:” If the rug is in the garage, then it is in the car”.
Cherry:” If the rug is not in the car, then it is not in the garage”.

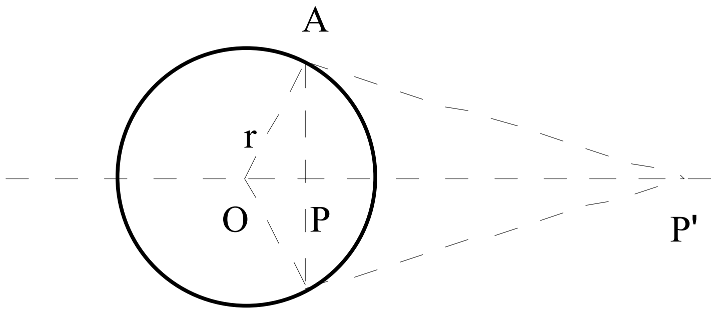
a) Lucy.
b) Sally.
c) Cherry.
d) None need have told the truth.

Part six: Mathematical proof (M).

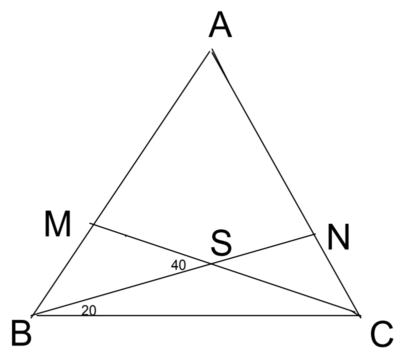
M1) Show that if" n" is divisible by 2, then n^2 is also divisible by 2.

M2) Prove that $\sqrt{2}$ can't be expressed as a fraction (in other words a/b, where a and b are integers and $b \neq 0$).

M3) On the adjacent shape, If (OPA) is a right angle, prove that m (OP') $\times$ m (OP) $= r^2$.



M4) In the ABC the altitudes BN and CM intersect at point S. The measure of $\angle$ MSB is 40°, and the measure of $\angle$SBC is 20°. Prove of the following statement: "ABC is isosceles".
Give geometric reasons for statement in your proof.



M5) Rule" If the lengths of the sides of triangle are 3, 4, 5, then the triangle is right "from that we deduce that".

a) The ratio between lengths of sides of every right triangle is 3:4:5.
b) The ratio between the lengths of the sides of some right triangle is 3:4:5.

c) Some of triangles that have the ratio between the lengths of their sides as 3:4:5 are not right triangles.
d) There are triangles that are not right triangles and the ratio between the sides of their side's 3:4:5.

تعليمات أختبار التفكير الرياضي.

عزيزي الطالب

لقد صمم هذا الاختبار ليقيس المظاهر الستة المختلفة ل قدرتك على التفكير الرياضي , و ي تالف هذا الاختبار من 30 سؤال, بواقع خمس اسئلة على كل مستوى. و تاخذ ثلاث ساعات.

- ارجو منك قراءة الاسئله بدقة و عنايه, و الاجابة عن كل سؤال بكل موضوعيه.
- حاول ان تستغل كل المعلومات المعطاه للاجابه عن السؤال.
- في اسئلة الاختيار من متعدد , لكل سؤال جواب واحد صحيح, يرجى منك كتابه المبررات عند اختيار البديل الذي تعتقد انه صحيح.
- في كل سؤال يطلب منك شرح اجابتك, بمعنى كتابة كيف فكرت و كيف توصلت للاجابه.
- الرجاء عدم كتابتة اي شيء على ورقة الأسئلة, وكتابة الاجابه على ورقة الاجابه.
- الرجاء كتابة رمز الطالب على ورقة الاجابة و ورقة الاسئله, لدراسة العلاقة بين التفكير الرياضي و تحصيل الرياضيات لطلبة الصف الاول الثانوي في الاردن
- ارجو منك أخذ الموضوع بكل جدية, لأن إجاباتكم ستبنى عليها نتائج هامه للدراسة التي اجريها.
- الرجاء اعادة ورقة الاسئلة و الاجابه بعد الانتهاء من الاختبار.
- إذا كان لدى اي منكم الرغبة في معرفة نتائج امتحاناته, يمكنه/ يمكنها الاتصال بالباحث لتزويده بذلك.
- اخيرا, عزيزي الطالب ان نتائج اختبارك ستعامل بسريه تامه, ولا تؤثر على علامتك المرسيه, وانها لاغراض البحث و الدراسة فقط.

اولا: التعميم.

(1) إذا كان ق اقتران معرف من ط*←ط*و حيث ط*={1، 2،3،---} ، وكان 1←1،2←8، 3←27 ، 4←64. فأن س ←---(حيث س ط*).

(2) اكمل الجملة الأخيرة:

1=1
4=3+1
9=5+3+1
1+2+3+5+7+----(2ن-1)=-------------.

(3) لاحظ كل عددين على يسار المعادلة بمجموعهما على الطرف الآخر في المعادلات التالية و اكتب ما تستقرئه.

6=3+3، 8=5+3 ، 10=5+5، 12=5+7، 14=7+7، 14=11+3 ، 16=11+5، 16=11+5.

(4) إكمل الجدول التالي:

عدد أضلاع الشكل	3	4	5	7	---	ن
عدد الأقطار	صفر	2	5	----	----	________

(اعتبر العلاقة بين عدد الأضلاع و الأقطار علاقة تربيعية).

(5) تأمل كل زوج من ازواج المتساويات التالية، و لاحظ العلاقة فيما بين الأعداد التي يمين المساواة و العددين الذين إلى يسارهما:

(أ) 6 × 6=36 (ب) 7 × 7=49 (ج) 8 × 8=64
3×9 = 27 5 × 11 = 55 5x11= 55

إذا كان س × س=289, فإن (س-3)(س+3) =-----------------.

ثانيا:الاستقراء.

(1) بكتيريا ما تنمو في مستعمرة بطريقة اسية ، حيث كان عددها الساعة الواحدة مساء اليوم ا لماضي 1000 ، و الساعة الثالثة مساء اليوم الماضي 4000 ، جد عددها في المستعمرة الساعة السادسة مساء اليوم الماضي.

(2) ظهرت مجموعة الأعداد على النحو التالي:
3 1/2 ، 5 1/3 ، 7 1/4 ، 9 1/5 ، -----
جد الحد العاشر.

(3) من ذ زمن بعيد لعبة الشطرنج، و احضرها للملك، سر الملك باللعبه وسأل الرياضي عن الجائزة التي يريدها. نظر الرياضي الى لوح الشطرنج، والذي يتكون من 64 مربع. و طلب ارز حسب القاعدة التالية.
حبة واحدة في المربع الاول من لوح الشطرنج، وحبتين في المربع الثاني، و 4حبات في الثالث، وهكذا حتى المربع 64. كم حبة من الارز يوجد في المربع 64؟ كم مجموع حبات الارز التي طلبها الرياضي ؟ اشرح النمط الذي استخدمته؟

(4) بناءً على تحليل المقادير الجبرية التالي ة، اكتب تحليل المقدار الجبري الأخير.

$(\text{س}-1)^2=\text{س}^2-2\text{س}+1$

$(\text{س}-1)^3=\text{س}^3-3\text{س}^2+3\text{س}-1$

$(\text{س}-1)^4=\text{س}^4-4\text{س}^3+6\text{س}^2-3\text{س}+1$.

$(\text{س}-1)^{\text{ن}}=$---.

(5) كتبت البطاقات الثلاث التالية حسب قاعدة ما ، إذا كان -----فان----------,في حين كتبت البطاقة الرابعة بصورة لا تتفق مع القاعدة .

البطاقات التي تتفق مع القاعدة البطاقة التي لا تتفق مع القاعدة

أ	ب	ج	د
○	14	م	ع
4	□	15	Δ

القاعدة هي:
أ) إذا ظهر شكل في النصف العلوي من البطاقة ، ظه ر عدد في النصف السفلي منها.
ب) إذا ظهر عدد في النصف العلوي من البطاق ة، ظهر شكل في النصف السفلي منها.
ج) إذا ظهر حرف في النصف العلوي من البطاق ة، ظهر عدد في النصف السفلي منها.
د) إذا ظهر حرف في النصف العلوي من البطاق ة، ظهر شكل في النصف السفلي منها.

ثالثا:الاستدلال.

(1) إذا كان (س ص) =1 و كان س اكبر من صفر ، أي من العبارات التالية صحيحة.

أ) عندما تكون س اكبر من 1 ، فان ص سالبة.

ب) عندما تكون س اكبر من 1 ، فان ص اكبر من 1.

ج) عندما تكون س اقل من 1 ، فان ص اقل من 1.

د) عندما تكون س متزايدة ، فان ص متزايدة.

هـ) عندما تكون س متزايدة ، فان ص متناقصة.

(2) جميع الأعداد في المجموعة ا تقبل القسمة على 5.

العدد 20 تقبل القسمة على 5 و ينتمي إلى المجموعة ب ، نستنتج من ذلك:

أ) المجموعة ا تساوي المجموعة ب.

ب) ا مجموعة جزئية من ب.

ج) ب مجموعة جزئية من ا.

د) لا شي مما ذكر.

(3) أقرا الفرضيتين التاليتين:

1) جميع طلبة الهندسة في جامعة العلوم و التكنولوجيا الأردنية أذكياء.

2) جميع طلبة العلوم في جامعة اليرموك أذكياء.

ما الاستنتاج الصحيح مما يلي:

أ) ج ميع طلبة الهندسة في الجامعتين أذكياء.

ب) ج ميع طلبة العلوم في الجامعتين أذكياء.

ج) جميع طلبة الهندسة والعلوم في الجامعتين أذكياء.

د) لا يمكننا استنتاج أي شي.

(4) رؤوس المثلث ا ب خ هي النقاط ا (2,1),ب (6,4), ج (12,4). أي الجمل التالية صحيحة بالنسبة ل المثلث ا ب ج ؟

أ) المثلث ا ب ج قائم الزاوية في ا.

ب) المثلث ا ب ج قائم الزاوية في ب.

ج) المثلث ا ب ج قائم الزاوية في ج.

د) المثلث ا ب ج ليس مثلث قائم الزاوية.

(5) بعض المثلثات المتساوية الساقين قائمة الزاوية.

جميع المثلثات تلتقي مستقيماتها المتوسطة في نقطة واحدة.

أ ب ج مثلث قائم الزاوية.

ماذا تستنتج عن المثلث أ ب ج ؟

أ) أ ب ج مث لث مستقيماتة المتوسطة متساوية.

ب) أ ب ج مثلث متساوي الساقين تلتقي مستقيماتة المتوسطة في نقطة واحدة.

ج) أ ب ج مثلث تلتقي مستقيماتة المتوسطة في نقطة واحدة.

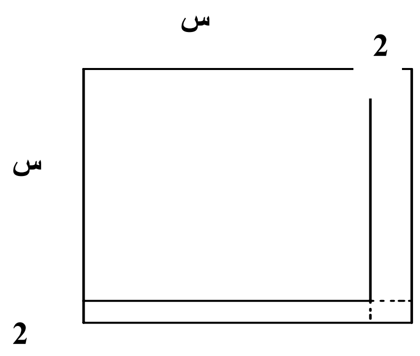
د) أ ب ج مثلث تلتقي مستقيماتة المتوسطة في نقطة واحدة و ليس متساوي الساقين .

رابعا: استخدام الرموز.

(1) صفان في مدرسة أ & ب، عدد الطلبة في الصف أ يزيد بمقدار 10 طلاب عن عددهم في ب، إذا انتقل خمسة طلاب من الصف ب إلى الصف أ، عندها يصبح عدد الطلبة في أ ثلاث أمثال العدد في ب. عبر عن السابق بالمعادلات.

(2) دائرة وحدة مركزها نقطة الاصل، قطعها المستقيم ل في النقطة (1,0) ، وميله (م). عبر عن معادلة الدائرة بالرموز (باستخدام المتغير (س) فقط.

(3) جد (س + 2)^2 باستخدام الشكل التالي.



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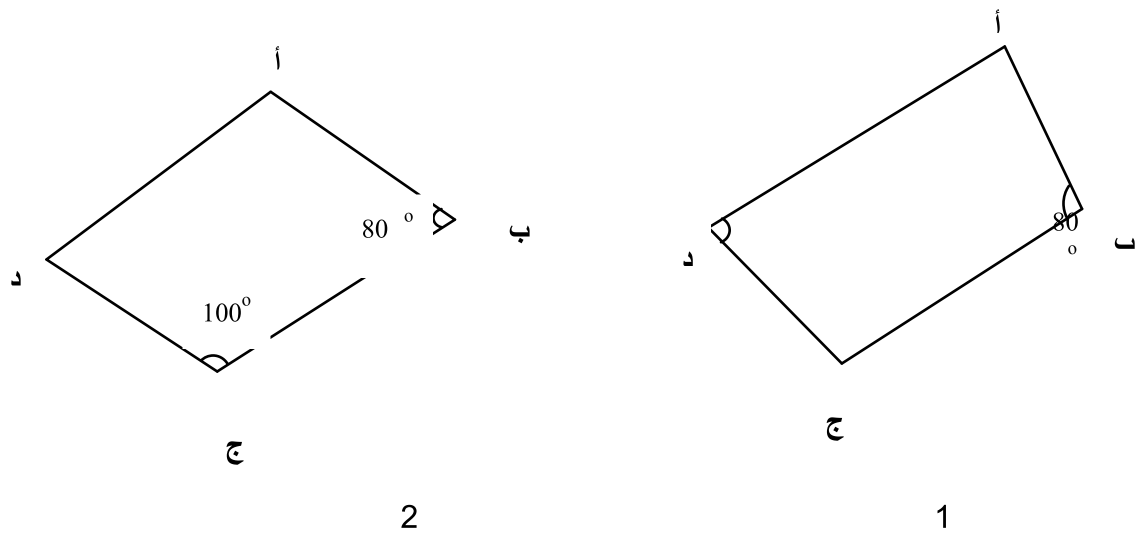
(4) إذا كان الشكل الرباعي دائريا فان مجموع كل زاويتين متقابلتين فيه = 180 والعكس صحيح.



ماذا نستنتج بالنسبة للشكلين "1" & "2".

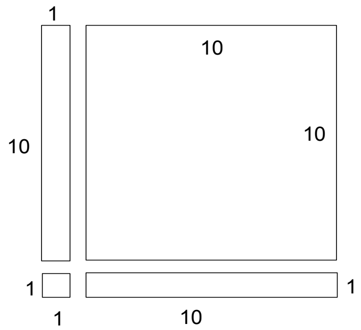
أ) الشكل "1" & "2" دائريان

ب) الشكل "1" دائري، والشكل "2" غير دائري.

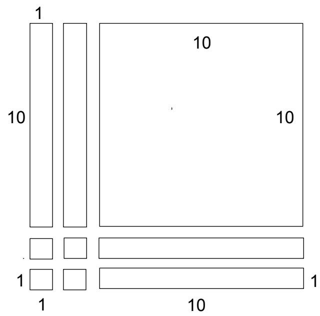
ج) الشكل "2" دائري، والشكل "1" غير دائري.

د) الشكل "1" دائري، والشكل "2" لا يمكن معرفته دائري ام غير دائري.

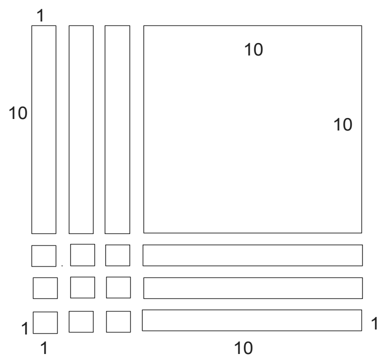
(5) عبر عن الاشكال التالية بالرموز.



شكل 1



شكل 2



شكل 3

وبناءا على تحليل الاشكال السابقة, فان ن^{2} = --------------.

خامسا: التفكيرالمنطقي .

(1) التماثل لمجموعتين أ & ب يعرف على النحو التالي:

أ Δ ب = (أ – ب) $\cup$ (ب- أ)

أ) أرسم باشكال فن توضيح أ Δ ب.

ب) أثبت أن (أ-ب)$\cup$ (ب-أ) = (أ$\cup$ ب)- (أ$\cap$ ب).

(2) انفي الجمل التالية, باستخدام كلمة "لا".

أ) يوجد عدد حقيقي ، حيث مربعه سالب.

ب) يوجد س ∈ ح ، حيث ق(س) < 100.

ج) لكل δ > صفر ، يوجد ن ∈ ط* ، حيث 1/ن < δ.

(3) **لا** " يضهر على البطاقة عدد أو شكل".

أ	ب	ج	د
د	○	م	ل
Δ	أ	ل	4

(4) " حرف **و** رقم يظهر على البطاقة".

أ	ب	ج	د
5	ل	○	م
Δ	4	9	□

(5) قالت أخت ماري الجملة التالية" إذا فيري قالت الحقيقة ، من غيرها تكون قد قالت الحقيقة.

سوزان " اذا كانت السجادة في ، فأنها ليست في الكراج".

سالي" إذا كانت السجادة ليست في السيارة ، فأنها في الكراج".

فيري" إذا كانت السجادةفي السيارة ، فأنها في الكراج".

ناديا" إذا كانت السجادة ليست في السيارة ، فأنها ليست في الكراج".

أ) سوزان

ب) سالي.

ج) نادي ا

د) لم يقل احد الحقيقه.

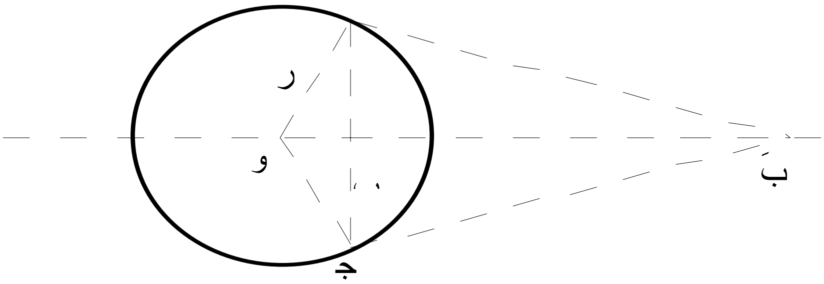
سادسا: البرهان الرياضي.

(1) بين أنه إذا كان" ن" يقبل القسمة على 2 ، فان $ن^2$ تقبل القسمة على 2 .

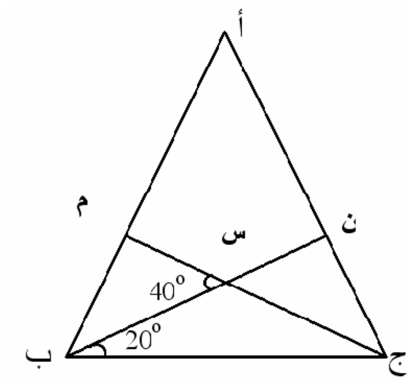
(2) اثبت انه لا يمكن التعبير عن $\sqrt{2}$ على شكل كسر (أي بصورة أ/ ب ،

حيث أ ، ب ∋ ص ، ب ≠ صفر).

(3) في الشكل التالي إذا كانت ∠و ب أ قائمة ، فأثبت إن(و ب) ×(و بَ)= $ر^2$.



(4) في المثلث أ ب ج ، تقاطع العمودان ب ن & ج م في النقطة س . إذا كان قياس ∠م س ب = 40° , و قياس ∠س ب ج =20° .فبرهن أن المثلث أ ب ج متساوي الساقين .
أعط المبررات الهندسية في الإثبات.



(5) قاعدة " إذا كانت أطوال أضلاع مثلث هي 3 ، 4 ، 5 فان هذا المثلث قائم الزاوية " نستنتج من ذلك أن

أ) كل مثلث قائم الزاوية تكون النسبة بين أطوال أضلاعه كنسبة 3 :4 :5 .
ب) بعض المثلثات القائمة الزاوية تكون النسبة بين أطوال أضلاعها كنسبة 3 :4 : 5.
ج) بعض المثلثات التي نسبة أطوال أضلاعها كنسبة 3 :4 :5 لا تكون قائمة .
د) هناك مثلثات ليست قائمة الزاوية و النسبة بين أطوال أضلاعها كنسبة 3 :4 :5 .

Appendix 2.2 Test of Mathematics achievement in both English and Arabic Languages

Test Instructions of Mathematics Achievement

Dear student,

This test is designed in order to measure your ability on mathematics achievement. This test consists of 4 questions from your math book. Each unit has one question. It should take two hours.

Please read each question carefully and accurately and answer every question objectively.

- Use the information given to answer the question.

- For each question, you are asked to interpret your answer. That is to write down the way thought and found the answer.

- Please don't write anything on the question sheet and use the answer sheet only to write down the answers.

