CHAPTER 3

CLOSURE PROPERTIES OF THE SPATIAL NUMERICAL RANGE

.1 INTRODUCTION AND PRELIMINARIES

The open unit disk is the spatial numerical range of the right shift operator on an infinite dimensional Hilbert space, [Halmos, 1, Solution 168(2)]. This shows that the spatial numerical range need not contain its boundary. In this chapter we examine some subsets of the boundary of the spatial numerical range and show that for particular operators on certain classes of Banach spaces these subsets are in the spatial numerical range.

The first result of this type is due to D. Hilbert [F. Riesz and B. Sz-Nagy, 1, p. 232] who proved that if T is a compact hermitian operator on a Hilbert space and $\lambda \in \overline{W(T)}$ is such that $|\lambda| = \sup\{|\alpha| : \alpha \in W(T)\}$, then $\lambda \in W(T)$. This is extended, in section 3, to a larger class of compact operators on a restricted family of Banach spaces. Hilbert's result was extended by B. A. Mirman [1], who proved that for a compact operator T on Hilbert space the set of exposed points of $\overline{W(T)}$, which do not lie on a line segment of the boundary of W(T) passing through the origin, are contained in W(T). This result also follows from Lemma 1.6.3. and the well known fact that for a compact operator T $\sigma(T) \setminus \{0\} = p\sigma(T)$. In section 2 we improve on this result, obtaining

necessary and sufficient conditions for a compact operator on Hilbert space to have closed spatial numerical range. We also obtain related results for compact operators on $\ell_p(\ 1 < p < \infty)$.

Operators which attain their numerical radius are studied in section 3. In particular, we show that the set of hermitian operators on a Hilbert space, which attain their numerical radius, is dense among all the hermitian operators. From this we give an alternative proof, for Hilbert spaces, of a result by J. Lindenstrauss [1].

In the course of this chapter we raise several general questions to which we give only partial answers, thus indicating areas for possible future research.

Another result concerning the closure of the spatial numerical range is due to S. K. Berberian [1] and S. K. Orland. [Berberian & Orland, 1]. They show that a given Hilbert space H can be embedded in another Hilbert space K and that there exists an isometric *-isomorphism $T \mapsto [T]$ of B(H) to a subalgebra of B(K) such that $\overline{W(T)} = W([T])$. We follow their argument and extend it to normed linear spaces, however, because the numerical range may not be convex, the result obtained is weaker.

1.1. **Proposition.** *For a given normed linear space E there exists a Banach space X such that*

  i) E *can be embedded in* X;

  ii) *there exists an isometric isomorphism* $T \mapsto [T]$ *of* $B(E)$ *onto a subalgebra of* $B(X)$;

  iii) $\overline{W(T)} \subseteq W([T]) \subseteq V(T)$.

Proof. Let B be the set of sequences of elements of E whose norms are uniformly bounded, that is $s \in B$ if $s = \{x_n\}$, $x_n \in E$ and $\|x_n\| \leq M$ for all n and some M $(0 < M < \infty)$ depending on s. B, with addition and scalar multiplication defined point wise, is a linear space.

Let μ be a "Banach-Mazur generalized limit" [Day, 1] on bounded sequences of real numbers. For $s \in B$, $s = \{x_n\}$, define $p(s) = \mu\{\|x_n\|\}$, then it is readily seen that p is a pseudonorm on B. Set $P = B/N$ where $N = \{s \in B : p(s) = 0\}$ and write s' for $s + N \in P$. Then P is a normed linear space. Further, if we denote by [x] the element of P arising from the sequence $\{x\}$ where $x \in E$ then $x \mapsto [x]$ is an isometric ismorphism of E onto a subspace of P, showing that E may be embedded in P.

Now if α_n is a bounded sequence of real numbers $|\alpha_n| \pm \alpha_n \geq 0$ and so $\mu(|\alpha_n| \pm \alpha_n) \geq \lim\inf(|\alpha_n| \pm \alpha_n) \geq 0$ or or $|\mu(\alpha_n)| \leq \mu |\alpha_n|)$.

If for a bounded sequence of complex numbers $\lambda_n = \alpha_n + i\beta_n$,

α_n, β_n real, we define $\mu(\lambda_n) = \mu(\alpha_n) + i\mu(\beta_n)$, then

$\mu(\lambda_n) = r\,e^{i\theta}$ for some real r and θ and

$|\mu(\lambda_n)| = e^{-i\theta}\mu(\lambda_n) = \mu(e^{-i\theta}\lambda_n)$ is real and so by definition

$$\mu(e^{-i\theta}\lambda_n) = \mu(\text{Re } e^{-i\theta}\lambda_n)$$
$$\leqslant \mu(|e^{-i\theta}\lambda_n|)$$
$$\leqslant \mu(|\lambda_n|).$$

Therefore $|\mu(\lambda_n)| \leqslant \mu(|\lambda_n|)$.

Hence if $k = \{f_n\}$ is a sequence of elements of E' such that

$\|f_n\| \leqslant M$ for all n and some $M > 0$, then for any $s = \{x_n\} \; \varepsilon \; B$

$k(s') = \mu(f_n(x_n))$ defines a linear functional on P and

$$|k(s')| = |\mu(f_n(x_n))|$$
$$\leqslant \mu(|f_n(x_n)|)$$
$$\leqslant M\,\mu(\|x_n\|)$$
$$= M\,p(s')$$

So $k \; \varepsilon \; P'$, with $\|k\| \leqslant M$.

Further if the f_n's are chosen so that

$f_n \; \varepsilon \; D(x_n)$, $\|x_n\| = 1$, then $k = \{f_n\}$ has $\|k\| \leqslant 1$ and

$k(s') = 1$ where $s = \{x_n\}$, so $k \; \varepsilon \; D(s')$.

Now for $T \; \varepsilon \; B(E)$ define $[T]$ by $[T]: B \to B: \{x_n\} \mapsto \{Tx_n\}$, then for

any $s = \{x_n\} \; \varepsilon \; B$ $p([T]s) = \mu(\|Tx_n\|) \leqslant \|T\|\,\mu(\|x_n\|)$

$$= \|T\|\,p(s)$$

and so $[T]: P \to P: s' \mapsto ([T]s)'$ is a continuous linear operator on

P with$\|[T]\| \leq \|T\|$. However for any $\varepsilon > 0$ there exists $x \varepsilon E$, $\|x\| = 1$, such that$\|Tx\| \geq \|T\| - \varepsilon$ and since $\|[T][x]\| = \|Tx\|$ we see that$\|[T]\| = \|T\|$. The mapping $T \mapsto [T]$ is therefore an embedding of $B(E)$ in $B(P)$.

If $\lambda \varepsilon \overline{W(T)}$ then there exists $\{x_n\}$, $\|x_n\| = 1$, and $f_n \varepsilon D(x_n)$ such that $f_n(Tx_n) \to \lambda$. Letting $s = \{x_n\}$ and $k = \{f_n\}$ we have $k([T]s') = \mu(f_n(Tx_n)) = \lambda$ and $k \varepsilon D(s')$, therefore $\lambda \varepsilon W([T])$.

The result now follows by taking $X = \overline{P}$, the completion of P, and extending $[T]$ to an element of $B(X)$ by continuity. That $W([T]) \subseteq V(T)$ follows since, by Corollary .2.5.2. and the fact that $T \mapsto [T]$ is an isometric isomorphism, $V(T) = V([T])$.   //

We now recover the result of Berberian & Orland from the above theorem.

