ON NUMERICAL RANGE AND ITS APPLICATION TO BANACH ALGEBRA

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CANDIDATE'S CERTIFICATE

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

(Signed) Brailey Sims

ABSTRACT

The spatial numerical range of an operator on a normed linear space and the algebra numerical range of an element of a unital Banach algebra, as developed by G. Lumer and F. F. Bonsall, are considered and the theory of such numerical ranges applied to Banach algebra.

The first part of the thesis is largely expository as in it we introduce the basic results on numerical ranges. For an element of a unital Banach algebra, the question of approximating its spectrum by numerical ranges has been considered by Bonsall and J. Duncan. We give an alternative proof that the convex hull of the spectrum of an element may be approximated by its numerical range defined with respect to equivalent renormings of the algebra. In the particular case of operators on a Hilbert space this leads to a sharper version of a result by J. P. Williams.

An element is hermitian if it has real numerical range. Such an element is characterized in terms of the linear subspace spanned by the unit, the element and its square. This is used to characterize Banach*-algebras in which every self-adjoint element is hermitian. From this an elementary proof that such algebras are B*-algebras in an equivalent norm is given. As indicated by T. W. Palmer a formula of L. Harris' is then used to show that the equivalent renorming is unnecessary, thus giving a simple proof of Palmer's characterization of B^*-algebras among Banach algebras.

The closure properties of the spatial numerical range are studied. A construction of B. Berberian is extended to normed linear spaces, however because the numerical range need not be convex, the result obtained is weaker than that of Berberian for Hilbert spaces.

A Hilbert space or an $\ell\rho$-space $(1 \leq \rho < \infty)$ is seen to be finite dimensional if and only if all the compact operators have closed spatial numerical range. The spatial numerical range of a compact operator, on a Hilbert space or a $\ell\rho$-space $(1 < \rho < \infty)$, is shown to contain all the non-zero extreme points of its closure. So for a compact operator on Hilbert space the spatial numerical range is closed if and only if it contains the origin.

Operators which attain their numerical radius are also considered. A result of Hilberts is extended to a class of Banach spaces. In a Hilbert space the hermitian operators, which attain their numerical radius, are shown to be dense among all the hermitian operators. This leads to a stronger form of a result by J. Lindenstrauss in the special case of operators on a Hilbert space.

CONTENTS

NOTATION

E a normed linear space with elements $x, y, \ldots$

E' the dual space of E with elements $f, g, \ldots$

X a Banach space.

H a Hilbert space with innerproduct $(-,-)$

A unless otherwise stated, is a Banach algebra with elements $a, b, \ldots$

All spaces and algebras are over the field of complex numbers, unless otherwise stated.

e if present is the unit element of A.

$\hat{E}$ is the image of E in E'' under the natural embedding $x \mapsto \hat{x}$.

$\bar{S}$ is the closure of the set S.

$\#S$ is the cardinality of the set S.

$\mathbf{R}$ the real number field.

$\mathbf{C}$ the complex number field.

C O M M E N T S   A N D   E R R A T A

Page	Line*	
10	4'	For "Banach" read "normed".
13	4',2',1'	For "sequence" read "net".
	2'	Replace "X'-weak convergent" by "$\sigma(X,X')$-convergent".
16	2-12	Should read:

and since $0 < \alpha < ||T||^{-1}$

$$\frac{\mathrm{Re} f(T\underline{x})+0(\alpha)}{1-\alpha||T||} \geq \frac{\mathrm{Re} f(T\underline{x})+0(\alpha)}{1-\alpha \mathrm{Re} f(T\underline{x})} \geq \alpha^{-1}\{||[I+\alpha T]y||-1\},$$

since $\mathrm{Re} f(T\underline{x}) \leq ||T||$.