- Please write down the code number in place on both the answer and the questions, to study the relationship between mathematical thinking and mathematics achievement of 1st secondary class scientific stream.

- You are kindly required to this test seriously since it will greatly affect the result of the study conducted.

- Kindly return both answers and questions sheet once you have finished the test.

- If any student wishes to know the results of their tests, they can contact the researcher and will be provided with their results.

- Finally, the result of this test will be treated with maximum confidentiality and will not affect school performance for students. They will be merely used for the purposes of study.

Test of Mathematics Achievement

1) a) If $a>b>0$, prove that $\sqrt{ab}$ is strictly between a and b (3 points).

b) Solve these inequalities:

1) $7x - 5 < 3x + 4$.
2) $|x-3| > 1$.
3) $(x^2-4) / (x^2-9) \leq 1 (x \neq \pm 3)$ (9 points).

2) a) Solve these equations for x:

1) $\text{Log } x^2 - 2x + 1=2$
2) $5^{(x-3)} + 5^{(2-x)} = 6/5$ (hint: considered $5^{(x-3)} = 5^x / 5^3$).

(4 points).

b) Given than Ln2 = 0.69 and Ln7 = 1.95, find

1) Ln28.
2) Ln98 (4 points).

c) Under which condition $4^{(a2-b2)} \times 16^{(a-b)} = 1$ (3 points).

d) A town now has a population of 5000. If it grows to 6000 in 1 year, when will the population reach 10,000? (3 points).

3) a) If $A = \begin{pmatrix} 3 & 2 \\ -2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 7 & 2 \end{pmatrix}$

Find:

1) a + 2b.
2) a × (-b). (4 points).

b) Prove or disprove, explain your answer.

1) If $A_{2\times2}$ matrix, then $(2A)^{-1} = ½(A)^{-1}$.
2) If $A_{n\times m}$, $B_{m\times k}$, and AB = 0, then A = 0 or B = 0. (4 points).

c) Solve this system of equation using Cramer's rule:

$2X + Y = Z – ½$.
$X – Y + 3Z = 9/2$.
$X = 2Y + 6 Z$ (4 points).

4) a) In club checkers tournament, each player played every other player exactly once. If there were 20 players, how many games were played? Explain your answer.

b) Solve these equations.

1) $P (n, 4) = P (2n, 2) + 30$.

2) $(n-1)! = 720$ (4 points).

c) Write this summation $(1)(2)/2 + (2)(3)/24 + (3)(4)/40320$ As a $\sum$.

(4 points).

تعليمات أختبار تحصيل الرياضيات

عزيزي الطالب

لقد صمم هذاالا ختبار لكي يقيس قدرتك على ال تحصيل في الرياضيات .هذا الاختبار يتكون من اربع اسئلة من كتاب الرياضيات, كل وحدة تحتوي على سؤال. و ياخذ ساعتين.

- ارجو منك قراءة الاسئله بدقة و عنايه, و الاجابة عن كل سؤال بكل موضوعيه.
- حاول ان تستغل كل المعلومات المعطاه للاجابه عن السؤال.
- في كل سؤال يطلب منك شرح اجابتك, بمعنى كتابة كيف فكرت و كيف توصلت للاجابه.
- الرجاء عدم كتابتة اي شيء على ورقة الأسئلة, وكتابة الاجابه على ورقة الاجابه.
- الرجاء كتابة رمز الطالب على ورقة الاجابة و ورقة الاسئله, لدراسة العلاقة بين التفكير الرياضي و تحصيل الرياضيات لطلبة الصف الاول الثانوي في الاردن
- ارجو منك أخذ الموضوع بكل جدية, لأن إجاباتكم ستبنى عليها نتائج هامه للدراسة التي اجريها.
- الرجاء اعادة ورقة الاسئلة و الاجابه بعد الانتهاء من الاختبار.
- إذا كان لدى اي منكم الرغبة في معرفة نتائج امتحاناته, يمكنه/ يمكنها الاتصال بالباحث لتزويده بذلك.

-اخيرا, عزيزي الطالب ان نتائج اختبارك ستعامل بسريه تامه, ولا تؤثر على علامتك المرسيه, وانها لاغراض البحث و الدراسة فقط.

اخ ت بار في تحصيل الرياضيات:

السؤال الاول:-

أ) إذا كان أ > ب > صفر, اثبت ان الجذر التربيعي ل أب (حيث يطلق عليه المتوسط الهندسي ل أ & ب) يقع بالضبط بين أ & ب.

ب) حل المتباينات التالية:

1) 7س - 5 > 3س +4.

2) |س-3| > 1.

3) ($س^2$-4)/($س^2$-9) ≥ 1 (س ≠ ±3). (12 علامة).

السؤال الثاني: -

أ) حل هذه المعادلات:

1) لو($س^2$-2س +1)=2.

2) $5^{(س - 3)}$+$5^{(2 - س)}$=5/6. (4 علامات).

ب) إذا كان لوهـ2=.69 & لوهـ7=1.95, جد .

1) لوهـ 28.

2) لوهـ 98. (4 علامات).

ج) متى $4^{(أ2-ب2)}$ × $16^{(أ-ب)}$=1. (3 علامات)

د) بلد عدد سكانها الان =5000 نسمة, إذا اصبح عدد سكانها بعد عام 6000 نسمة. متى يصل عدد السكان الى 10000 نسمة. (3 علامات).

السوال الثالث:-

أ) إذا كانت أ = [3 4 / -2 صفر] , ب= [-1 صفر / 7 2] , جد.

1) أ+2ب.

2) أ ×(- ب). (4 علامات).

ب) اثبت صحة او خطأ العبارات التالية:

1) إذا كانت أ مصفوفه 2 × 2 , فإن $(2أ)^{-1}$=1/2$(أ)^{-1}$.

2) اذا كان $أ_{م×ن}$, $ب_{ن×ل}$, وكان أ ب= صفر , فان ب= صفر او أ= صفر (4 علامات).

ج) حل نظام المعادلات التالية باستخدام طريقة كريمر.

2س+ص=ع-1/2.

س-ص+3ع=9/2.

س=2ص+6ع. (4 علامات).

السؤال الرابع:-

أ) في نادي لمباريات الشطرنج, كل فريق يلعب مع الاخر مرة واحدة بالضبط , إذا كان هناك 20 فريقا, كم عدد المباريات ستكون؟ (4 علامات).

Appendix 2.3

The Teacher Interviews in both English and Arabic Languages

Teaching of mathematical thinking in Jordanian schools

1) In your opinion, what does mathematical thinking mean? Do you think mathematical thinking is restricted to the domain of mathematical computation and formula (e.g. it is restricted to the use of numbers and formulas to find answers to specific problems) or can it be used like a game to explore mathematical processes? Do you think mathematical thinking is "effective thinking" or the basis of mathematics, and contributes to the development of the student through the study of mathematics, in particular, and other sciences in general?

2) What are the aspects of mathematical thinking? For example Generalization is one of the aspects; do you know what the others may be?

3) How important is each of the aspects in teaching mathematics? Rank these aspects according to level of importance for mathematics achievement.

4) Why do you consider the -----aspect the most important aspect? And how useful is it for the students to improve their progress in mathematical thinking?

5) What is the most difficult aspect for the students, and what is the easiest? Rank these aspects according to their level of difficulty?

6) How many weeks or hours (lessons) do you spend to teach them?

Aspects of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization			
Induction			
Deduction			
Use of Symbols			
Logical thinking			
Mathematical proof			

7) What are the most effective strategies you use when you teach mathematical thinking?

تدريس التفكير الرياضي في المدارس الاردنية:-

(1) حسب رايك, ماذا يعني التفكير الرياضي؟ هل تعتقد أن التفكير الرياضي مقيد في مجال الحساب الرياضي و الصيغ(اي انه مقيد باستعمال الأعداد و الصيغ لإيجاد حلول للمسائل الخاصة)؟ أونه يشبه لعبه------؟ هل تعتقد أن التفكير الرياضي هو تفكير فعال أو قوة الرياضيات وانه يتطور من قبل الطالب من خلال دراسته للرياضيات بشكل خاص و العلوم الاخرى بشكل عام؟

(2) ما هي مظاهر التفكير الرياضي؟ على سبيل المثال التعميم هو احدى المظاهر ما هي المظاهلر الاخرى؟

(3) كم هي مهمه في تدريس الرياضيات؟ ضعها في ترتيب حسب الاهميه بالنسبة للتحصيل الرياضي ؟

(4) ما هو أهم مظهر ؟ وكيف مهم للطلبة في تطوير تفكيرهم الرياضي؟

(5) ما هو اصعب مظهر بالنسبه للطلاب ؟ و ما هو اسهلها؟ ضعها في ترتيب حسب مستوى الصعوبة.

(6) كم اسبوع أو ساعه (حصة) تمضي في تدريس هذه المظاهر؟

تسيطيع استخدام الجدول التالي للإجابة عن الأسئلة 3, 5 و 6.

مظاهر التفكير الرياضي	مستوى الأهمية	مستوى الصعوبة	الزمن المستغرق
التعميم			
الاستقراء			
الاستدلال			
استخدام الرموز			
التفكير المنطقي			
البرهان الرياضي			

(7) ما هي اكثر ا لا ستراتيجي ات فعالية والتي تستخدمها عند تدريس التفكير الرياضي؟

Appendix 2.4

Consent Forms

(i) Teacher's Interview Consent Form in both English and Arabic languages

(ii) Students' Mathematical Thinking Test Consent Form in both English

(iii) Students' Mathematics Achievement Test Consent Form in both English and Arabic Languages

(iv) Students' Interview Consent Form in both English and Arabic Languages

Appendix 2.4

(i) Teacher’s Interview Consent Form in both English and Arabic Languages

Consent Form

Teacher's Interview

Consent form for Teachers volunteering to be involved in the research project entitled ″Mathematical thinking and mathematical achievement″

I agree to participate in the research, and give my consent freely. I understand that the study will be carried out as described in the information statement, a copy of which I have retained. I realize that I can withdraw from the interview at any time and do not have to give any reasons for withdrawing. All my questions about the study have been answered.

Name (Please print): ----------------------.

Signature :-------------------------.

Date:--------------------------.

المقابلات

موافقة المعلمين:

نموذج الموافقه للمعلمين المتطوعين الذين يشاركون في مشروع بحث بعنوان " التفكير الرياضي و التحصيل الرياضي".

أنا الموقع ادناه أوافق على المشاركة في بحث " أثر التفكير الرياضي على التحصيل لطلبة الصف الأول الثانوي في الأردن" , واعطيت موافقتي بحريتي مع السماح لطلبة الصف الاول الثانوي العلمي الذين ادرسهم على المشاركة في المقابلة واختبار التحصيل الرياضي والتفكير الرياضي, وانا افهم أن الدراسة ستتستخدم لأغراض علمية, و النسخة التي املكها مصونة و محفوظة.

اعرف اني أستطيع أن انسحب من المقابله في اي وقت دون أعطاء اي مبرر لانسحابي و سوف يقوم الباحث بالاجابة على اسئلتي المتعلقة بالمقابلات واختبارات الطلبة..

الاسم(راجيا كتابته):-----------------------.
التوقيع:-----------------------------------.
التاريخ:-----------------------------------.

Appendix 2.4

(ii) Students' Mathematical Thinking Test Consent Form in both English and Arabic Languages

Participation Consent Form

Test of Mathematical Thinking

Consent form for students volunteering to be involved in the research project entitled ″Mathematical thinking and mathematical achievement″.

I agree to participate in the research, and give my consent freely. I understand that the study will be carried out as described in the information statement, a copy of which I have retained. I realize that I can withdraw from the study at any time and do not have to give any reasons for withdrawing. All my questions about the study have been answered.

Name (Please print): ----------------------.

Signature :-------------------------.

Date :--------------------------.

موافقة الطلاب:

أختبار التفكير الرياضي

نموذج الموافقه للطلبه المتطوعين الذين يشاركون في مشروع بحث بعنوان " التفكير الرياضي و التحصيل الرياضي".

أناالموقع ادناه أوافق على المشاركة في البحث , واعطيت موافقتي بحريتي , وانا افهم أن الدراسة ستستخدم لأغراض الدراسة, و النسخة التي املكها مصونة و محفوظة.

اعرف اني أستطيع أن انسحب من الدراسة في اي وقت دون أعطاء اي مبرر لانسحابي و سوف يقوم الباحث بالاجابة على جميع ألأسئلة المتعلقة بهذه الدراسه.

الأسم(راجيا كتابته):-------------.
التوقيع:-------------------------.
التاريخ:--------------------------.

Appendix 2.4

(iii) Students' Mathematics Achievement Test Consent Form in both English and Arabic Languages

Participation Consent Form

Test of Mathematics Achievement

Consent form for students volunteering to be involved in the research project entitled ″Mathematical thinking and mathematical achievement″.

I agree to participate in the research, and give my consent freely. I understand that the study will be carried out as described in the information statement, a copy of which I have retained. I realize that I can withdraw from the study at any time and do not have to give any reasons for withdrawing. All my questions about the study have been answered.

Name (Please print): ----------------------.

Signature :-------------------------.

Date :-------------------------.

موافقة الطلاب:

أختبار تحصيل الرياضيات

نموذج الموافقه للطلبه المتطوعين الذين يشاركون في مشروع بحث بعنوان " التفكير الرياضي و التحصيل الرياضي".

أناالموقع ادناه أوافق على المشاركة في البحث , واعطيت موافقتي بحريتي , وانا افهم أن الدراسة ستستخدم لأغراض الدراسة, و النسخة التي املكها مصونة و محفوظة.

اعرف اني أستطيع أن انسحب من الدراسة في اي وقت دون أعطاء اي مبرر لانسحابي و سوف يقوم الباحث بالاجابة على جميع ألأسئلة المتعلقة بهذه الدراسه.

الأسم(راجيا كتابته):-------------.
التوقيع:--------------------------.
التاريخ:---------------------------.

Appendix 2.4

(iv) Students' Interview Consent Form in both English and Arabic Languages

Participation Consent Form

Student's Interview

Consent form for students volunteering to be involved in the research project entitled ″Mathematical thinking and mathematical achievement″.

I agree to participate in the research, and give my consent freely. I understand that the study will be carried out as described in the information statement, a copy of which I have retained. I realize that I can withdraw from the interview at any time and do not have to give any reasons for withdrawing. All my questions about the study have been answered.

Name (Please print): ----------------------.

Signature :-------------------------.

Date: :-------------------------.

المقابلات

موافقة الطلاب:

نموذج الموافقه للطلبه المتطوعين الذين يشاركون في مشروع بحث بعنوان " التفكير الرياضي و التحصيل الرياضي.

أنا أوافق على المشاركة في هذا البحث , واعطيت موافقتي بحريتي , وانا افهم أن الدراسة ستستخدم لأغراض علمية, و النسخة التي املكها مصونة و محفوظة.

اعرف اني أستطيع أن انسحب من المقابله في اي وقت دون أعطاء اي مبرر لانسحابي و سوف يقوم الباحث بالاجابة على جميع أسئلة الدراسه.

الاسم(راجيا كتابته):---------------.
التوقيع:---------------------------.
التاريخ:---------------------------.

(v) Appendix 2. 5

Information Letters

(i) Principals Letter in both English and Arabic Languages

(ii) Teachers Letter in both English and Arabic Languages

(iii) Students Letter in both English and Arabic Languages

Appendix 2.5

(i) Principals Letter in both English and Arabic Languages

Principal Letter

FACULTY OF EDUCATION & ARTS

Professor S.F. Bourke
Assistant Dean, Research & Research Training
Phone: 61 2 4921 5901
Fax : 61 2 4921 6895
Email : sid.bourke@newcastle.edu.au

xx December 2003

Dear principal,

I am inviting your school to participate in a research study called 'Mathematical thinking and mathematical achievement' being conducted by Mamoon Mubark a doctoral student in the School of Education at the University of Newcastle, Australia. Enclosed is a letter from the Ministry of Education.

The aim of this research is to study the relationships between different aspects of mathematical thinking and mathematics achievement, and to investigate whether male and female students differ in mathematical thinking and achievement. This could provide important information for assisting students in their mathematical learning and performance in examination.

Male and female students from the 1st secondary scientific stream in your district are being chosen to be invited to participate in the study. Your school have been selected at random to participate in this study.

If you give permission for your school to participate, the participation of individual teachers and students in this study is entirely voluntary. Both are free to withdraw at any time during the testing of after.

Participation would involve students in taking two tests, a test of mathematical thinking and, 15 days later, a mathematics achievement test. The tests would be done during mathematics lessons. Students deciding not to do the tests could work from their mathematics textbooks during these lessons. The test of mathematical thinking will take approximately three hours, so there is a break in the middle of the test of 15 minutes. The test of mathematics achievement will take two hours. A small number of students will also be invited to take part in a group interview about their test answers, but this would be additional to the tests. For each participating class I will be asking the teacher's advice about which students to interview – preferably two each of the highest, middle and lowest performing students in the class who have previously given their permission to be interviewed. Perhaps the interviews could take place in the library.

The study also includes a 30-minute interview for the teachers of the classes involved in the testing. The interview focuses on only one topic – how the teachers develop mathematical thinking in their students.

If you have any questions about any aspect of the study, please contact me.

The test and interview data will not have any names attached to them, but will be kept securely by the researchers until the information is accurately recorded in computer files, and will then be destroyed.

The results of the mathematics tests and interviews will be used in my thesis and possibly scientific journals. If you wish to know the results of the tests for your school, I will be happy to provide them. I will not be able to provide individual student results, however, because to preserve confidentiality, no names will be recorded.

Thank you for considering this invitation.

Yours sincerely,

Mamoon M. Mubark
Student Researcher
Email : mamoon.mubark@studentmail.newcastle.edu.au
Home phone : +96227310290

Professor Sid Bourke
Project supervisor I
Email:Sid.bourke@newcastle.edu.au

Dr Frances Rosamond
Project supervisor II
Email: fran@cs.newcastle.edu.au

If you have any concerns or complaints related to this research, you should first contact to Mamoon Mubark, or if an independent person is preferred, you should contact:

The Human Research Ethics Officer
Research & International Division,
The Chancellery,
Callaghan, NSW 2308, Australia
Tel.+61 2 49216333
Email: human-ethics@newcastle.edu.au

Or:
Dr. Farouq Almeqdadi
Assis. Prof of Math Education & Computers
Yarmouk University
Irbid.Jordan 21163
Email: farouq@yu.edu.jo

This project has been approved by the University's Human Research Ethics Committee, Approval No. H- [**Insert** approval number when known).

Should you have concerns about your rights as a participant in this research, or you have a complaint about the manner in which the research is conducted, it may be given to the researcher, or, if an independent person is preferred, to the Human Research Ethics Officer, Research Office, The Chancellery, The University of Newcastle, University Drive, Callaghan NSW 2308, telephone (02 49216333, email Human-Ethics@newcastle.edu.au.

كلية التربية و الاداب
أ.د سيد بورك
مساعد العميد,   لشؤون البحث.
تلفون : 61249215901+
فاكس : 61249216895+
Email: sid.bourke@newcastle.edu.au.

عزيزي مدير/مديرة المدرسة **التاريخ:** كانون الاول 2003م**

ادعو مدرستك للمشاركه في مشروع البحث بعنوان " التفكير الرياضي و التحصيل الرياضي" للباحث مأمون مبارك طالب دكتوراه , كلية التربيه جامعة نيو كاسل, استراليا. بناءا على كتاب وزارة التربية و التعليم.

الهدف من هذا البحث هو دراسة العلاقات بين المظاهر المختلفة للتفكير الرياضي و تحصيل الرياضيات, و التحقق من وجود اختلافات بين الذكور و الاناث في التفكير و التحصيل الرياضي. هذا البحث قد يساعد الطلبه في تطور تعليمهم الرياضي و ادائهم في الاختبارات.

لقد تم اختيارالذكور و الاناث من طلبة الصف الاول الثانوي العلمي في منطقتك ودعوتهم للمشاركة في هذه الدراسه. لقد تم اختيارمدرستك بطريقة عشوائية للمشاركة في هذه الدراسه.

اذا انت اعطيت الوافقة لمدرستك بالمشاركه, المشاركين من ا لمعلمين و الطلاب بشكل فردي لهم الحريه في المشاركه. كلاهم له الحق بالانسحاب في وقت خلال الاختبار او بعد اجرائه.