1.1.1. **Corollary.** *For a given Hilbert space H, there exist another Hilbert space K such that*

   i) H *can be embedded in* K;

   ii) *there exists an isometric isomorphism* $T \mapsto [T]$ *of* B(H) *onto a subalgebra of* B(K);

   iii) $\overline{W(T)} = W([T])$.

Proof. Identify H with E in Proposition 1.1. Then the normed linear space P, constructed in the proof of Theorem 1.1, is clearly an inner-product space with inner-product $(s',t') = \mu((x_n,y_n))$ where

$s = \{x_n\}$, $t = \{y_n\}$, and $K \equiv X$ is the Hilbert space completion of P.

Hence for $T \in B(H)$ we have $\overline{W(T)} \subseteq W([T]) \subseteq V(T)$. But $W(T)$ is convex

and so, by Corollary 1.3.1.3, $V(T) = \overline{W(T)}$.   //

For a finite dimensional normed linear space the spatial

numerical range of every operator is closed. Whether this property

characterizes finite dimensional spaces is an open question.

Certainly for Hilbert spaces it does, for at the beginning of this

chapter we saw that in an infinite dimensional Hilbert space the

right shift operator has open numerical range. In fact for Hilbert

spaces it is sufficient to require every compact operator to have

closed numerical range as Halmos [1, solution 168(1)] shows how to

construct a compact hermitian operator T, on an infinite dimensional

Hilbert space, with $\overline{W(T)} \backslash W(T)$ non empty. A similar construction

can also be carried out for $\ell\rho$-spaces $(1 < \rho < \infty)$.

1.2. Proposition. *Let* $X = \ell\rho$ *for some* ρ, $1 < \rho < \infty$, *then if the spatial numerical range of every compact operator on X is closed, X is finite dimensional.*

Proof. Take any sequence $\{\lambda_i\}$ where $0 < \lambda_i \leqslant 2^{-i}$, then we may

construct $K \in B(X)$ by $K(x_1, x_2, \ldots, x_n, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots,$

$\lambda_n x_n, \ldots)$. Let K_n be defined by

$K_n(x_1, \ldots, x_n, \ldots) = (\lambda_1 x_1, \ldots, \lambda_n x_n, 0, \ldots)$ then K_n is a finite

rank operator, and so compact.

Further $\|K - K_n\| \to 0$ as $n \to \infty$ so K is a compact operator.

Now for any $x = (x_1, \ldots, x_n, \ldots) \in X$, $\|x\| = 1$, the

$f \in D(x)$ is represented by

$f = (\text{sgn } x_1 |x_1|^{\rho-1}, \ldots, \text{sgn } x_n |x_n|^{\rho-1}, \ldots)$ where, for $\lambda \in C$

$\text{sgn } \lambda = \bar{\lambda}/|\lambda|$ for $\lambda \neq 0$

$\qquad = 0 \qquad$ for $\lambda = 0$.

Hence $f(Kx) = \sum_n \lambda_n |x_n|^\rho > 0$ so $0 \notin W(K)$, but $W(K)$ is closed by

assumption so by Theorem 1.4.2. $0 \notin \sigma(K)$ and hence K is regular.

Therefore $I = KK^{-1}$ is compact and so the unit sphere of X is compact

and therefore X is finite dimensional. $\qquad\qquad$ //

The above argument is easily extended to any normed

linear space E with a Schauder basis $\{b_i\}$, $\|b_i\| = 1$, if for each

$x = \sum x_i b_i$ and $f \in D(x)$, there exists scalars f_i such that

$f = \sum f_i \beta_i$ and $f_i x_i \geq 0$ where $\{\beta_i\}$ is the biorthogonal sequence to

$\{b_i\}$.

Proposition 1.2. enables us to construct examples of

compact operators, on $\ell\rho$-space $(1 < \rho < \infty)$, with spatial numerical

ranges that are not closed. A later example will show that for

c_0 (the space of all sequences converging to 0, with supremum norm)

even finite rank operators may fail to have closed numerical ranges.

It is tempting therefore to conjecture that finite dimensional spaces

may be characterized by the property that every compact operator

has closed spatial numerical range.

In the next section we make a more detailed study of the spatial numerical range of a compact operator over certain types of Banach spaces.

2. THE SPATIAL NUMERICAL RANGE OF A COMPACT OPERATOR

In this section we examine the closure properties of the spatial numerical range of a compact operator on various Banach spaces, which unless otherwise stated are infinite dimensional. As already mentioned Halmos [1, solution 168(1)] has given the example of the operator T on Hilbert space, l_2, defined by

$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, \tfrac{1}{2}x_2, \ldots, 1/nx_n, \ldots),$$

with $W(T) = (0,1]$ to show that the spatial numerical range of a compact operator may fail to be closed.

The following subset of the closure of the spatial numerical range of an operator will be of importance in our arguments.

2.1. DEFINITION. For a normed linear space E and $T \in B(E)$ the *extreme edge* of $W(T)$ is defined by

extreme edge $W(T) = \{\lambda \in \overline{W(T)} : r\lambda \notin \overline{W(T)}$ for any $r > 1\}$

Thus $\lambda \neq 0$ belongs to the extreme edge $W(T)$ if and only if λ is the point in $\overline{W(T)}$ farthest from the origin in the direction of λ. Note that $0 \notin$ extreme edge $W(T)$ even if $0 \in \overline{W(T)}$.

2.2. **LEMMA.** *For a normed linear space* E *and compact* T ε B(E) *if* $\lambda \neq 0$ *is an extreme point of* $\overline{W(T)}$ *then* λ ε *extreme edge of* W(T).

Proof. Since T is compact $0 \in \sigma(T) \subseteq \overline{W(T)}$, by Theorem 1.4.2. Assume $\lambda \notin$ extreme edge of W(T) then there exists r > 1 such that $r\lambda \in \overline{W(T)}$, but then λ lies in the interior of the straight line segment from 0 to $r\lambda$ which contradicts λ being an extreme point. Hence λ ε extreme edge W(T).   //

.2.2.1. **Corollary.** *For a normed linear space* E *and compact* T ε B(E) *we have* $\overline{co}$ W(T) = co(extreme edge W(T) $\cup$ {0})

Proof. Since $\overline{W(T)}$ is closed and bounded we have by Eggleston [1, Theorem 10], that

$co \overline{W(T)} = \overline{co}$ W(T). Now the set of extreme points of $\overline{W(T)}$ is contained in extreme edge W(T) $\cup$ {0} by Lemma 2.2, and so co (extreme edge W(T) $\cup$ {0}) = co $\overline{W(T)}$.   //

We will say that a normed linear space E has the *weak upper semi-continuity property (wusc-property)*, If $\{x_n\}, \|x_n\| = 1$, is a sequence $\sigma(E, E')$-convergent to x ε E and $\{f_n\}, f_n \in D(x_n)$, is a sequence $\sigma(E', E)$-convergent to f ε D', then $f(x) = \|f\| \, \|x\|$.

The importance of such spaces is indicated in the next lemma.

.2.3. **LEMMA.** *Let X be a reflexive Banach space with the wusc-property.*
Then for compact T ϵ B(X) *extreme edge* W(T) $\subseteq$ W(T).