Now $\mathrm{Re} f(T\underline{x}) \in \mathrm{Re} W_\phi(T)$ so

$$\begin{aligned}\frac{\sup \mathrm{Re}\, W_\phi(T)+0(\alpha)}{1-\alpha||T||} &\geq \alpha^{-1}\{||[I+\alpha T]y||-1\}, \text{ for all } y,\ ||y|| = 1\\ &\geq \sup\{\alpha^{-1}\{||[I+\alpha T]y||-1\}:\ y \in E,\ ||y|| = 1\}\\ &= \alpha^{-1}\{||I+\alpha T||-1\}.\end{aligned}$$

Since the limit as $\alpha \to 0+$ exists we get

$$\sup \mathrm{Re}\, W_\phi(T) \geq \lim_{\alpha\to 0+} \alpha^{-1}\{||I+\alpha T||-1\}.$$

Page	Line*	
24	1'	The correct proof reads:
25	1-6	

$||(T-\lambda I)'f|| \geq k||f||$ for some $k > 0$ and every f which attains its norm.

Hence let $f \in X'$, $||f|| = 1$, attain its norm at $x \in X$, $||x|| = 1$, then

$$\begin{aligned}||(T-\lambda I)'f|| &\geq |(T-\lambda I)'f(x)|\\ &= |f(Tx)-\lambda|\\ &\geq d, \text{ as before.}\end{aligned}$$

So as required $||(T-\lambda I)'f|| \geq d||f||$, for every f attaining its norm.

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* A dashed line number refers to lines from the bottom of the page.

Page Line*

29 I am indebted to one of my examiners for suggesting the following short proof of Lemma 4.4 which avoids any appeal to function theory.

By the Spectral Radius Formula there is δ, $0 < \delta < 1$, and n_0 such that $||a^n|| \leqslant \delta^n$ for $n \geqslant n_0$.

So $$\sum_{n=n_0}^{\infty} ||a^n||^2 \leqslant \sum_{n=n_0}^{\infty} \delta^{2n} < \infty$$

and hence $$\sum_{n=1}^{\infty} ||a^n||^2 = M^2 < \infty.$$

31 7' Should read:

... $||T^n|| \leqslant M$, for all n, and some $M > 0$, and for each ...

34 5 Should read:

If H is an infinite dimensional Hilbert space ...

48 In agreement with accepted usage (p.48, line 6;7) Definition 6.4 reads: "... $a \in A$ is a *normal element* if ..." and throughout the remainder of the thesis "normal type element" reads "normal element".

91 12 Should read:

... a monomorphism $a \to T_a$ of A to $B(H)$.

102 11-12 Should read:

"... since $B(E)$ is isometrically isomorphic to the subalgebra $A = \{[T]: T \in B(E)\}$ of $B(X)$ and so $V(T) = V_A([T])$, the algebra numerical range of $[T]$ relative to A. Further $[I] \in A$ is easily seen to be the identity operator of $B(X)$, so by Corollary 2.5.2 $V([T]) = V_A([T]) = V(T)$

Page	Line*	
102 to 103	13 to 3	Delete and replace by:
138	4	The paper of Koehler and Rosenthal [1] appeared in Studia Math. 36 (1970), pp.213-216.
139	1	The paper of Moore [1] has appeared under the amended title: "Hermitian Functionals on B-algebras and Duality Characterizations of C*-algebras" in Trans. Amer. Math. Soc. (1972).

Replacement text for page 102, line 13 to page 103, line 3:

1.1.1. COROLLARY. For a given Hilbert space H, there exists a Banach space X such that:

(i) H can be embedded in X;

(ii) there exists an isometric isomorphism $T \to [T]$ of $B(H)$ onto a closed subalgebra of $B(X)$;

(iii) $\overline{W(T)} = W([T])$ which is closed and convex.

Proof. Take E = H in Proposition 1.1 then $V(T) = \overline{W(T)} \subseteq W([T]) \subseteq V(T)$.//

This shows that the operator algebra of a Banach space may contain closed subalgebras all of whose elements have spatial numerical ranges equal to their algebra numerical ranges. It is also worth remarking that G. de Barra has recently extended Proposition 1.1. He considers norms on P given by $p_q(s') = (\mu\{||x_n||^q\})^{1/q}$, $1 \leq q < \infty$, and shows that these norms are not equivalent for different q's. He also uses the translational invariance of μ to obtain the stronger version of Proposition 1.1.(iii); viz.,

$$W([T]) = \overline{co}\, W(T) = V(T).$$

INTRODUCTION

In this introduction I aim to outline the development of my research on numerical range and its application to Banach Algebra. I will also briefly sketch the results of this research as set out in later chapters.