مشاركة الطلاب تضمن أختبارين , أختبار في التفكير الرياضي و بعد 15 يوم هناك أختبار تحصيل الرياضيات. وقت ألاختبارين خلال ساعات الدوام. الطلاب الذين ليس لديهم الرغبة في المشاركه في الاختبارين سوف يقوموا بحل واجب من كتاب الرياضيات. أختبار التفكير الرياضي يستغرق ثلاث ساعات تقريبا, لذا يوجد استراحه في منتصف الاختبار لمدة 15 دقيقه.أختبار تحصيل الرياضيات يستغرق ساعتين. سوف يقوم الباحث بدعوة عدد من الطلاب لاجراء مقابلة حول اجاباتهم, اضافة للاختبارين. لمشاركة هولاء الطلاب سوف أحذ بنصيحة المعلم حول كيفة اختيارهم, ومن الافضل طالبين من كل فئه عالي التحصيل, متوسط, ضعيف من ضمن الطلبة الذين وافقوا مسبقا على المقابلة. ربما تجرئ المقابلات في المكتبه.

تتضمن الدراسة ايضا اجراء مقابلة لمدة 30 دقيقة مع المعلمين الذين تشارك صفوفهم في الاختبار. المقابلة تتركز على موضوع واحد – كيف يطورا المعلمين التفكير الرياضي عند طلبتهم.

اذا لديك اي سؤال حول اي جزء من الدراسه, راجيا الاتصال بي.

لا يوجد اسم الطالب على ورقه الاجابه للاختبار و المقابله, لكن سوف تحفظ بسريه من قبل الباحثين حتى يتم ادخال المعلومات الى ملفات الكمبيوتر, و بعدها ستتلف اوراق الاجابة والمقابلة.

سوف تستخدم نتائج الامتحانات , و المقابلات في اطروحتي ومن الممكن استخدمها في المجلات العلميه. . اذا كان لديك الرغبه في في معرفة نتائج مدرستك , سأكون مسررا بتزويدك اياها. انا لا استطيع تزويدك بنتائج الطلبة بشكل فردي , ومن اجل سرية الاختبار و المقابلة, فلن يتم كتابة الاسم.

شكرا لتلبيتكم الدعوه.

المخلص,

مأمون مبارك
طالب دكتوراه
كلية التربية
Email: Mamoon.Mubark@studentmail.newcastle.edu.au
تلفون: +96227310290

أ.د سيد بورك
المشرف الاول
Email: Sid.Bourke@newcastle.edu.au

د. فرانسيس روزمند
المشرفه الثانيه
Email: fran@cs.newcastle.edu.au

إذا كان لديك شكوى متعلقة بهذا البحث , عليك ان تتصل بـ مأمون مبارك اولا. اذا كنت تحبذ شخصا مستقلا غير الباحث يرجى الاتصال به.

مكتب اخلاقيات البحوث التربوية
البحث & الشعبة الدولية , الرئاسه.
كالقن, نيو ساوث و يلز 2308, استراليا.
تلفون 61249216333 +
Email: human-ethics@newcastle.edu.au

أو
د. فاروق مقدادي
أستاذ مساعد اساليب تدريس الرياضيات و الحاسوب التعليمي.
جامعة اليرموك.
اربد- الاردن 21163.
Email: faroug@yu.edu.au

هذا المشروع حصل على الموافقة من لجبة اخلاقيات البحوث التربويه. رقم الموافقه هـ
[هـ -0903-655]

اذا كان لديك ما يقلقك فيما يتعلق بحقوقك كمشارك في هذا البحث او اذا كان لديك شكوى عن الطريقة التي اجري فيها البحث، فيمكن تقديمها للباحث.اما اذا كنت تحبذ شخص مستقلا, ميمكنك تقديم الشكوى الى مكتب اخلاقيات البحوث التربوية, مكتب البحث, الرئاسه, جامعة نيوكاسل, شارع الجامعه, كالقن نيو ساوث ويلز 2308, تلفون (0249216333).
Email: Human-Ethics@newcastle.edu.au.

Appendix 2.5

(ii) Teachers Letter in both English and Arabic Languages

Teacher Letter

FACULTY OF EDUCATION & ARTS

Professor S.F. Bourke
Assistant Dean, Research & Research Training
Phone: 61 2 4921 5901
Fax: 61 2 4921 6895
Email: sid.bourke@newcastle.edu.au

xx December 2003

Dear teacher,

You are invited to participate in a research study called 'Mathematical thinking and mathematical achievement' being conducted by Mamoon Mubark a doctoral student in the School of Education at the University of Newcastle, Australia.

The aim of this research is to study the relationships between different aspects of mathematical thinking and mathematics achievement, and to investigate whether male and female students differ in mathematical thinking and achievement. This could provide important information for assisting students in their mathematical learning and performance in examination.

Male and female students from the 1st secondary scientific stream in your district are being chosen to be invited to participate in the study. Your school and class have been selected at random to participate in this study.

Participation in this study is entirely voluntary. If you agree to participate, you are free to withdraw at any time during the testing or after. You will not be disadvantaged in any way if you decide that your class will not to participate or if you to withdraw.

I would like to include your class in my study. Participation would involve them in taking two tests, a test of mathematical thinking and, 15 days later, a mathematics achievement test. The tests would be done during mathematics lessons. Students deciding not to do the tests could work from their mathematics textbooks during these lessons. The test of mathematical thinking will take approximately three hours, so there is a break in the middle of the test of 15 minutes. The test of mathematics achievement will take two hours. A small number of students will also be invited to take part in a group interview about their test answers, but this would be additional to the tests. I would appreciate your advice about which students to interview – preferably two each of the highest, middle and lowest performing students in the class who have previously given their permission to be interviewed.

I am also asking for your permission to interview you for about 30 minutes about how you develop mathematical thinking in your students. I would like to audio-tape the interview. If you agree, you would have the right to listen to the tape and ask for its erasure or to edit any section of it you are not happy with.

If you are unclear about any aspect of the study, please contact me.

If you are willing to be interviewed please complete the form attached and leave it in the box provided in the staffroom. The interview would take place in the library at a time that suits you.

The test and interview data will not have any names attached to them, but will be kept securely by the researchers until the information is accurately recorded in computer files, and will then be destroyed.

The results of the mathematics tests and interviews will be used in my thesis only. If you wish to know the results of the tests for your class, I will be happy to provide them. I will not be able to provide individual student results, however, because to preserve confidentiality, no names will be recorded. A summary of school results would also be provided to the school principal.

Thank you for considering this invitation.
Yours sincerely,
Mamoon M. Mubark
Student Researcher
Email: mamoon.mubark@studentmail.newcastle.edu.au
Home phone: +96227310290

Professor Sid Bourke
Project supervisor I
Email:Sid.bourke@newcastle.edu.au

Dr Frances Rosamond
Project supervisor II
Email: fran@cs.newcastle.edu.au

If you have any concerns or complaints related to this research, you should first contact to Mamoon Mubark, or if an independent person is preferred, you should contact:

The Human Research Ethics Officer
Research & International Division,

The Chancellery,
Callaghan, NSW 2308, Australia
Tel.+61 2 49216333
Email: human-ethics@newcastle.edu.au

Or:
Dr. Farouq Almeqdadi
Assis. Prof of Math Education & Computers
Yarmouk University
Irbid.Jordan 21163
Email: farouq@yu.edu.jo

This project has been approved by the University's Human Research Ethics Committee, Approval No. H- [**insert** approval number when known).

Should you have concerns about your rights as a participant in this research, or you have a complaint about the manner in which the research is conducted, it may be given to the researcher, or, if an independent person is preferred, to the Human Research Ethics Officer, Research Office, The Chancellery, The University of Newcastle, University Drive, Callaghan NSW 2308, telephone (02 49216333, email Human-Ethics@newcastle.edu.au.

كلية التربية و الاداب
أ.د سيد بورك
مساعد العميد, لشؤون البحث.
تلفون : 61249215901+
فاكس : 61249216895+
Email: sid.bourke@newcastle.edu.au.

عزيزي المعلم, **التاريخ:** كانون الاول 2003م**

ادعوك للمشاركه في بحث بعبوان " التفكير الرياضي و التحصيل الرياضي" للباحث مأمون مبارك طالب دكتوراه في كلية التربية جامعة نيوكاسل, استراليا.

الهدف من هذا البحث هو دراسة العلاقات بين المظاهر المختلفة للتفكير الرياضي و تحصيل الرياضيات, و التحقق من وجود اختلافات بين الذكور و الاناث في التفكير و التحصيل الرياضي. هذا البحث قد يساعد الطلبه في تطور تعليمهم الرياضي و ادائهم في الاختبارات.

لقد تم اختيارالذكور و الاناث من طلبة الصف الاول الثانوي العلمي في منطقتك ودعوتهم للمشاركة في هذه الدراسه. لقد تم اختيارمدرستك و صفك بطريقة عشوائية للمشاركة في هذه الدراسه.

المشاركة في هذا البحث ا اختيارية. اذا انت وافقت على المشاركة, لك مطلق الحرية في انسحاب صفك في اي وقت من الاختبار أو بعد اجراء الاختبار . واذا ما قررت عدم مشاركة صفك أو الانسحاب فإن ذلك لم يؤثر عليك سلبا.

أرغب مشاركة صفك في الدراسة.مشاركة الطلاب تضمن أختبارين , أختبار في التفكير الرياضي و بعد 15 يوم هناك أختبار تحصيل الرياضيات. وقت ألاختبارين خلال ساعات الدوام. الطلاب الذين ليس لديهم الرغبة في المشاركه في الاختبارين سوف يقوموا بحل واجب من كتاب الرياضيات. أختبار التفكير الرياضي يستغرق ثلاث ساعات تقريبا, لذا يوجد استراحه في منتصف الاختبار لمدة 15 دقيقه.أختبار تحصيل الرياضيات يستغرق ساعتين. سوف يقوم الباحث بدعوة عدد من الطلاب لاجراء مقابلة حول اجاباتهم, اضافة للاختبارين. اقدر نصيحتك حول كيفة اختيار الطلبة, و من الافضل طالبين من كل فئه عالي التحصيل, متوسط, ضعيف من ضمن الطلبة الذين وافقوا مسبقا على المقابلة.

أسالك ايضا في سماحك لي بعمل مقابلة معك ستاخذ حوالي 30 دقيقه حول كيف تطور التفكير الرياضي عند طلبتك. ارغب في تسجيل المقابلة,اذا انت موافق, لك الحق ان تستمع الى الشريط و اضافة اي شئ جديد على المقابلة, وكذلك ازالة اي مقطع انت ليست مسرور به.

اذا كان هناك اي غموض حول اي جزء من الدراسه, راجيا الاتصال بي.

إذا كنت ترغب في المقابلة, يرجى منك تعبئة الموافقة المرفقة , و اتركها في الصند وق في غرفة المعلمين. على ان تجري المقابلات في مكتبة المدرسة حسب الوقت المناسب لك.

لا يوجد اسم الطالب على ورقه الاجابه للاختبار و المقابله, لكن سوف تحفظ بسريه من قبل الباحثين حتى يتم ادخال المعلومات الى ملفات الكمبيوتر, و بعدها ستتلف اوراق الاجابة والمقابلة.

سوف تستخدم نتائج الامتحانات , و المقابلات في اطروحتي ومن الممكن استخدمها في المجلات العلميه. . اذا كان لديك الرغبه في في معرفة نتائج صفك , سأكون مسررا بتزويدك اياها. انا لا استطيع تزويدك بنتائج الطلبة

شكرا لتلبيتكم الدعوه.

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إذاكان لديك شكوى متعلقة بهذا البحث , عليك ان تتصل بـ مأمون مبارك اولا. اذا كنت تحبذ شخصا مستقلا غير الباحث يرجى الاتصال به.

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هذا المشروع حصل على الموافقة من لجبة اخلاقيات البحوث التربويه. رقم الموافقه هـ
[هـ -655-0903]

اذا كان لديك ما يقلقك فيما يتعلق بحقوقك كمشارك في هذا البحث او اذا كان لديك شكوى عن الطريقة التي اجري فيها البحث، فيمكن تقديمها للباحث.اما اذا كنت تحبذ شخص مستقلا, ميمكنك تقديم الشكوى الى مكتب اخلاقيات البحوث التربوية, مكتب البحث, الرئاسه, جامعة نيوكاسل, شارع الجامعه, كالقن نيو ساوث ويلز 2308, تلفون (0249216333).
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Appendix 2.5

(iii) Students Letter in both English and Arabic Languages

Student Letter

FACULTY OF EDUCATION & ARTS

Professor S.F. Bourke
Assistant Dean, Research & Research Training
Phone: 61 2 4921 5901
Fax: 61 2 4921 6895
Email: sid.bourke@newcastle.edu.au

xx December 2003

Dear student,

You are invited to participate in a research study called 'Mathematical thinking and mathematical achievement' being conducted by Mamoon Mubark a doctoral student in the School of Education at the University of Newcastle, Australia.

The aim of this research is to study the relationships between different aspects of mathematical thinking and mathematics achievement, and to investigate whether male and female students differ in mathematical thinking and achievement. This will be important information in assisting students in their mathematical learning and performance in examination.

Male and female students from the 1st secondary scientific stream in your district are being chosen to be invited to participate in the study, your school and class have been selected at random to participate in this study.

Your participation in this study is entirely voluntary. If you agree to participate, you are free to withdraw at any time during the testing. You will not be disadvantaged in any way should you decide not to participate or to withdraw.

Participation would involve you taking two tests, a test of mathematical thinking and, 15 days later, a mathematics achievement test. The tests would be done during mathematics lessons. Students deciding not to do the tests will work from their mathematics textbooks during these lessons. The test of mathematical thinking will take approximately three hours, so there is a break in the middle of the test of 15 minutes. The test of mathematics achievement will take two hours. A small number of students will also be invited to take part in a group interview about their test answers, but this would be additional to the tests. The teacher will advise the researcher about who should be selected for interview, from those students who have previously given their permission. The groups will include students with a range of mathematical ability, and are designed to show different ways of thinking in mathematics, and not to indicate levels of ability.

If you are willing to take the tests as a part of this research project, please complete the consent form attached and leave it in the box provided in the library.

If you are also willing to take part in a 30-minute interview about your test answers as a part of this research project after the test of mathematical thinking (and lunch rest), please complete the other form attached and also leave it in the box provided in the library. I would like to audio-tape the interview. You have the right to listen to the tape and request the erasure or editing of your contribution to any section of the tape that you are not happy with.

This research will have no risks of harm for the participants. The test and interview procedure will be conducted in a supportive manner. Students should not be concerned about failing or doing poorly in either the test or the interview. Any student who feels anxious or distressed at any time may end their participation at that moment. The teacher will receive information about their class's performance in maths, but not about the performance of individual students. The class performance can be used by the teacher in teaching the class.

No student names are being collected. Only class and school identification will be recorded for the purpose of providing feedback to participants. Individual students' contribution will not be identified, during group interview. During the data collection period, Mr Mubrak will keep the data secure at his home in Jordan and on his return to Australia the School of Education will provide lockable storage.

If you need help to understand this information, you can either talk to your teacher or contact the researcher.

The test and interview data will not have your name on them, but will be kept securely by the researchers until the information is accurately recorded in computer files, and will then be destroyed.

The results of the mathematics tests and interviews will be used in my thesis and possibly in scientific journals. If you wish to know the results of the tests, you can ask your teacher or contact the researcher and the class results will be provided. You cannot receive your personal results because, to preserve confidentiality, I will not have your name recorded.

Thank you for considering this invitation.

Yours sincerely,
Mamoon M. Mubark

Student Researcher
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Home phone: +96227310290

Professor Sid Bourke
Project supervisor I
Email:Sid.bourke@newcastle.edu.au

Dr Frances Rosamond
Project supervisor II
Email: fran@cs.newcastle.edu.au

If you have any concerns or complaints related to this research, you should first contact to Ma'moon Mubark, or if an independent person is preferred, you should contact:

The Human Research Ethics Officer
Research & International Division,
The Chancellery,
Callaghan, NSW 2308, Australia
Tel.+61 2 49216333
Email: human-ethics@newcastle.edu.au

Or:
Dr. Farouq Almeqdadi
Assis. Prof of Math Education & Computers
Yarmouk University
Irbid.Jordan 21163
Email: farouq@yu.edu.jo

This project has been approved by the University's Human Research Ethics Committee, Approval No. H- [**Insert** approval number when known).

Should you have concerns about your rights as a participant in this research, or you have a complaint about the manner in which the research is conducted, it may be given to the researcher, or, if an independent person is preferred, to the Human Research Ethics Officer, Research Office, The Chancellery, The University of Newcastle, University Drive, Callaghan NSW 2308, telephone (02 49216333, email Human-Ethics@newcastle.edu.au.

كلية التربية و الاداب

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عزيزي الطالب, **التاريخ:** كانون الاول 2003**

ادعوك للمشاركه في بحث بعبوان " التفكير الرياضي و التحصيل الرياضي" للباحث مأمون مبارك طالب دكتوراه في كلية التربية جامعة نيوكاسل, استراليا.

الهدف من هذا البحث هو دراسة العلاقات بين المظاهر المختلفة للتفكير الرياضي و تحصيل الرياضيات, و التحقق من وجود اختلافات بين الذكور والاناث في التفكير و التحصيل الرياضي. هذا البحث سوف يساعد الطلبه في تطور تعلمهم للرياضيات و ادائهم في الاختبارات.

لقد تم اختيارالذكور و الاناث من طلبة الصف الاول الثانوي العلمي في منطقتك ودعوتهم للمشاركة في هذه الدراسه. لقد تم اختيارمدرستك و صفك بطريقة عشوائية للمشاركة في هذه الدراسه.

مشاركتك في هذا البحث ا اختيارية. اذا وافقت على المشاركة, لك مطلق الحرية في الانسحاب في اي وقت من الامتحان , واذا ما قررت عدم المشاركة أو الانسحاب فإن ذلك لم يؤثر عليك سلبا.

مشاركتك تتضمن أختبارين , أختبار في التفكير الرياضي و بعد 15 يوم هناك أختبار تحصيل الرياضيات. وقت ألاختبارين خلال ساعات الدوام. الطلاب الذين ليس لديهم الرغبة في المشاركه في الاختبارين سوف يقوموا بحل واجب من كتاب الرياضيات. أختبار التفكير الرياضي يستغرق ثلاث ساعات تقريبا, لذا يوجد استراحه في منتصف الاختبار لمدة 15 دقيقه.أختبار تحصيل الرياضيات يستغرق ساعتين.سوف يقوم الباحث بدعوة عدد من الطلاب لاجراء مقابلة معهم حول اجاباتهم. اضافة للاختبارين، سوف يقوم المعلم بمساعدة الباحث حول كيفية الاختيار الطلبة الذين سوف تجرى معهم المقابلات شزيطة ان يكونوا من ضمن الطلبة الذين وافقوا مسبقا على المشاركة .سوف تتضمن المجموعة طلاب ذوي قدرات رياضية مختلفة. و صمم هذا للتعرف على الطرق المختلفة في التفكير, و ليس على مستويات القدره.

إذا كنت ترغب في المشاركة في هذين الاختبارين كجزء من مشروع البحث هذا, يرجى منك تعبئة الموافقة المرفقه, و تركها في الصندوق في المكتبة.

إذا كنت ترغب ايضا في المشاركة في مقابلة لمدة 30 دقيقة حول اجابات اختبارك كجزء من مشروع البحث هذا بعد اختبار التفكير الرياضي (فترة الاستراحه), يرجى منك تعبئة الموافقة الاخرى المرفقة , وتركها في الصنوق في المكتبه. لدي الرغبه في تسيجل شريط خلال المقابلة, و لك الحق في الاستماع الى مشاركتك وا ضافة او شطب اي جزء من مشاركتك في اي مقطع من الشريط.

لاينطوي اجراء هذا البحث على مخاطر تؤذي المشاركين.سوف يتم اجراء الاختبار و المقابلة بطريقة تتضمن تقديم كل مساعدة ممكنة للمشارك. لا داعي لقلق الطلاب عند شعورهم بالفشل او الاداء الضعيف سواء في الاختبار او المقابله. بامكان اي طالب يشعر بالقلق في اي وقت من الاختبار او المقابلة انهاء مشاركته في تلك اللحظه. المعلم سوف يحصل على معلومات حول اداء صفه في الرياضيات. لكن ليس على اداء الطلبه بشكل فردي. اداء الصف يمكن ان يستخدم من قبل المعلم في تطوير تدريس صفه.

لن يطلب من الطلبة كتابة اسماءهم. سوف يكتفى فقط برمز للمدرسه و الصف و ذلك من اجل تزويد المشاركين بالتغذيه الراجعه. لن تعلن نتائج اداء الطلاب اثناء مجموعة المقابله. سوف يقوم السيد مبارك خلال مرحلة جمع المعلومات بحفظ البيانات بشكل مصون في بيته في الاردن وعند رجوعه الى استراليا سوف تحفظ كلية التربيه البيانات في مكان آمن.