Proof. Take λ ϵ extreme edge W(T), then $0 \neq \lambda$ ϵ W(T) and there
exists $\lambda_n = f_n(Tx_n)$, $\|x_n\| = 1$ and f_n ϵ D(x_n), such that $\lambda_n \to \lambda$.
Since X is reflexive we can choose a subsequence of the $\{x_n\}$,
$\{x_{n_k}\}$, converging in $\sigma(E,E')$ to x ϵ X, $\|x\| \leq 1$, similarly we can

choose a subsequence of the $\{f_{n_k}\}$, $\{f_{n_{k_1}}\}$, converging in $\sigma(E',E)$

to f, $\|f\| \leq 1$. Relabelling the $\{x_{n_{k_1}}\}$ as $\{x_m\}$ we have:

$$\left| f_m(Tx_m) - f(T_x) \right| \leq \left| f_m(Tx_m) - f_m(Tx) \right| + \left| f_m(Tx) - f(Tx) \right|$$

$$\leq \|Tx_m - Tx\| + \left| (f_m - f)(Tx) \right|$$

$$\to 0, \text{ since T is compact and}$$

$\{f_m\}$ is $\sigma(E',E)$-convergent to f.

Therefore $\lambda = f(Tx)$

or $\qquad \lambda = \|f\|\|x\| (\underline{f}(T\underline{x}))$ where $\underline{f} = f/\|f\|$

and $\underline{x} = x/\|x\|$, But f ϵ D(x) and so

$\qquad \lambda = k\lambda_1$ where k = $\|f\| \|x\| \leq 1$ and

$\qquad \lambda_1 = \underline{f}(T\underline{x})$ ϵ W(T).

Hence $k^{-1}\lambda$ ϵ W(T) and so since λ ϵ extreme edge W(T) we have

$\qquad k^{-1} \leq 1$,

therefore k = 1 and

$\qquad \lambda = \lambda_1$ ϵ W(T). $\qquad\qquad$ //

2.3.1. **Corollary.** *If* X *is a reflexive Banach space with the wusc-property and compact* T ε B(X) *has* 0 ε W(T) *then* coW(T) = $\overline{coW}$(T).

Proof. By Corollary 2.2.1. and the above lemma

$$\overline{co}\ W(T) = co(\text{extreme edge } W(T) \cup \{0\}) \subseteq coW(T) \subseteq \overline{coW}(T).$$

We now give some examples of reflexive Banach spaces which have the wusc-property.

2.4. **LEMMA.** *If* $\{x_n\}$, $\|x_n\| = 1$, *is a sequence of elements in* $\ell\rho$, $1 < \rho < \infty$, *and* $\{x_n\}$ *is* $\sigma(\ell\rho,\ell\rho')$-*convergent to* x, *then for* f_n ε $D(x_n)$ *the sequence* $\{f_n\}$ *is* $\sigma(\ell\rho',\ell\rho)$-*convergent to* f *where* $f(x) = \|f\|\ \|x\|$

Proof. For y ε $\ell\rho$ write $y = (y^{(1)}, y^{(2)}, \ldots, y^{(i)}, \ldots)$.
Then since $\{x_n\}$ converges to x in $\sigma(\ell\rho,\ell\rho')$ and $\|x_n\| \leq 1$ we have by [Taylor, 1, p. 210]

$$x_n^{(i)} \to x^{(i)} \text{ for all i.}$$

Now $f_n^{(i)} = \text{sgn } x_n^{(i)}\ |x_n^{(i)}|^{\rho-1}/\|x_n\|^{\rho-2}$

$= \text{sgn } x_n^{(i)}\ |x_n^{(i)}|^{\rho-1}$, since $\|x_n\| = 1$, and so

$f_n^{(i)} \to f^{(i)} = \text{sgn } x^{(i)}\ |x^{(i)\rho-1}$ for all i.

Again by [Taylor, 1, p. 210] $\{f_n\}$ is $\sigma(\ell\rho',\ell\rho)$-convergent to

$f = (f^{(i)}, \ldots, f^{(n)}, \ldots)$, and $f(x) = \Sigma|x^{(i)}|^\rho$. While

$\|f\|\ \|x\| = (\underset{i}{\Sigma}(x^{(i)}|^{\rho-1})^{\rho/\rho-1}\ \ \ \ ^{1/}(\underset{i}{\Sigma}|x^{(i)}|^\rho)^{1/\rho}$

$= \underset{i}{\Sigma}|x^{(i)}|^\rho.$

Therefore $f(x) = \|f\|\ \|x\|$.   //

Combining this lemma with Corollary .2.3.1. we have

proved the following theorem:

.2.5.   THEOREM. *For compact* T ε B($\ell\rho$), 1 < ρ < ∞, *we have*

extreme edge W(T) $\subseteq$ W(T),

further if 0 ε W(T) *then* coW(T) = $\overline{co}$ W(T).

We now restrict our attention to Hilbert spaces. As might

be expected, the convexity of the spatial numerical ranges leads

to more precise results than those of the last Theorem.

2.6.   LEMMA. *A Hilbert space* H *has the wusc-property.*

Proof. If $\{x_n\}$ converges in σ(H,H') to x then $(x_n,y) \to (x,y)$

for all y ε H. Therefore $(y, x_n) \to (y,x)$ for all y ε H and so

the functionals $f_n = (\cdot, x_n)$ ε D(x_n) converge in σ(H',H) to

f = $(\cdot,x)$, further f(x) = (x, x) = $\|x\|^2$.   //

The next theorem is a joint result of J. R. Giles and the

author [de Barra, Giles and Sims, 1].

.2.7.   THEOREM. *For a Hilbert space* H *and compact* T ε B(H) *we have*

        i)   *If* 0 ε W(T) *then* W(T) *is closed.*

        ii) *If* 0 $\notin$ W(T) *then* 0 *is an extreme point of* $\overline{W(T)}$, *and*

$\overline{W(T)} \setminus$ W(T) *consists at most of two line segments in* δW(T), *the*

boundary of W(T), *which contain* 0 *but no other extreme points of* $\overline{W(T)}$.

Proof. By Lemmas 2.2., 2.3. and 2.6., W(T) contains all the non-zero extreme points of $\overline{W(T)}$. The result now follows from the convexity of W(T).   //

2.7.1. **Corollary.** *If* T *is a finite rank operator on a Hilbert space* H, *then* W(T) *is closed.*

Proof. KerT ≠ {0} otherwise H would be in one to one correspondence with a finite dimensional space R(T). Hence $0 \in p\ \sigma(T) \subseteq W(T)$ and the result follows from Theorem 2.7.i).   //

2.7.2. **Corollary.** *If* T *is a compact operator on a non-separable Hilbert space* H, W(T) *is closed.*

Proof. Since R(T) is separable, it follows that Ker(T) ≠ {0} and so $0 \in W(T)$, Theorem 2.7.1) now completes the proof.   //

The author used the operator T on l_2 defined by $T(x_1, x_2,\ldots,x_n,\ldots.) = (ix_1, \tfrac{1}{2}x_2,\ldots,1/nx_n,\ldots.)$, which has $\overline{W(T)} \setminus W(T)$ equal to the line segment joining 0 and i, to illustrate the exceptional behaviour of W(T) when $0 \notin \overline{W(T)}$ for a compact operator T. A similar example was noted by A. M. Sinclair, who observed that examples illustrating the exceptional behaviour of W(T) could easily be constructed using the properties of compact normal operators. Such constructions follow readily from the next lemma due to G. de Barra [de Barra, Giles and Sims, 1].

