4

Frequency Response and Performance Limitations

In this chapter we develop a theory of inherent design limitations for sampled-data feedback systems wherein we consider full intersample behavior.

As pointed out in Chapter 1, a well-developed theory of design limitations is available for LTI feedback systems, both in continuous and discrete-time cases. Yet, this theory is insufficient to deal with hybrid systems, since they are periodically time-varying due to the action of the sampler. As explained in textbooks on sampled-data control, this fact implies that one cannot use transfer functions to describe system input-output properties[1]. However, it is possible to calculate the Laplace transform of the response of a hybrid system to a particular input, and hence one may evaluate the steady-state response of a stable hybrid system to a sinusoidal input of given frequency. For analog systems, the response to such an input is a sinusoid of the same frequency as the input, but with amplitude and phase modified according to the transfer function of the system evaluated at the input frequency. The response of a stable hybrid system to an input signal, on the other hand, consists of a sum of infinitely many sinusoids spaced at integer multiples of the sampling frequency away from the frequency of the input. We shall refer to that component with the same frequency as the input as the *fundamental*, and the other components as the *harmonics*[2]. Each of these components is governed by a *frequency-response function* with many properties similar to those of a transfer function. In particular, the response functions have sufficient structure to allow complex analysis to be applied to derive a set of formulas analogous to the Bode and Poisson integrals. As in the continuous-time case, these integrals describe tradeoffs between system properties in different frequency ranges.

Frequency response properties of hybrid systems have been discussed in several recent papers [e.g., Thompson et al., 1983, 1986, Leung et al., 1991, Araki and Ito, 1993, Araki et al., 1993, Yamamoto and Khargonekar, 1993, Goodwin and

[1] An interesting notion of transfer function defined using lifting techniques is developed in Yamamoto and Araki [1994], and Yamamoto and Khargonekar 1993, 1996.

[2] In fact, the fundamental corresponds to the first harmonic. Our denomination is motivated by the fact that the first harmonic will be predominant in most applications, since the anti-aliasing filter should be designed to suppress higher frequency components.

Salgado, 1994, Feuer and Goodwin, 1994, Yamamoto and Araki, 1994]. The frequency response of periodic analog feedback systems was treated in Wereley and Hall [1990]. Particularly related to our setting are the works of Goodwin and Salgado [1994], Araki and Ito [1993], and Araki et al. [1993]. Goodwin and Salgado first introduced the idea of sensitivity functions to describe the fundamental response of a sampled-data system, and so give insights into the analysis of its intersample behavior. A frequency-domain framework to analyze both the fundamental and the harmonics, was communicated in Araki and Ito [1993] and Araki et al. [1993]. This framework introduced the concept of FR-operators, which are a hybrid system counterpart of transfer functions, and emphasized on the study of the sensitivity and complementary sensitivity operators. In this chapter, we develop similar methods to analyze fundamental properties of the frequency response of a sampled-data system.

The chapter is organized as follows. In §4.1 we define the hybrid fundamental sensitivity, fundamental complementary sensitivity, and the harmonic response functions. These functions have many common properties with transfer functions and govern the steady-state response of the hybrid system to sinusoidal disturbance and noise inputs. Using these functions we discuss the use of high-gain feedback, and describe differential sensitivity properties of the sampled-data system. §4.2 is devoted to a catalogue of interpolation constraints for these hybrid functions. As with their analog counterparts, the values of these functions at points in the ORHP is constrained by poles, zeros, and time delays in the plant and controller. Unlike the analog case, the constraints imposed by the compensator manifest themselves differently than do those imposed by the plant, and this fact leads to interesting design interpretations. Some of this interpretations are given in §4.3 in terms of steady-state disturbance rejection properties of the hybrid system. In §4.4, these interpolation constraints are used to derive generalizations of the Bode and Poisson integrals to the hybrid response functions. Design implications of these integrals are discussed in detail. Of particular interest is the fact that non-minimum phase zeros of the *analog* plant impose inherent tradeoffs upon the values of the fundamental sensitivity function on the $j\omega$-axis. Non-minimum phase zeros of the *discretized* plant, on the other hand, do not. A summary discussion of the costs and benefits of sampled-data feedback is given in §4.5.

4.1 Frequency Response of a Sampled-data System

The steady-state response of a stable hybrid feedback system to a complex sinusoidal input consists of a fundamental component at the frequency of the input as well as additional harmonics located at integer multiples of the sampling frequency away from the fundamental. This well-known fact is discussed in textbooks [cf. Åström and Wittenmark, 1990, Franklin et al., 1990], and has been emphasized in several recent research papers[3] Araki and Ito [1993], Araki et al.

[3]Similar results for systems with periodically time-varying *analog* controllers were derived by Wereley and Hall [1990].

[1993], Goodwin and Salgado [1994], Yamamoto and Araki [1994], Feuer and Goodwin [1994].