إذا كنت بحاجة الى مساعده لفهم هذه المعلومات, يمكنك التحدث مع معلمك او مع الباحث.

لا يوجد اسمك على ورقه الاجابه للاختبار و المقابله, لكن سوف تحفظ بسريه من قبل الباحثين حتى يتم ادخال المعلومات الى ملفات الكمبيوتر, و بعدها ستتلف اوراق الاجابة والمقابلة.

سوف تستخدم نتائج الامتحانات , و المقابلات في اطروحتي ومن الممكن استخدامها في المجلات العلميه. اذا كان لديك الرغبه في في معرفة نتائجك , تستطع ان تسأل معلمك او الباحث و سوف تزود بنتائج الصف. لا تستطيع ان تعرف نتيجتك بشكل فردي. ومن اجل سرية الاختبار والمقابلة، فلن يتم كتابة اسمك .

شكرا لك على قبولك الدعوة

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إذا كان لديك شكوى متعلقة بهذا البحث , عليك ان تتصل بـ مأمون مبارك اولا. اذا كنت تحبذ شخصا مستقلا غير الباحث يرجى الاتصال به.

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هذا المشروع حصل على الموافقة من لجبة اخلاقيات البحوث التربويه. رقم الموافقه هـ
[هـ -655-0903]

اذا كان لديك ما يقلقك فيما يتعلق بحقوقك كمشارك في هذا البحث او اذا كان لديك شكوى عن الطريقة التي اجري فيها البحث، فيمكن تقديمها للباحث.اما اذا كنت تحبذ شخص مستقلا, ميمكنك تقديم الشكوى الى مكتب اخلاقيات البحوث التربوية, مكتب البحث, الرئاسه, جامعة نيوكاسل, شارع الجامعه, كالقن نيو ساوث ويلز 2308, تلفون (0249216333).
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Appendix Three

This appendix will relate to Chapter 4 “The mathematical Tests: Scoring, Reliability, and Validity”

(i) Appendix 3.1 Mathematical Thinking Answers

(ii) Appendix 3.2 Examples of Rubrics for Extended-Response items, Multi Choice items and Mathematics Achievement Score

(iii) Appendix 3.3 Mathematical Thinking Test Reliability

(iv) Appendix 3.4 Mathematical Thinking Test (Facility and Discrimination for Each Item)

(v) Appendix 3.5 Factor Analysis for Each Scale of Mathematical Thinking

(vi) Appendix 3.6 Mathematics Achievement Test (Facility and Discrimination for Each Item)

Appendix 3.1

Mathematical Thinking Answers

Part one: Generalization.

G1) If n➔n is a function as n= [1, 2, 3---), and 1➔1, 2➔8, 3➔27, 4➔64, then X➔--- (As $X \in n$).

Answers: 1➔1, 2➔8, 3➔27, 4➔64, then, X→ X^3 or any number goes to its cube.

G2) Complete the last statement.
1=1
1+3=4
1+3+5=9
1+3+5+7=16
1+3+5+7+---+ (2n-1) =---------.

Answer:
(2x1-1)= 1
1+ (2x2-1) =4
1+3+ (2x3-1) =9
1+3+5+ (2x4-1) =16
1+3+5+7+---+ (2n-1) =---------.

The observation of the outcome in each statement is equal to the square of the variable in the last term in each statement. Based on this the outcome for the last statement is equal to n^2. Or the observation of the outcome in each statement is equal to the square number of terms in each statement. Based on this the outcome for the last statement is equal to n^2.

G3) Notice the two numbers of the right of the equals mark and their totals to its left in the following, and then discuss any deduction that can be made.
6=3+3 8=5+3 10=5+5 12=5+7 14=7+7 14=3+11
16=11+5 16=13+3.

Answer: The conclusion is: every even number can be expressed as two odd prime numbers or every even number greater than 4 can be expressed as two prime numbers.

G4) Complete the table:

Number of sides of the polygon	3	4	5	7	-----	n
Number of diagonals	0	2	5		------	

(Hint: This square relation).

Answer: This is a square relation mean f (X) = a X^2 + b X + c,
f (3) = 0 = 9a + 3b + c --------1
f (4) = 2 = 16a + 4b + c --------2

$f(5) = 5 = 25a + 5b + c$ --------3

Then, solve this system of equations by any method until we find ($a= ½$, $b = -3/2$ and $c = 0$). Consequently, $f(x) = ½ X^2 - 3/2 X$.

The number of diagonals for a heptagon is = $½ . (7)^2 - 3/2 . 7 = 49/2 – 21/2 = 28/2 = 14$ diagonals.

The number of diagonals for the polygon for which the number of sides is $n = ½ . (n)^2 - 3/2 . n$.

Or we can find the number of diagonals by drawing only the heptagon.

Or we can find the pattern through a combination for the relationship between the number of sides of the polygon and the number of diagonals.

In general, the number of diagonals = $\binom{n}{2} - n = ½ n^2 – 3/2n$, where n is the number of sides of the polygon. The same relationship that is found by square relationship, so $f(7) = 14$ diagonals.

G5) Contemplate in each pair of the following equal numbers and notice the relation between the numbers to the left of equal mark and the two numbers to the right of the equal mark.

a) $6\times6=36$ b) $7\times7=49$ c) $8\times8=64$

$3\times9=27$ $4\times10=40$ $5\times11=55$.

If $X\times X=289$, then $(x-3) \times (x+3=$----------.

Answer: The observation is that in each individual case, the first two numbers are equal, as are the other two numbers, to the left hand, one of them is equal (first number -3) and the other = (second number + 3). Then, the difference between the first outcome and the second is = 9. Consequently, If $X\times X=289$, then $(x-3) \times (x+3= 289 – 9 = 280$. However, the traditional method without dependence on the previous statements that we know $(x-3) \times (x+3) = X^2 - 9 = 289 - 9 = 280$, because $X\times X= X^2 = 289$

Part second: Induction.

I1) The number of bacteria in a colony was growing exponentially. At 1 pm yesterday the number of bacteria was 1000, and at 3 pm yesterday it was 4000. How many bacteria were there in colony at 6 pm yesterday?
The number of bacteria was 1000 at 1PM.

Answer: The number of bacteria was 4000 at 3PM, because the bacteria was growing exponentially, it was necessary the number of bacteria at 2PM = 2000, that means the number of bacteria was doubling in every hour, consequently, the number of bacteria was 1000, 2000, 4000, 8000, 16000 and 32000 at 1, 2, 3, 4, 5 and 6 PM respectively, so the number of bacteria was 32000 at 6 PM.
Or the bacteria was growing exponentially, then $f(n) = a(r)^{n-1}$, (where (a) is constant, r is basic and n time in hours, n-1 not n, because the number of bacteria started with 1000 not 2000).
At 1 PM the number of bacteria = 1000, $f(1) = 1000 = a(r)^{n-1} = 1000(r)^{1-1} = 1000$, then a = 1000.
At 3 PM the number of bacteria = 4000, $f(3) = 4000 = 1000(r)^{3-1} = 1000(r)^{2} = 4000$, then r = 2. Then $f(6) = 1000(2)^{6-1} = 1000(2)^{5} = 1000(32) = 32000$.

I2) A group of the numbers appeared classified as follows:

3 ½, 5 1/3, 7 ¼, 9 1/5, ---.
What is the tenth number?

Answer: The observation that each number contains integer and fraction the integer was odd number and stared from 3 then the integers until tenth integer. So the integers were: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 and the fractions until tenth fraction were: ½, 1/3, ¼, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/11, then the tenth term was 21 1/11.
Or by finding the pattern, we know an odd number can be expressed as 2n + 1, where n = 0, 1, 2, --- if the first number started from 1, or n = 1, 2, ---. So the odd number here can be expressed as 2n + 1, n = 1, 2, 3, ----, then the tenth integer = 2(10) + 1 = 20 +1 = 21. However, the fraction number can be expressed as 1/ (n+1), because the first number started from 2 then 3 and so on, then the tenth fraction = 1/10+1= 1/11. Consequently, the tenth term = 21 1/11.

I3) A long time ago, a mathematician invented the game of chess and presented it to the king. The king was so pleased with the game that he asked the mathematician to name a reward. The mathematician looked at the chess board, consisting of 64 square, and asked for the some rice according to this rule:

One grain of rice on the first square of chessboard, two grains on the second square, four grains on the third square, and so on until the last square.

How many grains of rice are there on 64^{th} square? And how many grains of rice did the mathematician ask for in total? Explain the pattern you are using.

Answer: The number of rice grains on the first square = 1 = 2^0
The number of rice grains on the second square = 2 = 2^1
The number of rice grains on the third square = 4 = 2^2
Then, in general, the number of rice grains on the square number n = 2^{n-1}. Consequently, the number of rice in the last square (the 64 square) = 2^{63}.

The total of rice in the first square = 1 = 2-1 = 2^1-1
The total of rice up to second square = 3 = 4-1 = 2^2-1
The total of rice up to third square = 7 = 8-1 = 2^3-1
In general, the total of rice up to square n = 2^n-1. Consequently, the total of rice in the whole square = 2^{64}-1.
Or the total of rice grains up to first square = (the number of rice grains in the second square – 1)➔ 1 = 2 -1.
The total of rice grains up to second square = (the number of rice grains in the third square – 1)➔ 3 = 4 -1.
The total of rice grains up to third square = (the number of rice grains in the fourth square – 1)➔ 7 = 8 -1.
In general, the total of rice grains up to nth square = (the number of rice grains in the (n+1) square – 1).
Then, the total of rice grains up to 63rd square = (the number of rice grains in the 64 square – 1)➔ 2^{63} – 1, and the total of rice grains up to 64^{th} square = (the total of rice grains up to 63 + the number of rice grains in the 64^{th} square) = 2^{63} – 1 + 2^{63} = 2×2^{63} – 1= 2^{64} -1. Or the total of rice up to 64^{th} square = (the number of the rice in the 65 square – 1) = 2^{64}-1.
Or the total of rice in the whole square = 2^0 + 2^1 + 2^2 +---+ 2^{63} = a (r) n -1/r-1, (where a the first term, r the basic of series, and n the total of terms) = (2^{64}-1)/(2-1) = 2^{64}-1 (However, the researcher administered the test in first semester, and this formula available in students' curricula in the second semester, one student found the total of grains of rice using this formula).

I4) Contemplate an analysis of the following algebra statement and write an analysis of the last statement.
$(x-1)^2=x^2 -2x+1$.
$(x-1)^3= x^3 -3x^2 + 3x -1$.
$(x-1)^4 = x^4 -4x^3 +6x^2 -4x +1$.
$(x-1)^n$ = -----------------------.

Answer: $(x-1)^n= x^n - nx^{n-1} + (n)(n-1)/2x^{n-2} - (n)(n-1)(n-3) / (3)(2)(1) + ---+ nx - 1$ if n is odd or $(x-1)^n= x^n - nx^{n-1} + (n)(n-1)/2x^{n-2} - (n)(n-1)(n-3) / (3)(2)(1) + --- - nx + 1$ if n is even.

I5) The following three cards to the left are written according to a certain rule form “If-----, Then ------, where as the fourth cards is written in a form that does not correspond to that rule.

The cards that correspond to the rule The card that does not correspond

A	B	C	D
○	14	M	A
4	□	15	Δ

The rule is:

a) If a shape appears in the upper half of the cards, a number appears in its lower half.

b) If a number appears in the upper half of the card, a shape appears in its lower half.

c) If a letter appears in the upper half of the card, a number appears in its lower half.

d) If a letter appears in the upper half of the card, a shape appears in its lower half.

Answer: the choices (a) and (b) are incorrect, because they are included the cards that correspond to the rule and agree with them, but they did not include the card that does not correspond. However, the choice (d) is incorrect also, because agree with the card that does not correspond and did not agree with the cards that correspond to the rule. Consequently, the correct answer is (c), because it agrees with the cards that correspond to the rule and did not agree with the card that does not correspond to the rule.

Part three: Deduction.

D1) If (xy) =1, and x is greater than 0, which of the following statements is true?
a) When x is greater then 1, y is negative.
b) When x is greater than 1, y is greater than 1.
c) When x is less than 1, y less than 1.
d) As x increases, y increases.
e) As x increases, y decreases.

Answer: Since (xy) =1 and x > 0, then y must be greater than 0. Consequently, the alternative (a) is incorrect, because y is negative < 0.
The alternatives (b) and (c) are also incorrect, because if x > 1 and y > 1, then xy > 1 ≠ 1, and if x < 1 and y < 1, then xy < 1 ≠ 1. For Example, x > 1 = 2 and y > 1 = 3, then xy = 6 > 1, and the same when both x and y less than 1, the outcome will be less than 1. The alternative (d) is incorrect, because if we suppose x is 2 then y must be ½ to get 1. So suppose x is 3 (x increase from 2 to 3), then y must be 1/3 to get 1) in this case we found x increase, whereas y decrease. Consequently, the correct answer is (e) if x increase, y will be decreased. Or we can find the correct answer by drawing xy = 1 on the x, y plane, and observe the inverse relation between x and y.

D2) All of the numbers in group A are divisible by 5, number (20) is divisible by 5, and belongs to group B, we infer from that:
a) Group A is equal to group B.
b) A is a sub-group of B.
c) B is a sub-group of A.
d) Nothing from what is mentioned above.

Answer: For example, let A = {5, 10, 15} because all these numbers are divisible by 5, and B = {20, 7} because all we know here that 20 belongs to B. The choice (a) is incorrect, because number 7 belongs to B but not to A, and the numbers 10. 15 belong to A not to B. The choice (b) is incorrect, because the numbers 10 and 15 belong to A but not to B. The choice (c) is incorrect also because number 7 belongs to B not to A. Consequently, the correct choice is (d).

D3) Read both of the following hypotheses.
1) All engineering students in (J.S.T.U) are intelligent
2) All science students in Y.U are intelligent.
What is the correct deduction from the following?
a) All engineering students in both universities are intelligent.
b) All science students in both universities are intelligent.
c) All science and engineering students in both universities are intelligent.
d) We can’t induce anything from what has been mentioned above.

Answer: We can not conclude that all science students in J.S.T.U are intelligent or not intelligent, and all engineering students in Y.U are intelligent or not. Consequently, the correct answer is (d) we can't induce anything from what has been mentioned above.

D4) The vertices of the triangle PQR are the point p (1, 2), q (4, 6), r (-4, 12), which one of the following statements above triangle PQR is true?

a) PQR is a right triangle with the right angle$\angle$P.
b) PQR is a right triangle with the right angle$\angle$Q.
c) PQR is a right triangle with the right angle$\angle$R.
d) PQR is not a right triangle.

Answer: The distance between pq = 5 (using the distance formula between two points in a plane $((x_2-x_1)^2 + (y_2-y_1)^2)^{½}$. qr = 10 and rp = √125. We observe by Pythagoras theorem that $(rp)^2 = (qr)^2 + (pq)^2 = 125 = 100 + 25 = 125$. Consequently, the q is a right angle, then PQR is a right triangle with the right angle$\angle$Q and the correct answer is (b). Or by finding the slope for each line, $m_{pq} = (y_2-y_1)/ ((x_2-x_1)$ = 6-2/4-1 = 4/3, $m_{qr} = (y_2-y_1)/ ((x_2-x_1)$ = 12-6/-4-4 = 6/-8 = -3/4, because $m_{pq\ X}\, m_{qr}$ = 4/3 X -3/4 = -1, then pq and qr are lines perpendicular, and q is right angle, then PQR is a right triangle with the right angle$\angle$Q and the correct answer is (b). Or by accurate drawing for triangle PQR on xy plane.

D5) Some of the isosceles triangles are right triangles.
The medians of all triangles intersect at one point.
ABC is a right triangle.
What do you induce from triangle ABC?

a) ABC is a triangle whose medians are of equal length.
b) ABC is an isosceles triangle whose medians intersect at one point.
c) ABC is a triangle whose medians intersect at one point.
d) ABC is a triangle whose medians intersect at one point, but is not an isosceles triangle.

Answer: ABC is a right triangle, then we can not say if it is an isosceles triangle or not, because some isosceles triangles are right triangles, but not all of them. Consequently, b and d are incorrect answers. For example, ABC is right triangle whose side lengths are 3, 4 and 5. We can find by Pythagoras two of the medians of the triangle with length √13, √72/4 and they are different. Then the choice (a) is incorrect and the correct answer is (b).

Part four: Use of Symbols.

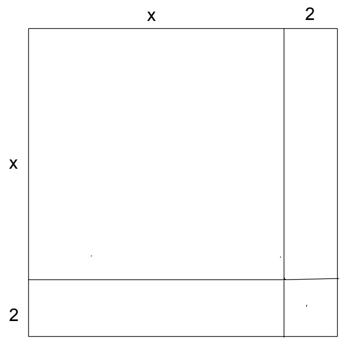
S1) There are two classes, A and B, in a school. The number of students in class A is ten more than in B. If five students move from class B to A, then the number of students in A becomes triple the number in B. Express the above in equations.

Answer: Let the number of students in class A = A, and the number of students in class B = B. The first equation is A = 10 + B or A-B=10 or A-10 = B. However, the second equation is A + 5 = 3 (B-5).

S2) Unit circle, its central (0, 0), a line L through (0, 1), with rational slope (m). Write an expression for the equation of the circle in terms of x.

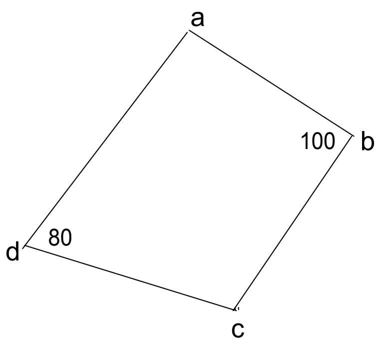
Answer: the circle equation $x^2 + y^2 = 1$ (where centre is original point and the radius is equal to1). The line equation is $y = mx + 1$, (where the line through (0, 1) and its slope is m), because the line cut $x^2 + y^2 = 1$ unit circle, then the line equation will satisfy the circle equation, then $x^2 + (mx + 1)^2 = 1$ by x only.

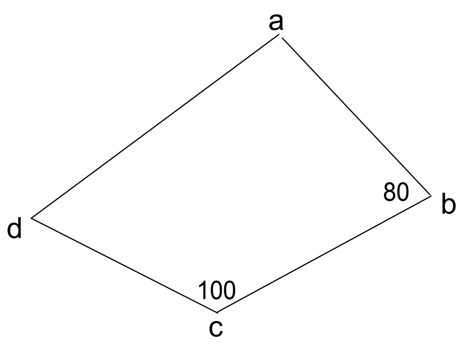
S3) Find $(x+2)^2$ using the following shape.



Answer: the area of the large square = x^2, the area of each rectangle = 2x, and the area of the small square = 2^2, then the area of the whole shape = $x^2 + 2x + 2x + 2^2 = x^2 + 4x + 4$.

S4) If the quadrilateral shape is cyclic, then the total of each opposite angle in it is equal to its two right angles (180)°and vice versa.



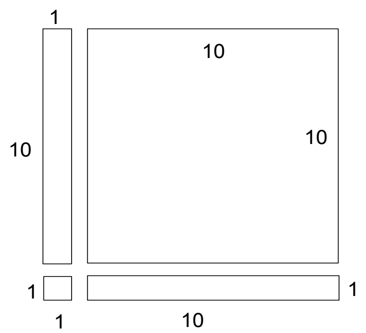


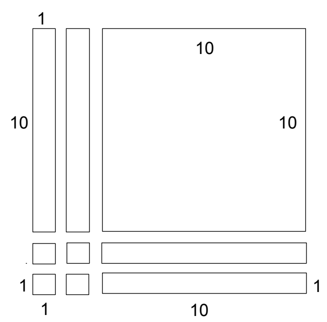
Shape 1 Shape 2

What can we induce in regards of the two shapes “1” and “2”
a) The shape “1” and “2” are cyclic.
b) Shape “1” is cyclic, “2” is not cyclic.
c) Shape “2” is cyclic, “1” is not cyclic.
d) Shape “1” is cyclic, “2” we don’t know.