2.9. LEMMA. *For a compact normal operator* T *on a Hilbert space*
W(T) = co(pσ(T)), *the convex hull of the point spectrum of* T.

Proof. Clearly co(pσ(T)) $\subseteq$ W(T) so it is sufficient to show that
W(T) $\subseteq$ co(pσ(T)). Suppose there exists a λ ε W(T) \ co(pσ(T)) then
0 $\notin$ co(pσ(T)) - λ ≡ A, say. Then for any z ε A, θ ≤ arg z ≤ θ + π
for some θ, so for any z ε $e^{-i\theta}$A, Im z ≥ 0. Now there exists
an x, $\|x\|$ = 1 such that λ = (Tx,x). By [Halmos, 1, p. 86] we may
choose an orthonormal basis for the space such that if
x = $\sum_{1}^{\infty} \alpha_n e_n$, Tx = $\sum_{1}^{\infty} \mu_n \alpha_n e_n$ where $\{\mu_n\} \subseteq$ pσ(T). So

$\lambda = \sum_{1}^{\infty} \mu_n |\alpha_n|^2$ where $\sum_{1}^{\infty} |\alpha_n|^2$ = 1, and $\sum_{1}^{\infty} e^{-i\theta}(\mu_n - \lambda) |\alpha_n|^2 = 0$.

If we write $e^{-i\theta}(\mu_n - \lambda) = \gamma_n + i\delta_n$ then $\sum_{1}^{\infty} \gamma_n |\alpha_n|^2 + i \sum_{1}^{\infty} \delta_n |\alpha_n|^2 = 0$

where $\delta_n \geq 0$ for all n. We may assume that $\alpha_n \neq 0$ for each n so
$\delta_n = 0$ for all n. We may choose n_1 and n_2 where γ_{n_1} and γ_{n_2}
have opposite signs to get λ ε $[\mu_{n_1}, \mu_{n_2}]$. But this contradicts
the convexity of co(pσ(T)). Therefore, W(T) $\subseteq$ co(pσ(T)).   //

2.9.1. Corollary. *For a compact normal operator on a Hilbert*
space $\overline{W(T)}$ \ W(T) = $\overline{co}\sigma(T)$ \ copσ(T)

Proof. The proof is immediate from Lemma 1.6.5. and the
above theorem.   //

2.10. **Examples.** Define operators T_1, T_2, T_3 on ℓ_2 by

$$T_1(x_1, x_2, \ldots, x_n, \ldots) = (\tfrac{1}{2}x_1, x_2, ix_3, \ldots, i/n\text{-}2\; x_n, \ldots)$$

$$T_2(x_1, x_2, \ldots, x_n, \ldots) = (x_1, ix_2, 1\text{+}i/2\; x_3, \ldots, 1\text{+}i/n\text{-}1\; x_n, \ldots)$$

$$T_3(x_1, x_2, \ldots, x_n, \ldots) = (x_1, ix_2, \ldots, 1/\tfrac{1}{2}n\text{-}2\; x_{2n\text{-}1}, i/2n\text{-}1\; x_{2n\text{-}1}, \ldots)$$

From Lemma 1.6.5. for $n = 1, 2, 3$ we have that $\overline{W(T_n)} = \overline{co}\sigma(T_n)$ is the triangle with verticies 0, 1 and i. Using Theorem 2.9. it is easily seen that $\overline{W(T_1)} \setminus W(T_1) = [0, \tfrac{1}{2})$

$$\overline{W(T_2)} \setminus W(T_2) = [0, i) \cup [0, 1)$$

and $\quad\quad \overline{W(T_3)} \setminus W(T_3) = \{0\}.$

where we have used an obvious notation for straight line segments in the complex plane.

These examples were first given by J. R. Giles and illustrate clearly the type of exceptional behaviour of $W(T)$ for a compact operator T when $0 \notin W(T)$.   //

The next example, obtained by G. de Barra and independently by the author, illustrates the same type of exceptional behaviour of $W(T)$ for a non-normal compact operator, and shows that Corollary 2.9.1. is not true for an arbitrary compact operator on a Hilbert space.

2.11. **Example.** Define T on ℓ_2 by

$$T(x_1, x_2, .., x_n, ...) = (x_1 + x_2, x_2, x_3, ..., 1/n\text{-}2\ x_n, ...)$$

This can be regarded as the direct sum of operators T_1 on ℓ_2^2 and

T_2 on ℓ_2 defined by

$$T_1(x_1, x_2) = (x_1 + x_2, x_2)$$

and $T_2\ (x_3, ..., x_n, ...) = (x_3, \tfrac{1}{2}x_4, .., 1/n\text{-}2\ x_n, ...).$

Now $\sigma(T_1) = \{1\}$ and $W(T_1)$ is the closed disk with centre 1 and

radius $\tfrac{1}{2}$, while $\sigma(T_2) = \{0, 1, \tfrac{1}{2}, ..., 1/n, ...\}$ and $W(T_2) = (0,1)$.

By [Halmos, 1, p. 113] we can write $W(T)$ as $W(T) = \text{co}\ [W(T_1) \cup W(T_2)]$,

from which we see that $\overline{W(T)} \setminus W(T)$ consists of the two half-open line

 segments containing 0 and tangent to the disk $W(T_1)$. However

$\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$ is real and $\overline{\text{co}}\sigma(T) \setminus \text{co}\sigma(T) = \{0\}$.   //

Returning to the general case of compact operators

on a Banach space, one may well ask whether the extreme edge of $W(T)$

is always contained in $W(T)$ for a compact operator T. The following

example, suggested by Giles, shows this need not be the case even

for finite rank operators.

2.12. **Example.** Define T on c_o by

$$T(x_1, x_2, .., x_n, ...) = (\sum_{n=1}^{\infty} x_n/2^n, 0, 0, ...).$$

Then

$$\|T\| = \sup\{|\Sigma\ x_n/2^n| : |x_n| \leq 1 \text{ for all } n\} \leq \Sigma \tfrac{1}{2}^n = 1.$$

Now choose the sequence $x^{(n)}$ such that

$$x_i^{(n)} = 1 \quad i \leq n$$

$$\phantom{x_i^{(n)}} = 0 \quad i > n,$$

then the function $f \in c_o{}'$, represented by $f = (1, 0, 0, \ldots.)$ is such that $f \in D(x^{(n)})$, for all n, and so $f(Tx^{(n)}) \in W(T)$, for all n, but $f(Tx^{(n)}) \to 1$ so $1 \in \overline{W(T)}$ and since $\|T\| \leq 1$, $1 \in$ extreme edge of $W(T)$. However $1 \notin W(T)$.   //

It would be of some interest to characterize those Banach spaces which have the wusc-property. We have already seen that Hilbert spaces and $\ell\rho$-spaces, for $1 < \rho < \infty$, do have this property. The space ℓ_1 also has the wusc-property, since if $\{x_n\}$, $\|x_n\| = 1$, is a sequence converging in $\sigma(\ell_1, \ell_1{}')$ to x and $\{f_n\}$, $f_n \in D(x_n)$, converges in $\sigma(\ell_1', \ell_1)$ to f, then by [Taylor, 1, p. 210] $\|x_n - x\| \to 0$ and so $\|x\| = 1$ and

$$|1 - f(x)| = |f_n(x_n) - f(x)|$$

$$\leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)|$$

$$\leq \|x_n - x\| + |(f_n - f)(x)| \to 0.$$

However $C[0,1]$, the space of continuous functions on the compact interval $[0,1]$ with the supremum norm, has not got the wusc-property.