My first acquaintance with the important extensions, by F. L. Bauer, F. F. Bonsall and G. Lumer, of Numerical ranges to operators on normed linear spaces was in 1969. At that time the basic results of numerical range, developed in chapter 1, were to be found, and in some cases only indicated, in a number of isolated papers. The first comprehensive account of numerical range was given in the excellent book of Bonsall and J. Duncan [1] which I first saw in June 1971. Much of the material given in Chapter 1 is also to be found in this book and I have only included that which forms an essential part of the theory developed in Chapters 2 and 3.

While working through various papers on numerical range I was fortunate to receive, in mid 1969, a preprint of a paper by J. P. Williams [2] in which he gave a formula, in terms of numerical range, for the closed convex hull of the spectrum of an operator on Hilbert space. A formula for the spectral radius obtained by H. F. Bohnenblust and S. Karlin [1], for a commutative Banach algebra, in 1955 and by Bonsall [2] for a general Banach algebra suggested that a similar result to William's should be true for any element of a Banach Algebra. I obtained such a result using a renorming simpler than that employed by Bohnenblust and Karlin or Bonsall to prove their spectral radius formula. Hence, I had also obtained a simpler proof of their formula (a similar proof using the same simpler renorming has also been given by Bonsall and Duncan [1]). When the Banach Algebra consists of all the operators on a Hilbert space I obtained, as a consequence of my result, a sharper version of William's result.

In 1970 I was interested in applying numerical range techniques to the proof of B. Yood's [1] characterization, up to equivalent renorming, of B*-algebras among Banach *-algebras. From this work I gave a simple proof of a numerical range characterization of B*-algebras among Banach *-algebras in terms of the squares of the self-adjoint elements. At this point J. Duncan drew my attention to the important characterization of B*-algebras among Banach Algebras obtained by T. W. Palmer [1] from an extension of a metric characterization of such algebras by I. Vidav [1]. I then saw that an alternative and simpler proof of Palmer's result could be obtained from my arguments. This proof was published in 1971 [Sims, 1]. Shortly after this a series of elementary calculations concerning the spectral radius of an element of a Banach *-algebra were published by V. Pták [1]. Adapting Pták's arguments to my purpose and using a characterization of hermitian elements which I had obtained, I was then able to give an elementary proof of Palmer's result. It is this proof which is developed in Chapter 2. I am also grateful to T. W. Palmer who pointed out the possibility of using a formula of L. Harris [1] to give a simple proof of an essential lemma first proved by Palmer in 1968.

Chapter 3 is concerned with the closure properties of the spatial numerical range of an operator on a normed linear space, and largely represents work carried out while I was at the University of Edinburgh from July to December in 1971. It is shown that for a compact operator on Hilbert space the numerical range is closed if and only if the origin is contained in the numerical range. The exceptional behaviour when the origin is not contained in the numerical range is then illustrated by a set of examples. These results have been published in a joint paper by G. de Barra, J. R. Giles and myself [1]. The argument is extended to operators on ℓ_ρ-spaces ($1 < \rho < \infty$) and the possibility of extending it further is considered. Using similar arguments a result of D. Hilbert [Riez-Sz-Nagy, 1] for compact operators on a Hilbert space is extended to a wider class of Banach spaces.

I then give two alternative proofs that the set of hermitian operators on a Hilbert space, which attain their numerical radius, is dense among all the hermitian operators. I am indebted to

A. M. Sinclair who suggested the first of these proofs and with whom I had several helpful conversations. For the particular case of operators on a Hilbert space I then use the second of these proofs to obtain a stronger version of a result by J. Lindenstrauss [1]. In this chapter some areas for further research have suggested themselves and I have indicated these where they arise.