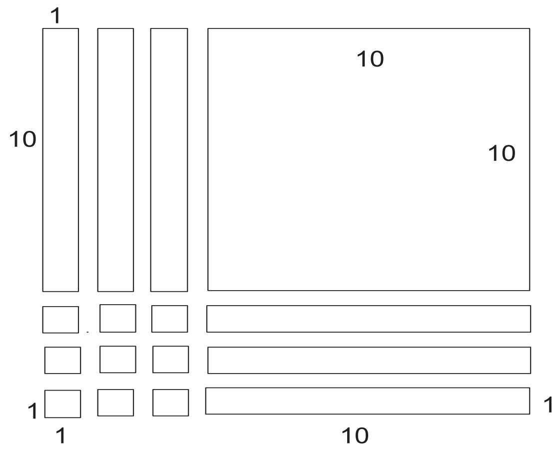
Because in shape 1 the total of < d and < b = 180°and they are opposite angles, it is compulsory that the total of < a and < c = 180° (since the total of any quadrilateral is equal to 360°), the shape 1 is cyclic. However, the known two angles in shape 2 are neighbours not opposite angles, so if the < d = 100° then the second shape will be cyclic, otherwise will be not cyclic, so from only this information we can not decide if shape 2 is cyclic or not, then the correct answer is d.

S5) Express the following shapes by symbols.





Graph1 Graph2



Graph3

Consequently, n^2=-----------.

$11^2 = 10^2 + 2 \times 10 \times 1 + 1^2$

$12^2 = 10^2 + 2 \times 10 \times 2 + 2^2$

$13^2 = 10^2 + 2 \times 10 \times 3 + 3^2$, we observed that 102, 2, 10 and the square are constant in each one. In contrast, the variables were 1, 2, and 3 when the numbers were 11, 12, and 13 respectively). Consequently, $n^2 = 10^2 + 2 \times 10 \times (n-10) + (n-10)^2$

Part five: Logical thinking:

L1) The symmetric difference of two sets A and B is defined to be.
$A\Delta B = (A-B) \cup (B-A)$.
a) Draw a Venn diagram to illustrate $A\Delta B$.
b) Prove that $(A-B) \cup (B-A) = (A\cup B)-(A\cap B)$.

Answer: a) by drawing correctly Venn diagram and shading $A-B \cup B-A$
B) $(A-B) \cup (B-A) = (A-B) + (B-A) - ((A-B) \cap (B-A)) = (A-B) + (B-A) - \varnothing = (A-B) + (B-A) = (A \cup B) - (A\cap B)$, (Note: $A-B \cap B-A = \varnothing$). Or we can prove it by Venn diagram.

L2) Negate the following statements in such a way your resulting sentence does not use the word "not".
a) There is a real number whose square is negative.
b) There exists $X\in\Re$ such that $f(x) > 100$.
c) For all $\delta > 0$, there exists $n\in N$ such that $1/n < \in$.

Answers:
a) There is a real number whose square is equal to zero or positive.
b) There exists $X\in\Re$ such that $f(x) \leq 100$.
c) For all $\delta > 0$, there exists $n\in N$ such that $1/n \geq \in$. Or for all $\delta \leq 0$, there exists $n\in N$ such that $1/n < \in$.

(**) In the questions 3, 4 that follow an <u>explicit rule</u> is written, and you are requested to choose <u>the card that corresponds to the rule</u>, from the following four cards written under the rule.

L3) A number <u>or</u> a shape doesn't appear on the card.

A	B	C	D
D / Δ	○ / A	M / L	L / 4

Answer:
A number is represented by p, and shape is represented by q

p	q	$p \cup q$	$-(p \cup q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

T in p means a number appear on the card, and vice versa
T in q means shape appear on the card, and vice versa. The correct answer is c (two letters appear on the card), because a number or a shape doesn't appear on the card, that means F on p and F on q, then - ($p \cup q$) is true statement.

L4) A letter and a number appear on the card.

5	L	○	M
Δ	4	15	□
A	B	C	D

Answer:
A letter is represented by p, and a number is represented by q, and (and means $\cap$).

p	q	$p \cap q$
T	T	T
T	F	F
F	T	F
F	F	F

The correct answer is B, because it contains both letter and number, and it is correct statement $p \cap q$ is true statement if only if p and q are correct.

L5) Mary's sister made these statements. If Vera told the truth, who else must have told the truth?
Lucy" If the rug is in the car, and then it is not in the garage".
Sally:" If the rug is not in the car, then it is in the garage".
Vera:" If the rug is in the garage, then it is in the car".
Cherry:" If the rug is not in the car, then it is not in the garage".

a) Lucy.
b) Sally.
c) Cherry.
d) None need have told the truth.

Correct answer is c, because Vera told the truth" If the rug is in the garage, then it is in the car". This means that the car in the garage and the rug in the car. Cherry is saying the same meaning" If the rug is not in the car, then it is not in the garage". - - (p) -→ p – means not.

Part six: Mathematical proof.

M1) Show that if" n" is divisible by 2, then n^2 is also divisible by 2.

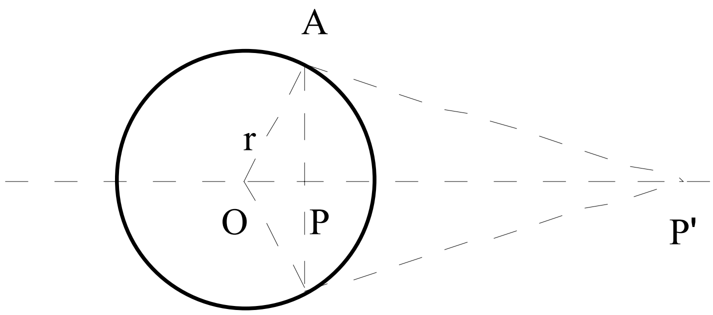
Answer: if n is divisible by 2, then we can express n as n = 2k, where k is integer number, then by squaring the two sides we will find $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, if k is integer, then k^2 is integer and $2k^2$ is integer as well, let $2k^2$ is other integer like m. Consequently, $n^2 = 2(2k^2) = 2m$, then n^2 is also divisible by 2.

M2) Prove that √2 can't be expressed as a fraction (in other words a/b, where a and b are integers and b≠0).

Proof by contradiction: Let √2 = a/b where a and b are both integers, and the largest common factor is equal to 1. If we square both sides then $2 = a^2/b^2 \Rightarrow 2b^2 = a^2$, we know that a^2 must be even (because double any number is even number) and a is even number as well (because if the square of a number is even, then the number itself must be even) . Since a is even it can be written in form a = 2r, where r is integer number, then $2b^2 = (2r)^2 = 4r^2$, divide both sides by 2 to get: $b^2 = 2r^2$, b^2 and b are both even number (the same arguments that a^2 and a are both even numbers. If this is the case, then b can be written as b = 2k, where k is integer number. Then, √2 = a/b = 2r/2k. However, this contradiction the original condition that the largest common factor between a and b is equal to 1. This contradicts our assumption, then √2 is can not be expressed as a/b.

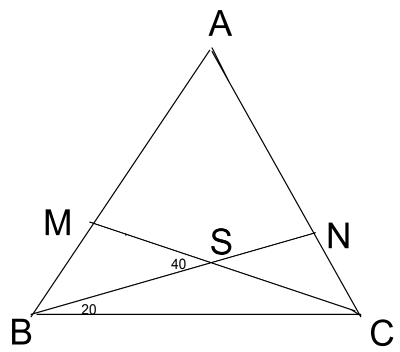
I

M3) On the adjacent shape, If (OṔA) is a right angle, prove that m (OṔ) ×m (OP) =r^2.



Proof: searching for similarity between Δ AṔO and Δ APO, < AOP (common angle), < ṔAO = < APO = 90°, then < AṔO = < PAO. Consequently, Δ AṔO similar to Δ APO, then PO/AO = AO/ṔO = $(AO)^2$ = PO X ṔO→ r^2 = PO X ṔO (where AO = r). Or < AṔO = < PAO, Sin < AṔO = r/OṔ = Sin < PAO = OP/r, then r/ OṔ = OP/r = r^2 = OṔ X OP.

M4) In the ABC the altitudes BN and CM intersect at point S. The measure of ∠ MSB is 40°, and the measure of∠SBC is 20°. Prove of the following statement: "ABC is isosceles".
Give geometric reasons for statement in your proof.



Proof: < MSB = 40°, < NSC = 40° opposite angle to <MSB. < CMB = <BNC = 90° (because altitudes BN and CM). Then, < NBM = < MCN = 90° -40° = 50° (the total of any triangle is equal to 180°), Δ CMB is right triangle in M < MCB = 90° - 70° = 20° (because Δ CMB is right triangle and < MBC = 20° + 50°, then < MBS + < SBC = < NCS + < SCB = 50° + 20° = 70°, then < B = < C. Consequently, Δ ABC is an isosceles triangle. . Other methods you can see appendix 5.2 student interviews.

M5) Rule” If the lengths of the sides of triangle are 3, 4, 5, then the triangle is right “from that we deduce that”.
a) The ratio between lengths of sides of every right triangle is 3:4:5.
b) The ratio between the lengths of the sides of some right triangle is 3:4:5.
c) Some of triangles that have the ratio between the lengths of their sides as 3:4:5 are not right triangles.
d) There are triangles that are not right triangles and the ratio between the sides of their side’s 3:4:5.

Answer: the choice (a) is incorrect, because if the lengths of the sides of triangle are 5, 12, 13, then the triangle is right, but the ratio between lengths of sides is not as 3:4:5. However, the choices (c) and (d) are incorrect, because all triangles that have the ratio between the lengths of their sides as 3:4:5 are right triangles. Then the correct answer is (b)

Appendix 3.2 Examples of Rubrics for Extended-Response items, Multiple Choice items and Mathematics Achievement Score

Extended –Response items:

Induction item 3: 3 point: the number grains of rice in each square as double in previous square 1, 2, 4, 8 and so on the first, second, third, fourth and so on respectively, then the pattern here as 2^0, 2^1, 2^2, 2^3 and so on, then the number grains of rice in 64^{th} square is 2^{63}. However, the total grains of rice up to first square, second, third, fourth and so on = 1, 3, 7, 15, and so on respectively, then the pattern is as 2^1-1, 2^2-1, 2^3-1, 2^4-1 up to first, second, third, fourth square and so on. Consequently, the total grains of rice up to 64^{th} or on the whole chessboard is 2^{64}-. Or the student knew the relationship between total grains of rice up to certain square and the number grains of rice in the next square and showing mathematically procedures.

2.5 if the student answered the number grains of rice in each square correctly and on the 64^{th} square and was able to find the number of grains rice in any square in general as described in 3 point responses above. And also the student answered correctly the total of grains rice up first, second, third, fourth square, and was able to find the pattern of the total grains of rice in general without adequate work, if the student found the pattern of total grains of rice up to n square 2^n rather than 2^n-1. Or if the student knew the relationship between total grains of rice up to certain square and the number grains of rice in the following square without showing correctly mathematically procedures.

2 if the student answered the number grains of rice in each square correctly, and was able to find the number of grains rice in any square in general and based on the number grains or rice on the 64^{th} square as described in 3 point responses above. And if the student was only to find the total of grains of rice up to certain squares and unable to find the pattern for the total or in the whole chessboard. Or if wrote the total of grains of rice up to 64^{th} square = $2^0 + 2^1 + 2^2 + --- + 2^{63}$, without the final answer.

1.5: if the student answer only the number grains of rice in each square correctly and on the 64^{th} square and any square in general as described in 3 point responses above.

1 if the student wrote the number grains of rice in each square is double in previous square and use the traditional method to do that 1, 2, 3, 4, 8, and so on.
.5 if the student wrote the number grains of rice in each square is double in each time and stop here.

0 no answer, or the number grains of rice in the first, second, third, fourth, and so on as 1, 2, 3, 4, and so on, then the number grains of rice in the 64^{th} square is 64 and the total is 1 + 2 + 3 +4 +---+ 64 = 2080.

Multiple Choice Items:

Deduction item 1:

3 point Since (XY) =1 and X > 0, then Y must be > 0. Consequently, the alternative (a) is incorrect, because Y is negative < 0.

The alternatives (b) and (c) are also incorrect, because if X > 1 and Y > 1, then XY > 1 ≠ 1, and if X < 1 and Y < 1, then XY < 1 ≠ 1. For Example, X > 1 = 2 and Y < 1 = 3, then XY = 6 > 1, and the same when both X and Y less than 1, the outcome will be less than 1. The alternative (d) is incorrect, because if we suppose X is 2 then Y must be ½ to got 1. So suppose X is 3 (X increase from 2 to 3), then Y must be 1/3 to got 1) in this case we found X increase, whereas Y decrease. Consequently, the correct answer is (e) if X increase, Y will be decreased. We can find the correct answer by drawing XY = 1 on the X, Y plane, or Y = 1/X and this is inverse relationship. Consequently, the correct answer is (e) if x increase, then y decrease.

2.5 Since (XY) =1 and X > 0, then Y must be > 0. Consequently, the alternative (a) is incorrect, because Y is negative < 0. The alternatives (b) and (c) are also incorrect, because if X > 1 and Y > 1, then XY > 1 ≠ 1, and if X < 1 and Y < 1, then XY < 1 ≠ 1. For Example, X > 1 = 2 and Y < 1 = 3, then XY = 6 > 1, and the same when both X and Y less than 1, the outcome will be less than 1. The alternative (d) is incorrect, because the correct answer will be (e) such as 2 (1/2) = 1 and 3 (1/3) = 1.

2 the correct answer is (e), because if x increase, then y will decrease. Or (XY) =1 and X > 0, then Y must be > 0. Consequently, the alternative (a) is incorrect, because Y is negative < 0. The alternatives (b) and (c) are also incorrect. The alternative (d) is incorrect, because the correct answer will be (e) for example 3 (/3) = 1.

1.5 giving the correct answer is (e), without given any explanation or justification.

1 The correct answer is (e), because the relationship between X and Y is direct. Or the student said the choice (a) is incorrect, because y is negative, and it should be positive. The alternative (b) is incorrect also, because for any two numbers greater than 1, the outcome will be greater than 1, and stop here. Or the correct answer is (d) if x increase, then y will increase. For example, 2 (1/2) = 1 and 3(1/3) = 1. X increase from 2 to 3 and Y increase from ½ to 1/3.

.5 For example, the student said the alternative (a) and (b) are incorrect and stop here, without any explanation or justification.

0 no answer, or incorrect choice without any Justification such that the correct answer is (a).

Mathematics Achievement Score

Write this summation (1) (2)/2 + (2) (3)/24 + (3) (4) /40320 as a ∑.

A full point response (4point): there are three terms, then the summation will be started from 1 to 3. The terms (1) (2), (2) (3), (3) (4) the first number in first, second, and third terms are 1, 2, 3 respectively and the summation started from 1 to 3, then in general, the first number in any term as (r). However, the second number in first, second, third terms are 2, 3, and 4 respectively and the summation started from 1 to three, then in general, the first number in any term as (r + 1). For numbers 2, 24, 40320 these factorial numbers for 2, 4, 8 numbers, we have to find the pattern between these numbers and 1, 2, 3 such as $2 = 2^1$, $4 = 2^2$ and $8 = 2^3$.

Consequently, $\sum_{r=1}^{3} (r)\,(r+1) / (2^r)!$ or $\sum_{r=2}^{4} (r-1)\,(r) / (2^{r-1})!$

(These samples of student responses).

75 % point response (3 point): If the student was able to know the summation started from 1 to 3 and find (r) (r+1) as described above. In addition, if he was also able to find these numbers 2, 4, 40320 as factorial to 2! 4! 8! respectively with writing 2, 4, and 8 as 2r r= 1 to 3, for example,

$$\sum_{r=1}^{3} (r)\,(r+1) / (2r)!$$

66% point response (2.5 point): If the student was able to know the summation started from 1 to 3 and find (r) (r+1) as described above. If the student knew only these numbers 2, 24, 40320 are factorial numbers. Or (from student answer)

$\sum_{r=1}^{3} (r)\,(r+1)$ and 2 = 1x2, 24 = 1x2x3x4, 40320 = 1x2x3x4x5x6x7x8.

A half point response 50% (2 point): If the student was able to know the summation started from 1 to 3 and find (r) (r+1) as described above. Or find the correct answer

$\sum_{r=1}^{3} (r)\,(r+1) / (2^r)!$ without showing any mathematically sound procedures.

33% point response (approximately 1.5 point): If the student was able to know the summation started from 1 to 3 and find (r) or (r+1) as described above (some samples of student responses)

$\sum_{r=1}^{3} (r-1)\,(r) / r^2$ or $\sum (r)\,(r+1)$ without writing r started from --- to----.

25% point response (1 point): If the student was able to know the summation started from 1 to three, because there are only three terms or if the student was able to express for the first or the second number r or r + 1 or if the student was able to know 2, 4, 40320 as factorial numbers.

0 point response: no answer, or completely incorrect, or irrelevant such as ∑ (n -1) /2n or 2/2 + 6/24 + 12/40320 = 1 + ¼ + 1/3360 (these some samples of student responses).

Appendix 3.3

Mathematical Thinking Test Reliability

The reliability for the mathematical thinking test calculated if each individual item deleted. The reliability of each of mathematical thinking scales also calculated separately. All information about reliability of mathematical thinking test in Table A3.1 below.

TABLE A3.1 MATHEMATICAL THINKING RELIABILITY IF EACH INDIVIDUAL ITEM DELETED

Mathematical thinking scale	Item	Reliability	Scale reliability if item deleted
Generalization		0.564	
	G1		0.824
	G2		0.822
	G3		0.825
	G4		0.824
	G5		0.824
Induction		0.607	
	I1		0.827
	I2		0.828
	I3		0.820
	I4		0.824
	I5		0.830
Deduction		0.578	
	D1		0.830
	D2		0.828
	D3		0.824
	D4		0.829
	D5		0.823
Use of Symbols		0.644	
	S1		0.823
	S2		0.820
	S3		0.821
	S4		0.827
	S5		0.820
Logical thinking		0.649	
	L1		0.820
	L2		0.822
	L3		0.826
	L4		0.826
Mathematical proof		0.603	
	M1		0.823
	M2		0.823
	M3		0.824
	M4		0.821
	M5		0.831

Appendix 3.4

Mathematical Thinking Test (Facility and Discrimination for Each Item)

The facility and discrimination for each individual item for the mathematical thinking test in Table A3.2 below.

TABLE A3.2 MATHEMATICAL THINKING DIFFICULTY AND DISCRIMINATION FOR EACH INDIVIDUAL ITEM

Item	Facility	Discrimination
G1	.87	.41
G2	.37	.48
G3	.40	.39
G4	.37	.42
G5	.63	.44
I1	.60	.38
I2	.73	.33
I3	.40	.54
I4	.57	.44
I5	.53	.31
D1	.53	.31
D2	.40	.36
D3	.63	.43
D4	.47	.33
D5	.43	.46
S1	.50	.46
S2	.33	.54
S3	.67	.51
S4	.43	.39
S5	.53	54
L1	.30	.55
L2	.40	.48
L3	.90	.37
L4	.83	.38
L5	.63	.12
M1	.30	.47
M2	.50	.45
M3	.07	.47
M4	.37	.51
M5	.37	.29

Appendix 3.5

Factor Analysis for Each Scale of Mathematical Thinking

All items in each separate aspect of mathematical thinking significantly loaded on that factor. The factor loadings for each item on the factor are shown in Table A3.3.

TABLE A3.3 FACTOR ANALYSIS FOR EACH MATHEMATICAL THINKING ASPECT
Component Matrix (a)

GENERALIZATION SCALE

Generalization	Component 1
G5	.686
G1	.676
G3	.609
G2	.599
G4	.589

INDUCTION SCALE

Induction	Component 1
I3	.706
I4	.704
I1	.651
I5	.616
I2	.604

DEDUCTION SCALE

Deduction	Component 1
D3	.705
D5	.640
D4	.608
D1	.594
D2	.575

USE OF SYMBOLS SCALE

Use of Symbols	Component 1
S5	.748
S1	.724
S3	.691
S2	.634
S4	.559

LOGICAL THINKING SCALE

Logical thinking	Component 1
L4	.760
L3	.757
L2	.721
L1	.670

MATHEMATICAL PROOF SCALE

Mathematical proof	Component 1
M4	.711
M1	.667
M2	.646
M3	.625
M5	.552

Appendix 3.6

Mathematics Achievement Test (Facility and Discrimination for Each Item)

The facility and discrimination for each individual item for the mathematics achievement test in Table A3.4 below.

TABLE A3.4 MATHEMATICS ACHIEVEMENT TEST DIFFICULTY AND DISCRIMINATION FOR EACH INDIVIDUAL ITEM

Item	Facility	Discrimination
Q1a	.57	.52
Q1b1	.90	.40
Q1b2	.73	.50
Q1b3	.40	.51
Q2a1	.65	.57
Q2a2	.15	.48
Q2b	.68	.65
Q2c	.40	.60
Q2d	.50	.59
Q3a	.80	.52
Q3b1	.30	.51
Q3b2	.20	.37
Q3c	.48	.63
Q4a	.50	.54
Q4b1	.30	.58
Q4b2	.55	.45
Q4c	.53	.54

Appendix Four

This appendix will relate to chapter 5 “Mathematical Thinking and Achievement

(i) Appendix 4.1: School Mean Scores

(ii) Appendix 4.2: Location Mean Scores

Appendix 4.1

School Mean Scores

The Mean Scores for Individual Schools are Shown Below for each of the aspects of mathematical thinking and mathematics achievement. In case schools with significantly different mean scores are included.