2.13.   **Example.** X=C[0,1] does not have the wusc-property.

Proof. Define the sequence $\{x_n\}$ of elements of X by

$$x_n(t) = (1 - 1/n)(1 - 4t) \qquad \text{for } 0 \leqslant t \leqslant 1/4$$

$$= 0 \qquad\qquad\qquad \text{for } 1/4 < t \leqslant 1 - 1/2n$$

$$= 4n(t - 1) + 2 \qquad \text{for } 1-1/2n \leqslant t \leqslant 1 - 1/4n$$

$$= 4n(1 - t) \qquad\qquad \text{for } 1 - 1/4n < t \leqslant 1$$

Then for any fixed $t \in [0,1]$ we see that

$$x_n(t) \to 1 - 4t \quad \text{for } 0 \leqslant t \leqslant 1/4$$

$$\to 0 \qquad \text{for } 1/4 \leqslant t \leqslant 1$$

$$= 0, \text{ for all } n, \text{ for } t = 1.$$

So by [Taylor, 1, p. 210] x_n is $\sigma(X,X')$-convergent to x_0 where

$$x_0(t) = 1 - 4t \text{ for } 0 \leqslant t \leqslant 1/4$$

$$= 0 \qquad \text{for } 1/4 < t \leqslant 1.$$

Also since $x_n(1 - 1/4n) = 1$ and $x_0(0) = 1$ we see that

$$\|x_n\| = \|x\| = 1.$$

Now from the known representation of X', [Taylor, 1, Theorem 4.32-C]

we may take $f_n \in D(x_n)$ to be represented by the step function

$$[f_n](t) = 0 \text{ for } 0 \leqslant t < 1 - 1/4n$$

$$= 1 \text{ for } 1 - 1/4n \leqslant t \leqslant 1.$$

The $\sigma(X',X)$ limit of f_n is f_0 where

$$[f_0](t) = 0 \text{ for } 0 \leqslant t < 1$$

$$= 1 \text{ for } t = 1,$$

since for any $x \in X$

$$f_n(x) = \int_0^1 x(t) \, d[f_n](t) = x(1 - 1/n)$$

$$\rightarrow x(1) \text{ since } x \text{ is continuous}$$

$$= \int_0^1 x(t) d[f_0](t).$$

Also $\|f_0\| = 1$, but

$$f_0(x_0) = \int_0^1 x_0(t) \, d[f_0](t) = x_0(1) = 0$$

so $f_0(x_0) \neq \|f_0\| \ \|x_0\|$ //

In this section we have examined the closure properties of the spatial numerical range of a compact operator. We introduced a weak upper semi-continuity property for Banach spaces and proved that for a compact operator T, over a reflexive Banach space with the wusc-property, $co(W(T)) = \overline{co}(W(T))$ whenever $0 \in W(T)$. In particular we showed this to be true for ℓ_p-space $(1 < p < \infty)$ and a Hilbert space. In the latter case we showed that $W(T)$ is closed if $0 \in W(T)$, clearly this is also a necessary condition. We then gave several examples to illustrate the different types of exceptional behaviour possible for $\overline{W(T)} \setminus W(T)$. The section was concluded with an example of a Banach space not having the wusc-property.

.3. OPERATORS WHICH ATTAIN THEIR NUMERICAL RADIUS

We have seen that any compact operator T, on a Hilbert space, with $0 \in W(T)$ has $W(T)$ closed, while, for several classes of Banach spaces, any compact operator T has the extreme edge of $W(T)$ in $W(T)$. A weaker condition than, either $W(T)$ being closed, or containing the extreme edge of $W(T)$ is that $v(T)$ be attained.

3.1. **Definition.** For a normed linear space E, we say the *numerical radius of* $T \in B(E)$ *is attained* if there exists $x \in E$, $\|x\| = 1$ and $f \in D(x)$ such that

$$|f(Tx)| = v(t).$$

Similarly we say an operator T attains its norm if there exists $x \in E$, $\|x\| = 1$, such that $\|Tx\| = \|T\|$.

In this section we examine and partially solve, the problem of determing those operators which attain their numerical radius.

In our investigation it is often sufficient to consider operators T with $v(T) = 1 \in \overline{W(T)}$, for if $T \in B(E)$ there exists $\lambda \in \overline{W(T)}$ with $|\lambda| = v(T)$ and then $T_1 = \lambda^{-1}T$ has $v(T_1) = 1 \in \overline{W(T_1)}$.

A Banach space X is said to be H_L [M. M. Day, 1] if whenever $\{x_n\}$ is $\sigma(X,X')$-convergent to x and $\|x_n\|$ converges to $\|x\|$, then $\{x_n\}$ is norm convergent to x.

The next proposition extends a result of Hilbert [Riesz and Sz.-Nagy, 1 p. 232]

3.2. **Proposition.** *If* X *is a smooth, reflexive,* H_L *Banach Space and* T ε B(X) *is compact with* $1 = v(T) = \|T\| \varepsilon \overline{W(T)}$, *then* $v(T)$ *is attained.*

Proof. There exists a sequence $\{x_n\}$, $\|x_n\| = 1$, and $f_n \varepsilon D(x_n)$ such that $f_n(Tx_n)$ converges to 1. Since X is reflexive there is a subsequence of the $\{x_n\}$, $\{x_m\}$ such that $\{x_m\}$ is $\sigma(X,X')$-convergent to $x \varepsilon X$ with $\|x\| \leqslant 1$. Therefore since T is compact $\|Tx_n - Tx\| \to 0$ and so

$$\left| \|Tx_n\| - \|Tx\| \right| \to 0 \quad \text{or} \quad \|Tx_n\| \to \|Tx\|$$

but $1 = \|T\| \geqslant \|Tx_m\| \geqslant \left| f_m(Tx_m) \right| \to 1$ so

$\|Tx\| = 1$ and therefore $\|x\| = 1$, otherwise if $\|x\| < 1$,

$\|T\| \geqslant \|Tx\| / \|x\| = \|x\|^{-1} > 1$.

Since X is H_L, we have $\|x_m - x\| \to 0$ and so by the smoothness of X

$$f_m(Tx_m) \to f_x(Tx) \varepsilon W(T)$$

therefore $1 = f_x(Tx) \varepsilon W(T)$ or $v(T)$ is attained.   //

That this proposition is not true for an arbitrary Banach space, even when T is taken to be of finite rank, is shown by Example 2.12.

The next lemma is due to Lumer [1].

3.3. LEMMA. *If* X *is a uniformly convex Banach space and*
T ϵ B(H) *has* $v(T) = \|T\|$, *then* $\rho(T) = \|T\|$.

Proof. We may assume $v(T) = \|T\| = 1 \epsilon \overline{W(T)}$, so there exists $\{x_n\}$,
$\|x_n\| = 1$, and $f_n \epsilon D(x_n)$ such that $f_n(Tx_n) \to 1$. Then

$$2 \geqslant \|T\| + \|x_n\| \geqslant \|Tx_n\| + \|x_n\|$$

$$\geqslant \|Tx_n + x_n\|$$

$$\geqslant |f_n(Tx_n + x_n)|$$

$$= |f_n(Tx_n) + 1| \to 2.$$

therefore $\|Tx_n + x_n\| \to 2$ and so by the uniform convexity of X,
$\|Tx_n - x_n\| \to 0$ or $1 \epsilon \sigma(T)$ and $1 \leqslant \rho(T) \leqslant \|T\| = 1$.   //

The next example shows that Lemma 3.3. is not true
for general Banach spaces. However, as we will show, it can be
extended for certain classes of operators, in particular operators
which attain their numerical radius.

3.4. **Example.** Let $X = \ell_\infty^2$, the space of ordered pairs of complex
numbers with the supremum norm, then the operator T on X, represented
by the matrix $\begin{bmatrix} \alpha & \bar{\beta} \\ 0 & 0 \end{bmatrix}$, where $\alpha, \beta \neq 0$, has $\rho(T) \neq v(T) = \|T\|$

$$\|T\| = \sup\{\|(\alpha\lambda_1 + \beta\lambda_2, 0)\| : \max \{|\lambda_1|, |\lambda_2|\} = 1$$

$$= \sup |\alpha\lambda_1 + \beta\lambda_2| \text{ where } \max \{|\lambda_1|, |\lambda_2|\} = 1$$

$$= |\alpha| + |\beta|.$$

Now choose $x \in X$ as $x = (\text{sgn } \alpha, \text{sgn } \beta)$, then $\|x\| = 1$

and for $f = (\text{sgn } \bar{\alpha}, 0)$, $f \in D(x)$, as $\alpha, \beta \neq 0$,

$$f(Tx) = \text{sgn } \bar{\alpha}(|\alpha| + |\beta|) \quad \in W(T)$$

or $v(T) \geq |\alpha| + |\beta|$

therefore $\|T\| = v(T) = |\alpha| + |\beta|$.

However $\sigma(T) = \{0, \alpha\}$

so $\rho(T) = |\alpha| < v(T)$, since $\beta \neq 0$.   //

The next lemma extends Lemma 3.3. to rotund Banach spaces for operators which attain their numerical radius. It is also proved by Bonsall and Duncan [1].

3.5. **LEMMA.** *If* X *is a rotund Banach space and* $T \in B(X)$ *has* $v(T) = \|T\|$ *and* $v(T)$ *attained, then* $\rho(T) = \|T\|$.

Proof. We may assume $1 = v(T) = \|T\| \in W(T)$, so there exists $x \in X$, $\|x\| = 1$, and $f_x \in D(x)$ with $1 = f_x(Tx) \leq \|Tx\| \leq 1$, so $f_x \in D(Tx)$ and hence by the rotundity of X, $x = Tx$ and $1 \in \rho\sigma(T)$ therefore $1 \leq \rho(T) \leq v(T) = 1$.   //

Combining this lemma with Proposition 3.2. we could obtain a slight extension of Lemma 3.3. for compact operators.

We now obtain equivalent conditions for $\dot{v}(T)$ to be attained, where T is an operator over a rotund Banach space which permits an hermitian decomposition. I am indebted to A. M. Sinclair, who suggested the following result to me.

3.6.   LEMMA. *For a rotund Banach Space* X, $x \in X$ ($\|x\| = 1$),

$f \in D(x)$, *and* $T \in B(X)$ *with* $v(T) = 1$ *and* $T = P + iQ$ *where*

P, Q *are hermitian, the following are equivalent:*

   i) $f(Tx) = 1$

   ii) $f(Px) = v(P) = 1$

   iii) $Px = x$

Proof. $1 = f(Tx) = \text{Re } f(Tx) = f(Px) \leqslant v(P) = \sup \text{Re } W(T) \leqslant v(T) = 1$,

so i) implies ii). That ii) implies iii) follows from Lemma 3.5.,

since P is hermitian and so by Sinclair [1], $v(P) = \|P\|$.

         If $Px = x$ then

$1 = f(Px) = \text{Re } f(Tx) \leqslant |f(Tx)| \leqslant v(T) = 1$, so iii) implies i).   //

.6.1.   **Corollary.** *For* X, x, f, T *as in Lemma 3.6., if*

$f(Tx) = 1$, *then* $[f(Qy)]^2 + [f(Py)]^2 \leqslant v(P)^2$ *for all* $y \in X$,

$\|y\| = 1$, *and* $f \in D(y)$. *In particular* $f(Qx) = 0$.

Proof. Since $f(Ty) \in W(T)$ we have by Lemma 3.6 ii) that

$$v(P)^2 = v(T)^2 \geqslant |f(Ty)|^2$$

$$= |f(Py) + if(Qy)|^2$$

$$= [f(Py)]^2 + [f(Qy)]^2.$$

Hence when $y = x$ we have by Lemma 3.6. ii) that

$v(P)^2 = v(P)^2 + [f(Qx)]^2$ or $f(Qx) = 0$.   //

Bishop and Phelps [1] have shown that the set of functionals on a Banach space which attain their norm is dense in the dual space while J. Lindenstrauss [1] has proved that the set of operators which attain their norm, is dense in all the operators on a reflexive Banach space. It is natural, therefore, to enquire whether the set of operators which attain their numerical radius is dense among all the operators on a Hilbert space. While this remains an open question, we now proceed to prove that the hermitian operators which attain their numerical radius, are dense among all the hermitian operators on a Hilbert space. This is readily seen to be a necessary condition for the set of operators, which attain their numerical radius, to be dense among all the operators. We use the result on hermitian operators to obtain an alternative proof of Lindenstrauss' result for the particular case of operators on a Hilbert space.

We will in fact, give two alternative proofs that for hermitian Q, on a Hilbert space, and $\epsilon > 0$ there exists hermitian Q_1 which attains its numerical radius and $\|Q - Q_1\| < \epsilon$. Each proof leads to slightly different conclusions. The first identifies the element x, $\|x\| = 1$, at which $|(Q_1 x, x)| = v(Q_1)$, while the second allows the form of $Q_1 - Q$ to be explicitly determined.

Lemma 3.6. shows that for an hermitian operator Q, with $v(Q) = 1 \ \epsilon \ \overline{W(Q)}$, the numerical radius is attained if and only if 1 is an eigenvalue of Q. An arbitrary hermitian operator may have no

eigenvalues (for example, a Toeplitz operator), one way in which this difficulty may be overcome is given by the next lemma.

3.8. LEMMA. *If* H *is a Hilbert space,* $\epsilon > 0$, *and* $Q \in B(H)$ *is hermitian with* $\|Q\| = 1 \in \overline{W(Q)}$, *then for any* $x \in H$, $\|x\| = 1$, *such that* $1 \geq (Qx,x) \geq 1 - \epsilon^2/16$, *there exists an hermitian* $Q_1 \in B(H)$ *such that* $\|Q - Q_1\| \leq \epsilon$ *and* x *is an eigenvector of* Q_1 *corresponding to the eigenvalue* λ *where* $\|Q_1\| - \lambda \leq \epsilon$.