Generalization
For generalization school No 1 was different from school No 11 (Means =10.7, 5.4 respectively, P<. 05).

TABLE A4.1a MEAN SCORES FOR INDIVIDUAL SCHOOLS IN GENERALIZATION

School No	N	Subset for alpha=. 05	
		1	2
11	29	5.4	
7	32	6.7	6.7
18	24	7.0	7.0
16	43	7.0	7.0
12	32	7.1	7.1
3	24	7.3	7.3
2	31	7.3	7.3
20	29	7.5	7.5
19	21	7.8	7.8
17	23	7.9	7.9
9	24	8.1	8.1
4	36	8.3	8.3
15	21	8.3	8.3
8	30	8.4	8.4
14	40	8.8	8.8
5	19	9.3	9.3
10	35	9.5	9.5
13	15	9.6	9.6
6	28	10.0	10.0
1	24		10.7
Prob		.191	.480

Induction
For Induction school No 1 was different from school No 10 (Means =11.6, 6.1 respectively, P<. 05).

TABLE A4.1b MEAN SCORES FOR INDIVIDUAL SCHOOLS IN INDUCTION

School No	N	Subset for alpha=.05	
		1	2
10	35	6.1	
7	32	6.4	6.4
8	30	6.4	6.4
11	29	6.6	6.6
20	29	7.3	7.3
16	43	7.5	7.5
2	31	7.5	7.5
14	40	8.5	8.5
19	21	8.9	8.9
13	15	8.9	8.9
18	24	9.1	9.1
5	19	9.3	9.3
17	23	9.4	9.4
12	32	9.6	9.6
15	21	9.7	9.7
3	24	9.8	9.8
6	28	9.8	9.8
4	36	9.8	9.8
9	24	11.1	11.1
1	24		11.6
Prob		.087	.064

Logical thinking

In Logical thinking the school No 5, 13, 19, 1, and 9 were different from school No 2 (Mean= 9.5, 9.4, 9.0, 9.0, 8.9, and 4.8, P<. 05).

TABLE A4.1c MEAN SCORES FOR INDIVIDUAL SCHOOLS IN LOGICAL THINKING

School No	N	Schools mean scores		
		1	2	3
2	31	4.8		
11	29	5.5	5.5	
8	30	5.8	5.8	5.8
16	43	6.5	6.5	6.5
12	32	6.5	6.5	6.5
14	40	6.8	6.8	6.8
10	35	7.1	7.1	7.1
4	36	7.3	7.3	7.3
7	32	7.4	7.4	7.4
6	28	7.5	7.5	7.5
18	24	7.5	7.5	7.5
3	24	7.8	7.8	7.8
20	29	7.9	7.9	7.9
17	23	8.2	8.2	8.2
15	21	8.6	8.6	8.6
9	24		8.9	8.9
1	24		9.0	9.0
19	21		9.0	9.0
13	15		9.4	9.4
5	19			9.5
Prob		.054	.055	.088

Mathematical Thinking (total)

In mathematical thinking (total) there is no significant difference with schools as show on Table A4.1d.

TABLE A4.1d MEAN SCORES FOR INDIVIDUAL SCHOOLS IN MATHEMATICAL THINKING (TOTAL)

School No	N	Schools mean scores 1
8	30	35.4
11	29	35.5
2	31	37.8
7	32	38.7
16	43	39.7
17	23	41.3
10	35	42.0
12	32	42.4
14	40	42.5
18	24	43.0
15	21	43.5
20	29	44.2
3	24	44.9
4	36	47.1
6	28	48.4
19	21	49.5
9	24	50.3
13	15	52.6
1	24	54.2
5	19	54.7
Prob		.132

Mathematics Achievement

In mathematics achievement schools No 1, 13,20,5 were different from school No 11(Means=33.4, 32.8, 32.3, 32.1, 19.3 respectively).

TABLE A4.1e MEAN SCORES FOR INDIVIDUAL SCHOOLS IN MATHEMATICS ACHIEVEMENT

School No	N	Schools mean scores	
		1	2
11	28	19.3	
12	30	22.0	22.0
16	39	22.7	22.7
8	30	23.4	23.4
18	24	24.9	24.9
7	29	25.8	25.8
15	24	26.0	26.0
4	37	26.7	26.7
10	33	27.2	27.2
2	29	27.3	27.3
17	23	28.1	28.1
9	24	28.1	28.1
14	38	28.3	28.3
6	28	28.5	28.5
3	22	29.3	29.3
19	22	30.8	30.8
5	18		32.1
20	25		32.3
13	16		32.8
1	24		33.4
Prob		.151	.154

Appendix 4.2

Location Mean Scores

The Mean overall Results for each Individual Location is shown below for each aspect of mathematical thinking and mathematics achievement. In each case locations with significantly different mean scores are included.

Generalization

For Generalization Suburban Students were different from Urban students (Means=8.7, 8.1, 7.7 respectively, P<.05). All Results for location with Generalization display on Table A4.2a

TABLE A4.2a MEAN SCORES FOR SCHOOL LOCATIONS IN GENERALIZATION

Location Urban(1) sub(2)Rural(3)	N	Subset for alpha = .05	
		1	2
1	244	7.7	
3	218	8.1	8.1
2	98		8.7
Prob		.585	.271

Induction

In Induction Suburban and Rural Students were different from Urban students (Means=9.8, 9.0, 7.5 respectively, P<.05). All Results for location with Induction display on Table A4.2b.

TABLE A4.2b MEAN SCORES FOR SCHOOL LOCATIONS IN INDUCTION

Urban(1) sub(2)Rural(3)	N	Location mean scores	
		1	2
1	244	7.5	
3	218		9.0
2	98		9.8
Prob		1.000	.191

Use of Symbols

In Use of Symbols Suburban students were different from Urban and Rural Students (Means= 8.6, 7.3, 7.1 respectively, P<.05). All results for locations with Use of Symbols display on Table A4.2c.

Table A4.2c MEAN SCORES FOR SCHOOL LOCATIONS IN USE OF SYMBOLS

Urban(1) sub(2)Rural(3)	N	Location mean scores	
		1	2
1	244	7.1	
3	218	7.3	
2	98		8.6
Prob		.920	1.000

Logical thinking

In Logical thinking Suburban students were different from Urban Students (Means= 7.7, 7.6, 6.9 respectively, P<.05). All results for locations with Logical thinking display on Table A4.2d.

TABLE A4.2d MEAN SCORES FOR SCHOOL LOCATIONS IN LOGICAL THINKING

Urban(1) sub(2)Rural(3)	N	Location mean scores	
		1	2
1	244	6.9	
3	218	7.6	7.6
2	98		7.7
Prob		.073	.918

Mathematical Thinking (total)

In mathematical thinking (total) Suburban students were different from Rural and Urban Students (Means= 48.2, 43.9, 41.5 respectively, P<.05). All results for location with mathematical thinking (total) display on Table A4.2e.

TABLE A4.2e MEAN SCORES FOR SCHOOL LOCATIONS IN MATHEMATICAL THINKING (TOTAL)

Urban(1) sub(2)Rural(3)	N	Location mean scores	
		1	2
1	244	41.5	
3	218	43.9	
2	98		48.2
Prob		.307	1.000

Mathematics achievement

In mathematics achievement Suburban students were different from Rural and Urban students (Means= 30.0, 26.8, 26.0 respectively). All Results for location with scale MA display on Table A4.2f.

TABLE A4.2f MEAN SCORES FOR SCHOOL LOCATIONS IN MATHEMATICS ACHIEVEMENT

Urban(1) sub(2)Rural(3)	N	Location mean scores	
		1	2
1	229	26.0	
3	219	26.8	
2	95		30
Prob		.651	1.000

Appendix Five

This appendix will relate to Chapter 6 “The Teacher and Student Interviews”

(i) Appendix 5.1 Semi Structured Interview Questions and Transcripts of the Teacher Interviews for each Individual Interview Conducted

(i) Appendix 5.2 Transcripts of each Student Group Interview conducted

Appendix 5.1

Semi Structured Interview Questions and Transcripts of the Teacher Interviews for Each Individual Interview Conducted

School No.1

Teaching of mathematical thinking in Jordanian schools

1) In your opinion, what does mathematical thinking mean? Do you think mathematical thinking is restricted to the domain of mathematical computation and formula (e.g. it is restricted to the use of numbers and formulas to find answers to specific problems) or can it be used like a game to explore mathematical processes? Do you think mathematical thinking is "effective thinking" or the basis of mathematics, and contributes to the development of the student through the study of mathematics, in particular, and other sciences in general?

> Mathematical thinking means: The ability to make inferences, connections and solve problems. It is not restricted to the use of numbers and formulas to find answers to specific problems, and looks like a game. It is development by the student through his study of mathematics in particular and other science in general.

(2) What are the aspects of mathematical thinking? For example Generalization is one of the aspects; do you know what the others may be?

- Generalization.
- Induction.
- Deduction.
- Symbols.
- Logic.
- Plane Geometry.

3) How important is each of the aspects in teaching mathematics? Rank these aspects according to level of importance.

4) Why do you consider the -----aspect the most important aspect? And how useful is it for the students to improve their progress in mathematical thinking? It is considered the Expression through symbols aspect the most important one because it contains the basis for the other aspects, when the students are able to express through symbols as the first step, then the other aspects become easier such as generalization, Induction, and mathematical proof.

5) What is the most difficult aspect for the students, and what is the easiest, and why? Rank these aspects according to their level of difficulty?
6) How many weeks or hours (lessons) do you spend to teach them?

TABLE A5.1a TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.1

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	4	3	10%
Induction	1	4	10%
Deduction	5	5	---
Use of Symbols	6	2	15%
Logical thinking	3	1	---
Mathematical proof	2	6	15%

7) What are the most effective strategies you use when you teach mathematical thinking?
-Search for pattern.
-Try a simple problem.

School No. 2

1) Mathematical thinking means: effective thinking and development by studying to lead the student to solve problems in mathematics and other sciences.

2) Generalization.
Specialization.
Problem Solving.
Applications on generalizations.

3) Generalization is considered the most important aspect, because arriving at general formulas from specific cases requires high level of thinking, so if the student is good on generalization, that means he will achievement highly in mathematics.

TABLE A5.1b TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.2

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	6	4	15%
Induction	3	3	15%
Deduction	1	2	10%
Use of Symbols	2	1	10%
Logical thinking	5	5	20%
Mathematical proof	4	6	20%

7) Look for a pattern.
Use a model
Draw a picture.

School No.3

1) Mathematical thinking means: effective thinking and power of mathematics and other sciences, and development by the students through their study for all types of sciences. This promotes student's ability for Induction and inferences.

2) Understanding the logical basis for mathematical cognition.
Mathematical proof.
Induction.
Deduction.
Generalization.
Using symbols and mathematical expression.

4) It is considered the Logical thinking aspect the most important one because it is the first step for all other aspects of mathematical thinking.

TABLE A5.1c TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.3

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	5	1	10%
Induction	2	5	15%
Deduction	1	4	25%
Use of Symbols	3	3	10%
Logical thinking	6	2	15%
Mathematical proof	4	6	20%

7) Write an Equation
(Generalization) Look for a pattern.
Try a simple problem.
Challenges.

School No.6

1) Mathematical thinking means: It is thinking which improves with practice and reflection. It helps us in understanding the world and ourselves. Some of the problems look like games and challenges.

2) The ability of translation from words to equations.
Mathematical proof.
Induction.
Deduction.
Generalization.
Logic.

4) Mathematical proof is the most important aspect because it is the most difficult aspect that requires connections between theorems, understanding the mathematical concepts, and justifications, so, if the student in particular has a high ability in proofs, that means he will have a high ability in mathematics in general.

TABLE A5.1d TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.6

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	5	1	10%
Induction	2	5	15%
Deduction	1	4	25%
Use of Symbols	3	2	10%
Logical thinking	4	3	15%
Mathematical proof	6	6	20%

7) Draw a picture.
Try and adjust.

School No.7

1) Mathematical thinking means: Logical thinking which depends on mathematical proof and logical thinking, and development of the students by their practice on mathematical problem and problem solving in the other sciences.

2) Generalization & Induction.
Specialization.
Symbols.
Deduction.
Mathematical proof.

4) Symbols is considered the most important aspect, because the use of symbols has a fundamental role in many mathematical areas, such as generalization, logic, algebra, geometry.

TABLE A5.1e TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.7

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	5	5	20%
Induction	1	2	15%
Deduction	2	1	5%
Use of Symbols	6	4	10%
Logical thinking	3	3	10%
Mathematical proof	4	6	20%

7) Search for patterns.
Challenges.

School No.8

1) Mathematical thinking means: There are two types of mathematical thinking; analytic thinking and anticipative thinking. Anticipative thinking involves direct experience and for the student and their treatment with things to build self-confidence and willingness. Analytical thinking is conclusion thinking. And mathematical thinking does not restrict the domain of mathematical computation and formula, and it is important for applied sciences.

2) Generalization.
Induction.
Deduction.
Mathematical proof.
Challenges.

3) Mathematical proof is considered the most important aspect, because it plays a fundamental role in mathematics. Proofs are prevalent in the topic of geometry, and learning geometry has an important role in understanding the environment and the world.

TABLE A5.1f TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.8

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time (Per week)
Generalization	4	3	1 hour
Induction	5	2	1 hour
Deduction	3	1	½ hour
Use of Symbols	1	4	½ hour
Logical thinking	2	5	1 hour
Mathematical proof	6	6	2 hours

7) Analyse the figure.
Translations from words to the symbols.
Making a sketch.

School No.9

1) Mathematical thinking means: Using laws and generalizations and theories to solving certain problems. It is not restricted to the use of numbers and formulas to find answers. Some problems need connections and reasoning, and sometimes look like games, especially in series and sequences, geometry, and algebra.

2) Generalizations.

Logical analysis
Mathematical proof.
Using the pattern.

4) It is considered the Expression through symbols, because after Expression through symbols for a certain problem, the student will use theories and generalizations and definitions. And symbols are the basis for mathematical thinking to use the theory and law.

TABLE A5.1J TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO 9

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time (Per week)
Generalization	1	1	1 hour
Induction	2	5	2 hour
Deduction	3	4	1 hour
Use of Symbols	6	3	2 hour
Logical thinking	5	2	Each mathematical topic needs logical thinking
Mathematical proof	4	6	The theories in maths curriculum are limited, and each theory needs 1 lesson

7) Logical analysis.
Use the figures.
Use the patterns.

School No.11

1) Mathematical thinking means: Using mathematical concepts in problem solving and mathematical applications.

2) Induction.
Investigation.
Proof.
Find the optimal solution.

4) The Induction aspect was considered the most important aspect, because with this method students can arrive at conclusions through observing specific cases.

TABLE A5.1h TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.11

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	4	1	10%
Induction	6	2	20%
Deduction	3	3	40%
Use of Symbols	5	5	15%
Logical thinking	2	4	10%
Mathematical proof	1	6	5%

7) Induction and deduction

School No.13

1) Mathematical thinking means: Power of mathematics, and development of the students through their study of mathematics in particular, and other sciences in general.

2) Reasoning.
Mathematical proof.
Translation from mathematical sentences into equations.
Ability to apply.
Ability to check the answer.
Ability to analyse and discuss.

4) Mathematical proof was the most important aspect, because it makes connections among mathematical ideas. If the student is skilled at mathematical proof that means they are skilled in mathematical achievement.

TABLE A5.1i TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.13

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	3	1	15%
Induction	2	5	15%
Deduction	1	3	5
Use of Symbols	5	4	15%
Logical thinking	4	2	10%
Mathematical proof	6	6	15%

7) Write an equation.
Connections among mathematical ideas.
Make an organized list.

School No.14

1) Mathematical thinking means: The best method, which leads the students to solving problems, using their knowledge and strategies.

2) Generalization.
Mathematical proof.
Logic.
Ability to express by using symbols.
Deduction.
Specializing.

3) Deduction aspect was the most important aspect, because it is the first step in the transformation to abstract thinking, and it's the basic point.

TABLE A5.1j TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.14

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	4	1	15%
Induction	3	2	15%
Deduction	6	5	10%
Use of Symbols	5	4	20%
Logical thinking	2	3	---
Mathematical proof	1	6	20%

7) Optimization.
Inferences from premises.
Looking for a pattern.

School No.16

1) Mathematical thinking means: The thinking, which is development by the student through his study of mathematics in particular and other sciences in general.

2) Generalization.
Induction.
Deduction.
Symbols.
Logic.
Mathematical proof.

4) Generalization is considered the most important aspect, because it leads the student to arrive at mathematical laws and certain rules that are common in mathematics.

TABLE A5.1k TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.16

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	6	1	20%
Induction	3	5	15%
Deduction	2	2	15%
Use of Symbols	1	4	15%
Logical thinking	4	3	20%
Mathematical proof	5	6	15%

7) Ability to discuss results.
Sketch pictures, figures.
Looking for a pattern.

School No.18

1) Mathematical thinking means: The ability to build or seriatim ideas to arrive at the decision to solve problems or mathematical idea.

2) Mathematical inferences.
Generalization.
Ability to improve mathematical concept.
Mathematical interpretation and accept or reject the solution.
Symbols.
Logic.
Mathematical proof.

4) The Symbols aspect is considered to be the most important aspect, because it is the basis of mathematics, and it's the first step to solve many practical problems such as Volume, area, and applications on maximum and minimum values.

TABLE A5.1l TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.18

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	2	1	15%
Induction	3	2	15%
Deduction	1	3	15%
Use of Symbols	6	5	10%
Logical thinking	4	4	25%
Mathematical proof	5	6	20%

7) Ability to discuss results.

Sketch pictures, figures.
Generalization.

School No.20

1) Mathematical thinking means: The thinking, which is based on problem solving using inferences and proof. It is effective thinking, and it is the development of the students through their study of mathematics in particular and other sciences in general.

2) Generalization.
Specialization.
Inductive thinking
Deductive thinking.
Symbols.
Logical analysis.
Mathematical proof.

4) Generalization aspects is considered the most important one, because it is important in arriving at a generalization, because it develops inductive thinking skills.

TABLE A5.1m TEACHER'S RESPONSE IN QUESTIONS 3, 5 AND 6 IN SCHOOL NO.20

Type of mathematical thinking	Level of importance	Level of difficulty	Spend time
Generalization	6	1	15%
Induction	3	4	15%
Deduction	2	3	10%
Use of Symbols	4	2	20%
Logical thinking	1	5	10%
Mathematical proof	5	6	15%

7) Logical analysis.
Prove the results.
Looking for a patterns.

Appendix 5.2

Transcripts of Each Student Group Interview Conducted

School No.1
Question 1
I: What is the known?
S: Two sides. The first one presents even numbers, whereas the second side presents the sum of two numbers.

I: What is the unknown?
S: What is the nature of the two numbers which is given even numbers?

I: What is the condition?
S: The given number is an even number.

I: Is the known sufficient to determine the unknown? Or insufficient?
S: It is sufficient, because the number 14 repeated twice gives us sufficient information.

I: Have you seen it before?
S: No.

I: Have you seen the same problem in a slightly different form?
S: (Just one student answers) Yes and the problem was the sum of two odd numbers is an even number.

I: Do you know a theorem that could be useful? Or generalization?
S: No.

I: What is the initial idea for this question? When you read this question for the first time?
S: Any even number can be expressed as the sum two odd numbers.

I: If you look at the numbers on the right side, what are those numbers?
S: Even numbers.

I: If you look at the numbers on the left side, what are those numbers?
S: Odd numbers.

I: If so, why we did not write 12 = 3+ 9, although 3 and 9 are odd numbers?
S: (The same student answers) I have no idea.

I: Do you know what those numbers are?
S: Different student answers the numbers on the left side are prime numbers?

I: If so, why we did not write 4 = 2 +2, although 2 is a prime number.
S: Maybe, those numbers are odd prime numbers?

I: That is great.
I: What is your generalization now?
S: Any even number can be expressed as the sum of two odd prime numbers.

Question 2
I: What are the data?
S: A chess board, consisting of 64 squares, one grain of rice on the first square, two grains on the second square, four grains on the third square, and so on.

I: What is the unknown?
S: Find how many grains of rice are on the 64^{th} square. And how many grains of rice did the mathematician ask in total.

I: Are the data sufficient to determine the unknown?
S1: Yes
S2: No, I think if the question gave us how many grains of rices on the fourth square, to make it easy to find the pattern.