Proof. Let $\delta = \epsilon/4$, choose $x \in H$, $\|x\| = 1$ such that
$1 \geq (Qx,x) \geq 1 - \delta^2$ then
$$\|Qx + x\| \geq |(Qx + x,x)| = |(Qx,x) + 1| \geq 2 - \delta^2.$$
So by the parallelogram rule
$$\|x - Qx\|^2 = 2\|Qx\|^2 + 2\|x\|^2 - \|Qx + x\|^2 \leq 4 - (2 - \delta^2)^2 = 4\delta^2 - \delta^4 \leq 4\delta^2$$
that is $\|x - Qx\| \leq 2\delta$.

Let $x_o = x - Qx$, $\|x_o\| \leq 2\delta$ and construct $K_1 \in B(H)$ by
$K_1(y) = (y,x)x_o$ for all $y \in H$, then $\|K_1\| = \|x_o\| \leq 2\delta$.
Let $K = K_1 + K_1^*$ then K is hermitian and $\|K\| \leq 4\delta = \epsilon$.
Now define $Q_1 = Q + K$ then
$$\|Q_1 - Q\| = \|K\| \leq \epsilon.$$
Further
$$Q_1(x) = Q(x) + K(x)$$
$$= Q(x) + x_o + (x,x_o)x$$
$$= Q(x) + x - Qx + (x,x_o)x$$
$$= (1 + (x,x_o))x.$$

That is $1 + (x,x_o)$ is an eigenvalue of Q_1 with eigenvector x.

Now $(x,x_o) = (x,x - Qx) = 1 - (x,Qx)$

therefore, since $1 \geqslant (Qx,x) = (x,Qx) \geqslant 1 - \delta^2$,

we have $0 \leqslant (x,x_o) \leqslant \delta^2$ and so $1 \leqslant \lambda \leqslant 1 + \delta^2$ and

$\|Q_1\| - \lambda \leqslant 1 + \epsilon - 1 = \epsilon.$ //

3.9. THEOREM. *For a Hilbert space* H, $\frac{1}{2} > \epsilon > 0$, *and hermitian*

$Q \in B(H)$, *if* $x \in H$, $\|x\| = 1$, *is such that*

$\|Q\| \geqslant |(Qx,x)| \geqslant (1 - \epsilon^2/256)\|Q\|$ *then there exist hermitian* $K \in B(H)$

such that $\|Q - K\| \leqslant \epsilon$ *and* $|(Kx,x)| = v(K)$.

Proof. We may assume without loss of generality that $\|Q\| = 1 \in \overline{W(Q)}$.

Then by Lemma 3.8. we can find Q_1 with $\|Q_1 - Q\| \leqslant \epsilon/4$, and with x

as an eigenvector corresponding to the eigenvalue λ_1 where

$\|Q_1\| - \lambda_1 \leqslant \epsilon/4$

Let $Q_2 = Q_1/\|Q_1\|$ then

$\|Q_2 - Q\| \leqslant \epsilon/2$ while $\lambda_2 = \lambda_1/\|Q_1\|$ is an eigenvalue of Q_2, with

eigenvector x, and $1 - \lambda_2 \leqslant \epsilon/2$.

Now there exists a continuous real valued function f on $\sigma(Q_2)$ such

that $f(\lambda_2) = 1 = \text{Max } \{|f(\alpha)| : \alpha \in \sigma(Q_2)\}$

and $|\alpha - f(\alpha)| \leqslant |\lambda_2 - f(\lambda_2)|$ for all $\alpha \in \sigma(Q_2)$.

So by the functional calculus, (see Section 1.6.) we can define

$K = f(Q_2)$ where:

i) K will be hermitian, by Lemma 1.6.7.

ii) $\|Q_2 - K\| = \|(I - f)Q_2\|$

$\qquad = \|(I - f)\|$

$\qquad = \text{Max } \{|\alpha - f(\alpha)| : \alpha \in \sigma(Q_2)\}$

$\qquad = |\lambda_2 - f(\lambda_2)|$

$\qquad = 1 - \lambda_2$

$\qquad \leqslant \varepsilon/2$

iii) $\|K\| = \|f\|$

$\qquad = \text{Max } \{|f(\alpha)| : \alpha \in \sigma(Q_2)\}$

$\qquad = 1$

iv) Since x is an eigenvector of Q_2 with eigenvalue λ_2, x is an eigenvector of K with eigenvalue $f(\lambda_2) = 1$.

Combining iii) and iv) with Lemma 3.6., we have that $(Kx,x) = v(K)$ and by ii) $\|Q - K\| \leqslant \|Q - Q_2\| + \|Q_2 - K\|$

$\qquad \leqslant \varepsilon/2 + \varepsilon/2$

$\qquad = \varepsilon.$ $\qquad\qquad\qquad //$

3.9.1. **Corollary.** *For a Hilbert space, the set of hermitian operators, which attain their numerical radius, is dense in the set of all hermitian operators.*

Proof. The proof is immediate from Theorem 3.9. $\qquad //$

The possibility of using the functional calculus as in Theorem 3.9., to prove Corollary 3.9.1., was first suggested to me by A. M. Sinclair.

Since the functional calculus can be developed for a normal

operator on a Hilbert space, if we had a counterpart to Lemma 3.8.

for normal operators, the result of Theorem 3.9. would then extend

in a straight forward way to normal operators. This may well be

worth further investigation.

The second proof of Corollary 39.1. uses the following

theorem due to Weyl [Halmos, 1, problem 143.].

3.10. **THEOREM.** *If* X *is a Banach Space and* T, T_1 ε B(X) *with* $T - T_1$

a compact operator, then $\sigma(T) \setminus \sigma(T_1) \subsetneq p\sigma(T)$.

Proof. If λ ε $\sigma(T) \setminus \sigma(T_1)$ then $V = T - \lambda I$ is singular while

$V_1 = T_1 - \lambda I$ is regular.

Now $V = V_1 + (V - V_1) = V_1(I + V_1^{-1}(V - V_1))$

so, since V is singular, we see that $(I + V_1^{-1}(V - V_1))$

must also be singular, or -1 ε $\sigma(V_1^{-1}(V - V_1))$.

But $V - V_1 = T - T_1$ is compact, and so therefore is $V_1^{-1}(V - V_1)$.

Hence -1 is an eigenvalue of $V_1^{-1}(V - V_1)$ and so there exists

x ε X, $\|x\| = 1$ such that $V_1^{-1}(V - V_1)x = -x$ or $Vx = 0$.

Therefore $(T - \lambda I)x = 0$ or λ ε $p\sigma(T)$.

3.10.1. **Corollary.** *For a Banach space* X, *the set of* $T \in B(X)$, *such that there exist* $\lambda \in p\sigma(T)$ *with* $|\lambda| = \rho(T)$, *is dense in* $B(X)$.

Proof. For any $T \in B(X)$ and $\varepsilon > 0$ there exists $\lambda \in \pi \sigma(T)$ such that $|\lambda| = \rho(T)$, since $\delta\sigma(T) \subseteq \pi\sigma(T)$ and $\sigma(T)$ is closed. Hence there exists $\{x_n\}$:, $\|x_n\| = 1$, such that $\|Tx_n - \lambda x_n\| \to 0$. Choose x_o, $\|x_o\| = 1$, to be such that $\|Tx_o - \lambda x_o\| \leqslant \varepsilon/2$ and let $y_o = (1 + \varepsilon/2|\lambda|)\lambda x_o - Tx_o$ then

$\|y_o\| \leqslant \|\lambda x_0 - Tx_0\| + \varepsilon/2 \leqslant \varepsilon$.