I: Have you seen it before?
S: No.

I: Have you seen the same problem in a slightly different form?
S: (One student answers) Yes.

I: What was the problem?
S: A chessboard of 64 squares, one grain of wheat on the first square , two grains on the second square, four grains on the third square, and so on.

I: Have you answered that question?
S: No.

I: What do you expect for the answer?
S: Huge number.

I: We need to simplify this question by considering that chessboard consisting of 4 squares. How many grains on each square? What is the total? Draw a figure.
S: One grain on the first square, two grains on the second, four grains on the third, and eight grains on the fourth.

I: Now, Do you know what the pattern is?
S: We can write these numbers 1, 2, 4, 8 as number 2 to the power $1 = 2^0$, $2=2^1$, $4=2^2$, $8=2^3$.

I: Based on this pattern, how many grains of rice on the 5^{th} square, 20^{th} square, and finally 64^{th} square.
S: There are 2^4 grains of rice on the 5^{th} square.
There are 2^{19} grains of rice on the 20^{th} square.
There are 2^{63} grains of rice on the 64^{th} square.

I: Do you have another way?
S: (One student speaks) By using double the amount each time, because two is the double of one, four is the double of 2, and so on.

I: Is this logical way? And what was your answer?
S: I have no idea, and I did not complete my answer, because it was long way to keep doubling.

I: Return to the chessboard consisting of 64 squares. What the total of rice up to 1, 2, 3, 4 squares?
S: The total of rice on the first square = 1.
The total of rice up to the second square = 3.
The total of rice up to the third square = 7.
The total of rice up to the fourth square = 15.

I: Could you please rewrite these numbers using 2 to the power the number of the square?
S: One student was able to rewrite

I: Now, what is the total of rice in the original question?
S: $2^{64}-1$.

I: Could you please explain your answer?
S: No answer.

I: Do you have any different answer?
S: Yes.

I: What is this way?
S: $2^0 + 2^1 + 2^2 + --- + 2^{63}$.

I: Do you know how much this amount comes to?
S: No answer?

I: If we suppose each 50 grain of rice = 1 gram, in your opinion how much rice do we need to answer this question?
S: A huge number.

I: In your opinion, do you think we can solve this problem practically? And why?
S: NO, because we need huge amount.

I: Do you have any idea of the amount?
S: (One student speaks) I think we need the world production of rice for ten years to solve this problem.

After that the interviewer gave the students the exact amount we need to solve this problem.

Question 3
I: What is the known?
S: Two numbers, multiplication of them is equal to 1.

I: What is the condition?
S: The first number is greater than 1.

I: What is the unknown?
S: What is the relationship between these two numbers (choose the correct answer.

I: Have you seen such a problem it before?
S: (answer as a group) No.

I: Have you seen the same problem in a slightly different form?
S: (answer as a group) No.

I: What is the condition for the second number (y)? and why?
S :(one student answer) y> 0, because multiplication of them equals 1, and 1 is a positive number, and because x>0, then y must be a positive number, because the two numbers must be the same sign to give a positive number.

I: are you agreeing with your friend answer?
S: Of course, yes.

I: What does that mean?
S: The choice (a) is incorrect, because in this choice Y is negative.

I: Is choice (b) correct? and why?
S1: No, because if x=2>1, and y =3>1, then the income will be greater than 1.
S2: In general, if any two numbers are greater than 1, the multiplication of them must be greater than 1.

I: Is choice (c) correct? and why?
S: (the highest achieving student answer): Yes, because for example, if x=0.2, and y=0.5, then xy=1.
Other students said: No

I: (for the highest achieving student) Could you please recalculate the multiplication of .02 ×.05 again?
S: 2/10 × 5/10 = 10/100 = 1/10, Oh I am sorry.

I: Is there any explanation for this choice?
S: In general, for any two numbers less than 1, the multiplication must be less than 1.

I: Now, we have just two choices, which choice do you think is correct? and why?

S: The correct answer is e, because if x=5, then y=1/5, and if x=2, then y=1/2, we observed that x is decreasing 5 to 2, and Y is increasing 1/5 to ½.

I: Have you remembered any relationship or rule that indicates the relationship between the numbers for their multiplication to equal 1.
S: (one student answer) the number and its conversion.

I: Then, what is the relationship between these two numbers?
S: Inverse relationship.

I: Do you have any answer?
S: (the highest achieving student answer) I think the answer (e) is incorrect because why do we match 3 with 1/3. If we match 3 with ½, then the answer will still not equal 1.

I: Again what is the multiplication of these numbers from the known?
S: Equals 1.

I: That is great, then for example, 2 must be matched with which number?
S: ½.
I: That is it.

Question 4
I: What is the known?
S: There are three shapes, each shape involves numbers of areas with the length of its dimensions.

I: What is the unknown?
S: Express the following shapes by numbers (symbols).

I: Have you seen such a problem before?
S: No.

I: Have you seen the same problem in a slightly different form?
S: (answer as group) Yes for previous classes we faced some problems from the mathematics curricula which requested us to find the whole areas for the shapes.

I: What is representing the first shape?
S: Square and its area is equal to 11^2.

I: What is representing the second and the third shapes?
S: Squares and their areas are equal to 12^2 and 13^2.

I: Express these shapes using the summation of the areas?
S: $11^2 = 10^2 + 2\times1\times10 + 1^2$
$12^2 = 10^2 + 2\times2\times10 + 2^2$
$13^2 = 10^2 + 2\times3\times10 + 3^2$

I: Then, could you please express for the shape area that its side length is n.

S: (one student was able to answer) $n^2 = 10^2$ (constant) + 2 (constant) × (n-10) + $(n-10)^2$

I: Could you please check if your answer is correct?
S: (the same student answer), let n = 13, then $13^2 = 10^2 + 2\times3 + 3^2 = 100 + 6 + 9 \neq 169$ Oh I am sorry, I missed to multiplication of the second term by10.

I: (other students questioned about the value of n^2) What is your answer?
S: $n^2 = 10^2 + 2\times10\times$ (constant) × something+ (something) ^{2}, and they were unable to find what is the value of the something.

I: What is the variable?
S: 1, 2, 3 when the areas were 11^2, 12^2, and 13^2, respectively.

I: Then, if the area was n^2, then what will the variable be?
S :(different student answer) n-10.

I: Then, Could you please express the shape area if its side length is n.
S: Of course, $n^2 = 10^2 + 2\times (n-10) \times 10 + (n-10)^2$.

I: Could you please check your answer?
S: let n=12, then $12^2 = 144 = 10^2 + 2\times2\times10 + 2^2 = 144$ (it is correct).

I: Is checking the answer to see whether it is correct or not an important step in mathematical problem solving?
S: Yes, but there is a problem that in most cases particularly in time limited tests, there is not enough time to check our answers.

Question 5
I: What is the known?
S: ABC triangle, the altitude BN and CM intersect at point S, and
$\angle$ MSB= (40) °, $\angle$ NBC = (20) °.

I: What is the unknown?
S: Prove that ABC is an isosceles triangle.

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.

I: If the $\angle$ NBC is unknown? Is the known still sufficient? And why?
S: Yes, because we know $\angle$ MSB, and if this angle is exterior to the Δ BSC, then $\angle$ MSB equals the two opposite interior angles in ΔBSC.

I: Do you know what the relationship between $\angle$ MCB and $\angle$ NBC is?
S: (one student answer) Yes.

I: How?
S: Because the Δ BSC is an isosceles triangle.

I: How do you know that Δ BSC is an isosceles triangle?

S: I have no idea.

I: Then, is the $\angle$ NBC is necessary to prove the unknown?
S: Yes.

I: Have you seen such a problem before?
S: We have seen the same idea, with different angles.

I: What are the required theorems to prove?
S: Isosceles triangle must have equal base angles.
The exterior angle of any triangle is sum of the two interior opposite angles.
Opposite angles are equal.
I: What is the measure of $\angle$ BNC? And why?
S: 90°, (this is another theorem) the altitudes make 90° with the intersection lines.

I: What else?
S: The total of the angles for any triangle is 180°.

I: What is the relationship between $\angle$ MSB and $\angle$BSC?
S: 180° (the measure of the straight angle).

I: Could you please first prove that the ΔBSC is isosceles, giving your geometric reasons.
S: $\angle$ MCB = 20°, because $\angle$ MSB = 40° is the exterior ΔBSC, and this angle is equal to the two opposite interior angles in this triangle, and $\angle$ NBC=20° (known angle), then because the base angles are equal, then Δ BSC is an isosceles triangle.

I: That is great, then what is measurement of $\angle$ NSC and $\angle$ SNC? And why?
S: $\angle$ NSC = 40° (opposite to the $\angle$ MSB), and $\angle$ SCN = 50° (the total of angles = 180° in Δ NSC).

I: What is the $\angle$ ACB?
S: $\angle$ ACB = $\angle$ NCS + $\angle$ SCB = 20° + 50° = 70°.

I: What is the $\angle$ MBS? And why?
S: $\angle$ MBS= 50° (the same reason for $\angle$ SCN).

I: Then, what is the $\angle$ ABC? And why?
S: $\angle$ ABC = 70° (the same reason for $\angle$ ACB).

I: What is the name of Δ ABC? And why?
S: Δ ABC is an isosceles triangle, because the measurement of $\angle$ ABC and $\angle$ ACB are equal.
I: Are there any other ways to prove the unknown?
S: (different student answer) Δ MCB is a right triangle in $\angle$ M, then $\angle$ ABC = 180° - (90° + 20°) = 70°, and Δ BNC is a right triangle in $\angle$ N, then, $\angle$ ACB

=180° - (90° + 20°) = 70°, then 180° - (90° + 20°) = 70. ∠ ABC = ∠ ACB (this is an isosceles triangle).

I: That is great, what is the measurement of ∠ BAC?
S: ∠ BAC = 180° - (∠ ABC + ∠ ACB) = 180°- 140° = 40°.

I: Is this triangle an equilateral triangle? And why?
S: No, because the three angles are not equal.

I: What is the relationship between an isosceles triangle and an equilateral triangle?
S: Every equilateral triangle is isosceles triangle.

I: What does that mean?
S: Equilateral is a specific case of an isosceles triangle.

School No.2
Question 1
I: What is the known?
S: The set of even numbers are expressed as the sum of two numbers.

I: What is the unknown?
S: What is our deduction about these numbers?

I: What is the condition?
S: No answer.

I: Is the known sufficient or insufficient to determine the unknown? Or is additional information required?
S: (answer as group) It is sufficient.

I: Have you seen such a problem before?
S: No.

I: Have you seen the same problem in a slightly different form?
S: No.

I: Do you know a theorem that could be useful?
S: No.

I: What is the initial idea for this question?
S: The sum of any two odd numbers is a given even number.

I: If so, why did we not write 14 = 5+ 9, although 5 and 9 are odd numbers?
S: (one students answer) No, these numbers are not odd numbers they are prime numbers, because 9 is an odd number but not a prime number.

I: What is your conclusion now?
S: every even number can be expressed as the sum of two prime numbers.

I: 2 Is prime number?
S: Yes.
I: why did we not write 4 = 2 +2?
S: (one student answer) any even number can be expressed as two odd prime numbers.

I: Is there just one expression for any even numbers?
S: No, because 14 is expression as two expressions, 14 = 7 + 7, and 14 = 3 + 11.

Question 2
I: What is the known?
S: Large square, containing 64 squares, we put one grain of rice in the first square, two grains on the second square, four grains on the third square, and so on.

I: What is the unknown?
S: How many grains of rice are in the 64^{th} square, and how many grains of rice on the whole square.

I: What is the condition?
S: The first square has one grain, the second square has two grains, the third square has fourth grains, and so on.

I: What is the type of relationship?
S: Exponential relationship.

I: Is the know sufficient, insufficient, or is additional information necessary to determine the unknown?
S: It is sufficient.

I: Have you seen such a problem before?
S: (one student answer) yes, I read this question from the book Fun Mathematics, without any answers, but it took the scientist the day to find the answer.

I: What is the length of this square?
S: Eight units.

I: Could you please draw the square?
S: Yes.
(Most of students were able to draw the diagram correctly).

I: How many grains of rice are there the in first, second, third, and fourth squares?
S: 1, 2, 4, 8, respectively.

I: Could you please rewrite these number as 2 to the power number?
S: $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$.

I: Then, how many grains of rice are there in the nth square?
S: 2^n.

I: If so, how many grains in the first square and the second square?
S: $2^1 = 2$, $2^2 = 4$.

I: But, how many grains of rice actually in these squares?
S: 1, 2 respectively.

I: Then, what is the relationship between the number of rice grains in each square and the square number?
S: The number of rice = $2^{\text{square number -1}}$.

I: Then, how many grains of rice are on the 64th square?
S: $2^{64-1} = 2^{63}$.

I: That is great; could you please give me the total number of grains of rice on the chessboard?
S: (one student answer) I found a relationship between the total of the rice up to a certain square, and the number of the rice on the following square.
The total of rice grains up to 1 square = (the number of rice grains in the second square – 1)➔ 1 = 2 -1.
The total of rice grains up to 2 square = (the number of rice grains in the third square – 1)➔ 3 = 4 -1.
The total of rice grains up to 3 square = (the number of rice grains in the fourth square – 1)➔ 7 = 8 -1.
In general the total of rice grains up to n square = (the number of rice grains in the (n+1) square – 1).
Then, the total of rice grains up to 63 square = (the number of rice grains in the 64 square – 1)➔ $2^{63} – 1$, and the total of rice grains up to 64 square = (the total of rice grains up to 63 + the number of rice grains in the 64 square) = $2^{63} – 1 + 2^{63} = 2×2^{63} – 1= 2^{64} -1$. Or the total of rice up to 64 square = (the number of the rice in the 65 square – 1) = $2^{64}-1$.

I: Do you have any different ways?
S: No.

I: What is the total of the rice grains up to squares 1, 2, 3?
S: The total of rice grains up to the first square = 1.
The total of rice grains up to the second square = 3.
The total of rice grains up to the third square = 7.

I: Could you please say, what are the numbers that could be written as a 2 to the power numbers?
S: even numbers.

I: If so, could you please rewrite 6 as a 2 to the power number?
S: No.

I: Then, What are these numbers?

S: 2, 4, 8, and so on.

I: What is the relationship between the set of numbers 1, 3, 7 and 2, 4, 8?
S: 1 = 2-1, 3 = 4-1, 8 = 7-1.

I: That is great; could you please rewrite these numbers using a 2 to the power number?
S: NO answer?

I: For example 1 = 2^1-1.
S: 3 = 2^2-1, 7 = 2^3-1.

I: Now, what is the total number of rice grains in the chessboard?
S: (one student answer) 2^{64}-1.

I: If we suppose each 50 grains of rice = 1 gram in your expectation, how much rice do we need to solve this problem?
S: A big number, they said if we farm the whole land with rice, we still need more for this problem.

After that, the interviewer gave the students the exact amount we need to solve this problem.

I: In your opinion, do you think our answer was correct?
S: Yes, but we can not solve it practically.

Question 3
I: What is the known?
S: The multiplication of two numbers is equal to 1, and x > 0.

I: What is the unknown?
S: Find out the correct answer through the known. .

I: What is the condition?
S: The same to the known.

I: Is the known sufficient or insufficient to determine the correct answer?
S: It is sufficient.

I: Is choice (a) correct? And why?
S: No, because if Y is negative, then positive times negative = negative, and the answer is equal 1, and 1 is positive number.

I: Is there any explanation?
S: No answer.
I: Is choice (b) correct? and why?
S: No, because if x=2>1, and y =2>1, then the outcome will equal 4 not 1, and the relation between x and y is if x increases then y decrease and vice versa.
I: Is there any explanation?
S: In general, if x > 1 and y > 1, then (xy) >1.

I: Is choice (c) correct? and why?
S: Yes, because for example if $x < 1 = ½$, and $y < 1 = ½$, then $(xy) = 1$.

I: Does $½ \times ½ = 1$?
S: Oh sorry, it is equal ¼.

I: In your opinion, which is the correct choice? And why?
S: The correct choice is e, when x is increasing then y will be decreasing, which means that when x increases, y will decrease to get 1.

I: What is the relationship between x and y?
S: Inverse relationship.

I: Do you have any answer?
S: x is the inverse of y.

Question 4
I: What is the known?
S: There are three shapes, known their length and width.

I: What is the unknown?
S: Express the following shapes by symbols and finally express for n^2.

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.

I: What is representing the first shape?
S: $11^2 = 10^2 + 1^2 + 2 \times 10 \times x$.

I: What is representing the third shape?
S: (different student answer) $13^2 = 10^2 + 3^2 +3 \times 3 \times 10$.

I: How many rectangles in each side?
S: Three rectangles.

I: How many sides are there?
S: Two.

I: Then, express again for 13^2.
S: $13^2 = 10^2 + 3^2 +3 \times 2 \times 10$.

I: That is great, based on your expression of these shapes, express for n^2.
S: $n^2 = 10^2 + (n-10) \times 10 \times 10 + (n-10)^2$ (where (n-10) is the area of small squares).

I: Are there any comments?
S: (different student answer) $n^2 = 10^2 + (n-10)^2 + (n-10) \times 2 \times x$.

I: What is the value of x?

S: No answer.

I: Again, $11^2 = 10^2 + 1^2 + 1 \times 2 \times 10$.
$12^2 = 10^2 + 2^2 + 2 \times 2 \times 10$.
$13^2 = 10^2 + 3^2 + 3 \times 2 \times 10$, then show what is the constant and what is the variable to express again n^2.
S: $n^2 = 10^2 + (n-10)^2 + (n-10) \times 2 \times 10$.

I: Then, what is the value of x now?
S: 10

I: If we analyse n^2 as its expression, what can we find?
S: n^2.

I: That is great, could you please check if it is correct?
S: one student tries n = 17 and he found this formula to be correct.

Question 5
I: What is the known?
S: ABC triangle, $\angle$ MSB and $\angle$ SBC are known angles. BN and CM are intersecting at point S.

I: What do we mean by altitude?
S: The line divides the base into two equal parts.

I: Are MC and CB dividing AC and AB into two equal parts?
S: No.

I: Then, what do we mean by altitude?
S: Make a right angle.

I: What is the unknown?
S: The Δ ABC is an isosceles triangle.

I: Have you seen such a problem before?
S: Yes, but slightly different.

I: Do you know any theorem that will be help us to prove it?
S: Isosceles triangle must have equal base angles.
The total of the angles in any triangle is equal to180°.

I: What is the measurement of any supplementary angles?
S: 180°.

I: What is the relationship between any two opposite angles?
S: They are equal.

I: Then, what is the measurement of $\angle$ NSC, and why?
S: $\angle$ NSC = 40° (opposite to the $\angle$ MSB).

I: What is the measurement of $\angle$ CSB, and why?
S: $\angle$ CSB = 180°-40° = 140° (supplement to $\angle$ MSB).

I: What is the measurement of $\angle$ MCB, and why?
S: $\angle$ MCB = 20° (the total of angles Δ BSC = 180°).

I: What is the measurement of $\angle$ NCS, and why?
S: $\angle$ NCS = 50° (the total of angles Δ CNS = 180°).

I: Then, what is the measurement of $\angle$ ACB, and why?
S: $\angle$ ACB = $\angle$ NCM + $\angle$MCB = 50° + 20° = 70°.

I: What is the measurement of $\angle$ MBS, and why?
S: $\angle$ MBS = 50° (the total of angles Δ MSB = 180°).

I: Then, what is the measurement of $\angle$ ABC?
S: $\angle$ ABC = $\angle$ ABN + $\angle$NBC = 50° + 20° = 70°.

I: What is the name of this triangle, and why?
S: Δ ABC is an isosceles triangle, because the base angles are equal.

I: Do you know the different ways to prove?
S: No answer.

School No.5
Question 1
I: What is the unknown?
S: The number of sides of the polygons are known, and the number of diagonals is known for the three polygons the triangle, quadrilateral, and pentagon.

I: What is the unknown?
S: The number of diagonals for the polygon for which the number of sides are seven and nine?

I: What is the condition?
S: The relationship between two variables is a quadratic relation, and for this relation we will find the number of diagonals.

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.
I: Have you seen such a problem before?
S1: Yes, I faced a problem in which I needed to find the number of diagonals for the hexagon polygon.
(Other student said) No.

I: Is there any pattern between the number of sides of the polygon and the number of diagonals?

S: Yes, the pattern is the number of diagonals = $ax^2 + bx + c$, where a, b, and c are constant and x is the number of sides of the polygon.

I: Are there any other ways to find the solution?
S: Yes, drawing the polygon and finding the number of diagonals.

I: Is it possible to find the number of diagonals for the polygon for which the number of sides is n?
S: No, because n is unknown.

I: Then, what is the optimal way?
S: Using the quadratic relation.

I: Are you able to find the number of diagonals for the heptagon polygon using the drawing?
S: (The students tried using paper and pen to find the answer, but only one student was able to find the correct answer) 14 diagonals.

I: What is the number of diagonals for the polygon for which the number of sides is n?
S: (Using quadratic relation, two students were able to find the correct relation) $f(n) = ½ n^2 - 3/2n$.

I: Do you know whether your answer is correct or not?
S: Yes, we know the number of diagonals for the pentagon polygon equals 5(the students tried f (5) to find whether it is 5 or not, then they find f (5) = 5, which proved the quadratic relations were correct).

Question 2
I: What is the known?
S: A sequence in which the first four terms are known.

I: What is the unknown?
S: The tenth number.

I: Have you seen such a problem before?
S: Yes, I faced a problem to find the base of the sequence then had to find any term.

I: Is this sequence arithmetical or geometrical?
S: If the difference between any two following terms is equal, then the sequence will be an arithmetical sequence, and if the ratio between any two following terms is equal, then the sequence will be a geometrical sequence. But, if neither the difference nor the ratio is equal, then this sequence is not arithmetical or geometrical.

I: Each number consists of an integer and a fraction, therefore, can any of them be considered an arithmetical or geometrical sequence?
S: Integer numbers are odd numbers and considered an arithmetical sequence. In contrast, the fraction number is not an arithmetical or geometrical sequence.

I: What was your answer when you saw the question for the first time?
S: I found the integer number for the tenth term using the arithmetical sequence since the integer numbers are odd numbers.

I: What is the integer number for the first, second, and third terms?
S: 3, 5, 7.

I: What is the integer number for nth term?
S: (2n-1).

I: Again, what is the integer number in the first term, using your formula?
S: 2n-1 = 2×1-1= 2-1=1? Oh sorry. It should be 2n+1.

I: That is great, what is the integer number for the 10^{th} term?
S: 2n-1 = 2×10 + 1= 20 + 1 = 21.

I: What is the fraction number in the first, second, third, terms?
S: ½, 1/3, ¼.

I: What is the fraction number for nth term?
S: 1/n.

I: If so, please check the second term using your formula?
S 1/n = ½ (oh sorry). It should be 1/ (n +1).

I: What is the fraction number for 10^{th} term?
S: 1/ (n +1) = 1/ (10+1) = 1/11.

I: Are there any different ways of solving this problem?
S: (Different student answer) I found the answer using the sequence until I found the tenth term.

I: Does anyone have any different ideas or different answers?
S: No answer.

Question 3
I: What is the known?
S1: A chessboard, consisting of 64 squares, we put one, two, three, --- grains of rice on the first, second, third, -- squares and so on, which means that each following square increases by one grain.
(Other students said) No this is the incorrect answer.

I: Again, what is the known?
S: A chessboard, consisting of 64 squares, we put one, two, four, eight grains of rice on the first, second, third, fourth squares and so on.

I: What is the unknown?
S: How many grains of rice are there in the 64^{th} squares? And what is the total of amount rice in the chessboard?

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.

I: Have you seen such a problem before?
S: (Just one student answer) I read a problem like this from the book *The Human between Science and Religion*, but the chessboard consisted of wheat grains.

I: Was there any answer to that problem?
S: I think yes.

I: If you remember, what was the solution?
S: I think a very large number.

I: (The interviewer asks the other students) Have you seen the same problem in a slightly different form?
S: No.

I: If we considered that the chessboard consisted of four squares, how many grains of rice are there on the first, second, third, and fourth squares?
S: (The students answered as a group) There are 1, 2, 4, and 8 grains on the first, second, third, and fourth squares, respectively.

I: Could you please rewrite these numbers 1, 2, 4, and 8 as exponential numbers?
S1: 1 = $1^{0,}$ then the student stops at this point.
S: (Other students) 2^0, 2^1, 2^2, 2^3, respectively.

I: Now, we want to return to the original problem, how many grains of rice are there on the 64th square?
S: 2^{63}.

I: In a chessboard with four squares, how many grains are there up to the first, second, third, fourth squares?
S: (Group answer) 1, 3, 7, and 15.

I: Could you please find how many grains there are up to any square? Is there any pattern?
S: The total number of rice grains up to the first square1 = 2^0.
The total number of rice grains up to the second square = 2^1 –1.
The total number of rice grains up to the third square = 2^2-1.

I: Could you please check your answer up to the second square?
S: 2^1 –1 = 2-1 = 1, (Oh it is incorrect).
(Different student said) $2^{\text{square number}}$ – 1, Then, the total number of rice grains up to first square = 2^1-1 = 1.
The total number of rice grains up to the second square is 2^2-1 = 3.
The total number of rice grains up to the third square is 2^3-1 = 7.

I: Then, how many grains of rice are there on the chessboard in total?
S: $2^{64} - 1$.

I: If we suppose each 50 grains of rice = 1 gram, in your expectation, how much rice do we need to solve this problem?
S: (Different answers from different students) 1 kilogram, 2 kilograms, 5 kilograms, and no answer.

I: In your opinion, do you think we can answer this problem practically?
S: Yes, but we need a long time.

After that, the interviewer gave the students the exact amount we need to solve this problem.

I: Is our answer logical?
S: Yes, but it is not practical.

I: Do you always believe that mathematics is true?
S: Not always.

Question 4
I: What is the known, unknown, condition?
S: Known: There are three squares, with the length of their dimensions.
Unknown: Express the area of the following squares and n^2 in general.

I: Have you seen such a problem before?
S: No.

I: Have you seen the same problem in a slightly different form?
S: Yes, we faced a problem which required us to find the whole area through addition of the areas which represent the whole shape.

I: What is representing the first, second, and third shapes?
S: They are squares.

I: These are squares for what numbers?
S: 11^2, 12^2, 13^2.

I: Could you please express these numbers using shapes?
S: $11^2 = 10\times10 + 1 \times 10 + 1 \times 10 + 1 \times 1$.
$12^2 = 10\times10 + 2 \times 10 + 2 \times 10 + 4 \times 1 \times 1$.
$13^2 = 10\times10 + 2 \times 1 \times 10 + 2 \times 1 \times 10 + 2 \times 1 \times 10 + 9 \times 1 \times 1$.

I: Could you please rewrite these numbers using exponential and repeated summation?
S: $11^2 = 10^2 + 2 \times 1 \times 10 + 1^2$.
$12^2 = 10^2 + 2 \times 2 \times 10 + 2^2$.
$13^2 = 10^2 + 2 \times 3 \times 10 + 3^2$.

I: Observe the constants and variables in each expression, then rewrite n^2?

S: $n^2 = 10^2 + 2 \times (n-10) \times 10 + (n-10)^n$.

I: Again, the area of the first shape was the area of the small square 1^2; the area of the second shape was the area of the small squares $=2^2$; and the area of the third shape was the area of the small squares $=3^2$. Then, what is the area of the small squares, if the length of the whole shape is n?
S: Oh I am sorry; the area of the small squares will be $(n-10)^2$.

I: That is great, now, what is the value of n^2.
S: $n^2 = 10^2 + 2 \times (n-10) \times 10 + (n-10)^2$.

I: Are there any other ways to find the solution?
S: Using analysis of the shapes.

I: Is checking whether your answer is correct or not important in mathematical problem solving?
S1: If there are a number of students who found the same answer, we consider our solution to be correct.
S2: I think it is not an important step if I solved the problem with concentration.
S3: I think it is not an important step if I repeated my solution once again.
I: Do your teachers focus on checking the answers?
S: Sometimes.

Question 5
I: What is the known?
S: ABC triangle, $\angle$ MSB = 40°, and $\angle$ SBC = 20°, BN and CM are altitudes which intersect at point S.

I: What is the unknown?
S: That Δ ABC is an isosceles triangle, which means AB = AC.

I: What is the condition?
S: Give geometric reasons in the proof.

I: Is the known sufficient or insufficient to prove the unknown?
S: (One student said) It is insufficient, because we need to know whether MS = BS, and SN = SC or not.
(Other students said) It is sufficient, because we can prove the unknown with this information.

I: (The interviewer asks the first student) What is the relation between the lengths and the proof?
S: I have no idea.

I: Have you seen such a problem before?
S: Yes, but slightly different in year 9.

I: What types are of the Δ CNB and Δ CMB?
S: They are congruent.

I: My question is what types are these triangles, not what the relation between them is?
S: They are right triangles.

I: What do you mean by altitude CM?
S: $\angle$ CMB = 90°.

I: Do you know any theorems that will help us to prove this?
S: Isosceles triangle must have equal base angles.
In this triangle, the altitude drop from the vertex triangle will divide the base into two equal parts, and divide the vertex angle into two identical angles.

I: Are you agreeing with what your friend said?
S: No, because we can not consider this theorem is true until we prove that the triangle is isosceles.

I: That is great, are there any theorems we can consider?
S: $\angle$ MSB is exterior the Δ BSC, and it is equal to the two opposite interior angles in this triangle, then $\angle$ MCB = 20°.

I: Then, what type is Δ BSC, and why?
S: Isosceles, because the base angles are equal.

I: What is the total of the angles in any triangle?
S: The total of the angles in any triangle is equal to180°.

I: What is the measurement of any supplementary angles?
S: 180°.

I: What is the relationship between any two opposite angles?
S: They are equal.

I: Then, what is the measurement of $\angle$ NSC, and why?
S: $\angle$ NSC = 40° (opposite to the $\angle$ MSB).

I: What is the measurement of $\angle$ NCM, and why?
S: $\angle$ NCM = 50° (the total of angles Δ CNS = 180°).

I: What is the measurement of $\angle$ ACB, and why?
S: $\angle$ ACB = 70° (the total of $\angle$ NCM + $\angle$ MCB).

I: What is the measurement of $\angle$ MBS, and why?
S: $\angle$ NCS = 50° (the total of the angles of Δ MSB = 180°).

I: Then, what is the measurement of $\angle$ ABC, and why?
S: $\angle$ ABC = $\angle$ MBN + $\angle$NBC = 50° + 20° = 70°.

S: Δ ABC is an isosceles triangle, because the base angles are equal.

I: Are there any other ways to prove the unknown?
S: We prove that Δ CSB is an isosceles triangle, ∠ BSC = 140° (straight angle with ∠ MSB, and it is considered exterior Δ MSB and Δ NSC and equal to the two opposite interior angles. Then, ∠ NCS = 140° - 90° = 50°). In the same way we can find ∠ MBS = 50°. ∠ ABC = ∠ ABN + ∠NBC = 50° + 20° = 70°, and ∠ ACB = ∠ ACM + ∠MCB = 50° + 20° = 70°. ∠ ABC = ∠ ACB, then ABC is isosceles.

School No.6
Question 1
I: What is the known?
S: 1 =1, 1 +3 = 4, 1 + 3 +5 = 9, 1+ 3 + 5 + 7 = 16.

I: What is the unknown?
S: Find the summation for the first n for odd numbers.

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.
I: Have you seen such a problem before?
S: No.

I: Have you seen the same problem in a slightly different form?
S: Yes, the problem was to find the summation for the first n numbers.

I: What are the numbers on the left side?
S: Odd numbers.

I: What are the numbers on the right side?
S: Even numbers.

I: Again, what are the numbers 1, 4, 9, 16?
S: Perfect squares.

I: What is the primary answer for this problem?
S: I found the difference between the first outcome and the second is 3, and between the second and the third is 5, then 7; that means the difference between the fourth outcome and the fifth will be 9, then the summation of the fifth statement will be 16 +9 = 25. Unfortunately, I was unable to find the last statement because I do not know the summation for the previous statement.

I: Could you please write the last term in the numbers on the left side as 2n -1?
S: No answer.

I: For example 1 = 2 ×1-1.
S: 3 = 2 ×2-1, 5 = 2 ×3-1, 7 = 2 ×4-1.

I: That is great, what is the variable in each term?
S: 1, 2, 3, 4, respectively.

I: Good, what is the relation between these numbers and the total?

S: The total = the variable in each term.
I: That is great, what is the variable in the last statement?
S: n.

I: Then, what is the summation for the last statement?
S: n^2.

I: Are there any other different ways or ideas to solve this problem?
S: Yes, I linked the number of the terms in each statement with the total, the numbers of the term in the first, second, third, and fifth statements are 1, 2, 3, 4, and the square of these numbers will give the total for each statement respectively. Then, there are n terms in the final statement, so the total of the last statement will be n^2.

Question 2
I: What is the known?
S: On a chessboard having 64 squares, there is one grain of rice on the first square, two grains of rice on the second, four on the third, and so on.

I: What does that mean?
S: There is double the number of grains of rice in the subsequent square.

I: What is the unknown?
S: How many grains of rice there are in the 64^{th} square, and how many grains of rice there are in chessboard.

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.

I: Have you seen such a problem before?
S: (Just one student answer) I faced this problem in the book; *General Information*; but I did not try to solve this problem.

I: When you read this problem for the first time, how much rice did you think we would need?
S: (one students answer) I thought we would need a large amount, because if this problem needed a limited amount of rice, the mathematician would not ask the King.

I: Do you know how many grains are in the fourth and sixth squares?
S: 8, 32.

I: Do you know how we can rewrite these numbers as exponential?
S: $8 = 2^3$, $32 = 2^5$.

I: That is great, what is the relation between the power and the square number?
S: Power = square number – 1.

I: Then, how many grains of rice are in the 64^{th} square?

S: 2^{64} -1 = 2^{63}.

I: If we consider the chessboard involves nine squares, how many grains of rice are there up to 1, 2, 3, and 4?
S: Total rice grains up to the first square = 1
Total rice grains up to the second square = 3
Total rice grains up to the third square = 7
Total rice grains up to the fourth square = 15.

I: Then, how many rice grains are there on this chessboard?
S: No answer.

I: We know 1 = 2 -1, 3 = 4 -1, 7 = 8 -1, and so on, then, please rewrite these numbers as exponential?
S: No answer.

I: For example, 1 = $2^1 – 1$, then rewrite the other numbers.
S: 3 = $2^2 – 1$, 7 = $2^3 – 1$.

I: That is great, what is the relation between the power and the square number?
S: Power = square number.

I: Then, how many grains of rice are there are in a chessboard having nine squares?
S: 2^9-1.

I: Then, how many grains of rice there are in the original problem?
S: 2^{64}-1.

I: How many grains of rice do you think there are in 2^{64}-1?
S1: I think this is a large number, because the budget of this country is not enough to buy the rice for this problem.
S: (Other students) no answer.

After that the interviewer gave the students the exact amount we need to solve this problem.

I: Is this answer correct?
S: Of course, it is correct.

I: Do you believe that mathematics always gives a correct answer? Yes, but we can not solve this problem practically.

Question 3
I: What is the known?
S: If the square shape is cyclic, then the total of each opposite two angles in it is equal to 180°, the first shape $\angle$ b and $\angle$ d are known, and in it the second shape $\angle$ b and $\angle$ c are known.

I: What is the relationship between the two known angles in the first and in the second shape?
S: In the first shape the two known angles are opposite, and the two known angles in the second shape are neighbours.

I: What is the unknown?
S: What is our induction of the two shapes (choose the correct answer)?

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.

I: Have you seen such a problem? Or have you seen the same problem in a slightly different form?
S: (The answer was a group answer) Yes, in year ten, we faced a problem that required us to determine if the square shape is cyclic or not.

I: What does it mean that the square shape is cyclic?
S: Each two opposite angles contains 180°.

I: What is your primary answer for this problem?
S: I thought the first shape is cyclic, because the two known angles are opposite, and the second shape is not cyclic, because the two known angles are neighbours.

I: Is the first shape cyclic, and why?
S: Yes, because the two known angles are 180°, and the total of the angles in any quadrilateral is 360°, then the total of the other two angles will 180°.

I: Is the second shape cyclic, and why?
S: I think the second shape is not cyclic, because the two other angles are unknown.

I: Do you agree with your friend's answer, and why?
S: No, we do not know, if it is cyclic or not, because if $\angle$ d = 100°, then the shape will be cyclic, and if the $\angle$ d ≠ 100°, then the second shape will not be cyclic, and both choices are possible, therefore, we do not know.

I: That is great, what is the correct answer?
S: d.

Question 4
I: What is the known?
S: There are three shapes, representing squares.

I: What is the unknown?
S: The area of each shape, and in general, the area of n^2.

I: Is the known sufficient or insufficient to determine the unknown?
S: It is sufficient.

I: Have you seen such a problem before?
S: Yes, but slightly different without find the general n^2.

I: What is representing the first, the second and the third shapes?
S: Squares with lengths of 11, 12 and 13, respectively.

I: Could you please rewrite 11^2, and 13^2 using these shapes?
S: $11^{2} = 10^2 + 2 \times 10 \times 1 + 1^2$
$13^{2} = 10^2 + 2 \times 2 \times 2 \times 1 + 3^2$

I: Do you know what 13^2 is?
S: Of course, it is 169?

I: Could you please check whether your answer is correct or not?
S: $13^{2} = 10^2 + 2 \times 2 \times 2 \times 1 + 3^2 = 100 + 8 + 9 = 117$ ("oh there is something missing" and he stopped at this point).

I: How many rectangles are in each side, and what is the area of each rectangle?
S: Three, 10.

I: That is great, what are the areas of the rectangles in each side?
S: 30 or 10×1.

I: Then, could you please write the expression for 13^2?
S: $13^{2} = 10^2 + 2 \times 3 \times 10 + 3^2$

I: Then, what is the expression for n^2?
S: (Different student answers) $n^2 = 10^2 + 2 \times n \times 10 + n^2$

I: Could you please check whether your answer is correct or not?
S: $n^2 = 10^2 + 2 \times n \times 10 + n^2 = 100 + 20n + n^2$

I: Is it correct, and why?
S: No, the right side does not equal n^2.

I: Do you know what is incorrect with this formula?
S: No answer.

I: Could you please tell me what the variables are in 11^2 and 13^2?
S: 11 linked with 1 and 13 linked with 3.

I: That is great, and then n will link with which number?
S: n is liked with n-10.

I: Now, rewrite n^2
S: $n^2 = 10^2 + 2 \times (n-10) \times 10 + n^2$.

I: Are there any other different ways or ideas to solve this problem?
S: No answer.

I: Have checked your answer for this question?
S: No.

I: Is checking whether your answer correct or not important in mathematical problem solving?
S: Of course.

I: If it is an important step, why didn't you check your answer?
S: Not enough time, nervousness during test time.

I: Does your teacher encourage you to check your answers or not?
S: No, he just tells that the important thing is the answer.

Question 5
I: What is the known?
S: Δ ABC with two known angles.

I: What is the unknown?
S: Prove that Δ ABC is isosceles.

I: Is the known sufficient or insufficient to prove the unknown.
S: It is sufficient.

I: Have you seen such a problem before?
S: We faced many problems like this in our mathematics curricula.

I: Do you know any theorems that will be help us to prove this problem?
S: Isosceles triangle must have equal base angles.
If we drop altitude from the vertex triangle it will divide the base into equal parts, and divide the vertex angle into equal angles.

I: Can you use the second theorem in the proof, and why?
S: No, because first we have to prove that the triangle is isosceles.

I: Do you know other theorems related to the proof?
S: No answer.

I: What is the total of any two supplementary angles?
S: 180°.

I: What is the relation between any two opposite angles?
S: They are equal.

I: What is the total of the angles of any triangle?
S: 180°.

I: What is the measurement of $\angle$ SMB and why?
S: 90°, (because CM is the altitude on AB).

I: What is the measurement of $\angle$ MBN, and why?
S: 50°, (the total of the angles of Δ MSB = 180°).

I: Then, what is the measurement of $\angle$ ABC?
S: $\angle$ ABC = 50° + 20° = 70°.

I: What is the measurement of $\angle$ SCB, and why?
S: $\angle$ SCB = 20° (because $\angle$ MSB is exterior Δ CSB, and it is equal to the two opposite interior angles in Δ CSB).

I: What is the measurement of $\angle$ CNB, and why?
S: $\angle$ CNB = 90°, (BN is the altitude on AC).

I: What is the measurement of $\angle$ NSC, and why?
S: $\angle$ NSC = 40°, (opposite to $\angle$ MSB).

I: What is the measurement of $\angle$ NCM, and why?
S: $\angle$ NCM = 50°, (the total of the angles of Δ NCS = 180°).

I: Then, what is the measurement of $\angle$ ACB?
S: $\angle$ ACB = 50° + 20° = 70°.

I: What type is the Δ ABC, and why?
S: ABC is an isosceles triangle, because the base angles are equal.

I: Do you known other ways to prove?
S: I do not know.