Now define $T_1 \in B(X)$ by

$T_1(x) = f(x)y_o + T(x)$ where $f \in D(x_o)$, then $T - T_1$ is compact and $\|T - T_1\| = \|y_o\| \leqslant \varepsilon$, further $T_1(x_o) = Tx_o + y_o = (1 + \varepsilon/2|\lambda|)\lambda x_o$, so $(1 + \varepsilon/2|\lambda|)\lambda$ is an eigenvalue of T and

$\rho(T_1) \geqslant |(1 + \varepsilon/2|\lambda|)\lambda| > |\lambda| = \rho(T)$.

Hence if $\alpha \in \sigma(T_1)$ is such that $|\alpha| = \rho(T_1)$ then $\alpha \notin \sigma(T)$ and so by Theorem 3.10.

$\qquad \alpha \in \sigma(T_1) \setminus \sigma(T) \subseteq p\sigma(T_1)$.　　　　　//

Less precisely, Corollary 3.10.1. states that the set of operators, which attain their spectral radius at eigenvalues, is dense among all the operators over a Banach space.

Restricting our attention to Hilbert spaces we have the following theorem:

3.11. THEOREM. *For a Hilbert space* H, $\varepsilon > 0$, *and hermitian* $Q \in B(H)$, *there exists hermitian* $Q_1 \in B(H)$ *with* $v(Q_1)$ *attained,* $\|Q - Q_1\| < \varepsilon$ *and* $Q_1 - Q$ *is a hermitian rank one operator.*

Proof. Assume, without loss of generality, that $v(Q) = 1 \in \overline{W(Q)}$. Then there exists $x \in H$, $\|x\| = 1$, such that $(Qx,x) > 1 - \varepsilon$. Let $Q_1 y = Qy + \varepsilon(y,x)x$ for all $y \in H$, then $Q_1 - Q$ is a positive hermitian rank one operator and $\|Q_1 - Q\| \leq \varepsilon$. Further $(Q_1 x,x) > 1 - \varepsilon + \varepsilon = 1$ so $\rho(Q_1) = v(Q_1) > 1$ and $\rho(Q_1) \in \sigma(Q_1)$ but $\rho(Q_1) \notin \sigma(Q)$, as $\rho(Q) = v(Q) = 1$, so $\rho(Q_1) \in \sigma(Q_1) \setminus \sigma(Q) \subseteq p\sigma(Q_1)$, by Theorem 3.10., and hence by Lemma 6. $v(Q_1)$ is attained.   //

Before proceeding to the proof of Lindenstrauss' result, for operators on Hilbert spaces, we need the following lemma:

3.12. LEMMA. *The operator* Q_1 *of Theorem* 3.11 *can be chosen such that* $R(Q_1) \subseteq (\text{Ker } Q)^{\perp}$.

Proof. It is sufficient to show that the x in the proof of Theorem 3.11 could have been chosen so that $x \in (\text{Ker } Q)^{\perp}$.

Hence let x be as in Theorem 3.11, then $x = x_0 - x_1$ for some $x_0 \in \text{Ker } Q$ and $x_1 \in \text{Ker}(Q)^{\perp}$,

So $1 = \|x\|^2 = (x,x)$

$$= (x_o + x_1, x_o + x_1)$$
$$= (x_o,x_o) + (x_o,x_1) + (x_1,x_o) + (x_1,x_1)$$
$$= \|x_o\|^2 + \|x_1\|^2, \text{ since } (x_1,x_o) = 0.$$

Therefore $\|x_1\| \leqslant 1$. Now

$$1 - \varepsilon < (Qx,x) = (Q(x_o + x_1), (x_o + x_1))$$
$$= (Qx_1, x_o + x_1)$$
$$= (Qx_1, x_1), \text{ as } R(Q) \subseteq (\text{Ker } Q)^\perp$$

so $x_1 \neq 0$ and letting $\underline{x}_1 = x_1/\|x_1\|$ we have

$1 - \varepsilon \leqslant (Q\underline{x}_1, \underline{x}_1)$. So in the proof of Theorem 3.11. x may be

replaced by $\underline{x}_1$ as required.   //

3.13. **THEOREM.** *For a Hilbert space H, $T \in B(H)$ and $\varepsilon > 0$,*
there exists $T_1 \in B(H)$ and $x \in H$, $\|x\| = 1$, such that $\|T - T_1\| \leqslant \varepsilon$
and $\|T_1 x\| = \|T_1\|$.

Proof. Any $T \in B(H)$ admits the polar decomposition $T = UP$ where P
is a positive hermitian operator, U is a partial isometry, that is
$\|Ux\| = \|x\|$ for all $x \in (\text{Ker } U)^\perp$, and Ker P = Ker U [Halmos, 1,
problem 105]. So for any $\varepsilon > 0$, Theorem 3.11. and Lemma 3.12.
show there exists an hermitian (in fact positive hermitian) operator
P_1 and $x \in H$, $\|x\| = 1$ such that $\|P - P_1\| < \varepsilon$, $(\text{Ker } P)^\perp \supseteq R(P_1)$ and
$\|P_1\| \geqslant \|P_1 x\| \geqslant (P_1 x, x) = v(P_1) = \|P_1\|$, or $\|P_1 x\| = \|P_1\|$.
Let $T_1 = UP_1$, then

$$\|T - T_1\| \leqslant \|U\| \ \|P - P_1\| \leqslant \epsilon \text{ and}$$

$$\|T_1 x\| = \|UP_1 x\| = \|P_1 x\| = \|P_1\| \text{ but}$$

$$\|T_1\| \leqslant \|P_1\| \text{ so } T_1 \text{ attains its norm at } \mathbf{x}. \qquad\qquad //$$

From his proof Lindenstrauss observed that the T_1 of the above theorem could be chosen so that $T - T_1$ is compact. For operators on a Hilbert space our proof yields an even stronger result.

3.13.1. Corollary. *The T_1 of the above theorem may be chosen so that $T_1 - T$ is a rank one operator.*

Proof. In the same notation as used in the proof of Theorem 3.13. we see that $T_1 - T = U(P_1 - P)$ where, by Theorem 3.11, $P_1 - P$ is a rank one operator, and so therefore is $T_1 - T$. $\qquad //$

We have seen that for a class of Banach spaces, compact operators attain their numerical radius. For an operator, which permits an hermitian decomposition on a rotund space, we obtained necessary and sufficient conditions for the numerical radius to be attained. We then gave two proofs that the hermitian operators which attain their numerical radius are dense among all the hermitian operators on a Hilbert space. The first proof enabled us to determine the element, at which the numerical radius is attained, while the second

revealed the exact nature of the perturbation necessary to produce an operator which attained its numerical radius. Using this latter result, we gave a new proof of Lindenstrauss' result, that the set of operators which attain their norm is dense among all the operators, for the particular case of operators on a Hilbert space. The general question, of whether the set of operators on a Hilbert space, which attain their numerical radius, is dense among all the operators, is still an open question and may well be worth further investigation.