We now present expressions for the response of y in Figure 2.4 to disturbances and noise. Analogous expressions may be stated for the response to the reference input, and for the response of the control u to these signals. When evaluated along the $j\omega$-axis, the following expressions are identical to those derived in Goodwin and Salgado [1994, Theorem 2.1] using Fourier transform techniques.

Recall the notation introduced in (2.9) on page 17, i.e., we write $F_k(s)$ to represent $F(s + jk\omega_s)$, for $k = 0, \pm 1, \pm 2, \ldots$

Lemma 4.1.1
Denote the responses of y to each of d and n by y^d and y^n respectively. Then the Laplace transforms of these signals are given by

$$\begin{aligned} Y^d(s) &= \left[I - \frac{1}{T} P(s)\, H(s)\, S_d(e^{sT})\, C_d(e^{sT})\, F(s) \right] D(s) \\ &\quad - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{1}{T} P(s)\, H(s)\, S_d(e^{sT})\, C_d(e^{sT})\, F_k(s) \right] D_k(s), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} Y^n(s) &= -\left[\frac{1}{T} P(s)\, H(s)\, S_d(e^{sT})\, C_d(e^{sT})\, F(s) \right] N(s) \\ &\quad - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{1}{T} P(s)\, H(s)\, S_d(e^{sT})\, C_d(e^{sT})\, F_k(s) \right] N_k(s). \end{aligned} \tag{4.2}$$

Proof: These formulas may be derived using standard techniques from sampled-data control theory [e.g., Franklin et al., 1990, Åström and Wittenmark, 1990]. We present only a derivation of (4.2). Assume that r and d are zero. Block diagram algebra in Figure 2.4 and Lemma 2.1.1 yield

$$Y^n(s) = P(s)\, H(s)\, U_d(e^{sT}) \tag{4.3}$$

and

$$U_d(z) = -S_d(z)\, C_d(z)\, V_d(z). \tag{4.4}$$

The sampled output of the antialiasing filter can be written as

$$\begin{aligned} V_d(z) &= \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{V(s)\}\}\} \\ &= \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{F(s)N(s)\}\}\}. \end{aligned}$$

The assumptions that F is strictly proper and that n satisfies Assumption 2 allow Corollary 2.1.3 to be applied, yielding

$$V(e^{sT}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F_k(s) N_k(s). \tag{4.5}$$

Substituting (4.4) - (4.5) into (4.3) and rearranging yields the desired result. $\square$

If the feedback system is stable, then the preceding formulas may be used to derive the steady-state response of the system to a periodic input. As noted above, the response will be equal to the sum of infinitely many harmonics of the input frequency. The magnitude of each component is governed by a function analogous to the usual sensitivity or complementary sensitivity function for FDLTI systems.

Definition 4.1.1 (Hybrid Sensitivity Functions)
We define the *fundamental sensitivity* and *complementary sensitivity functions* by

$$S^0(s) \triangleq I - \frac{1}{T} P(s)\, H(s)\, S_d(e^{sT})\, C_d(e^{sT})\, F(s) \tag{4.6}$$

and

$$T^0(s) \triangleq \frac{1}{T} P(s)\, H(s)\, S_d(e^{sT})\, C_d(e^{sT})\, F(s) \tag{4.7}$$

respectively. For $k \neq 0$ define the *k-th harmonic response function* by

$$T^k(s) \triangleq \frac{1}{T} P_k(s)\, H_k(s)\, S_d(e^{sT})\, C_d(e^{sT})\, F(s) \tag{4.8}$$

◇

These hybrid response functions are not transfer functions in the usual sense, because they do not equal the ratio of the transforms of output to input signals. Moreover, note that they are not even rational functions, since their definition involves functions of the variable e^{sT}, like $H(s)$, $C_d(e^{sT})$, and $S_d(e^{sT})$. However, as the following result shows, these functions do govern the steady-state frequency response of the sampled-data system. We note that (4.6) and (4.7) are identical to the disturbance and reference gain functions defined in Goodwin and Salgado [1994].

From now and for the rest of this chapter, we confine our analysis to the case of a SISO system.

Lemma 4.1.2 (Steady-State Frequency Response)
Suppose that the hypotheses of Lemma 2.2.2 are satisfied and assume that $d(t) = e^{j\omega t}$, $t \geq 0$, *and* $n(t) = e^{j\omega t}$, $t \geq 0$. *Then as* $t \to \infty$, *we have that*

$$y^d(t) \to y^d_{ss}(t) \quad \textit{and} \quad y^n(t) \to y^n_{ss}(t),$$

where

$$y^d_{ss}(t) = S^0(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T^k(j\omega)e^{j(\omega + k\omega_s)t}, \tag{4.9}$$

and

$$y^n_{ss}(t) = -T^0(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T^k(j\omega)e^{j(\omega + k\omega_s)t}. \tag{4.10}$$

Proof: The proof is a straightforward but tedious contour integration, so it is deferred to Subsection A.3.1 of Appendix A. □

Note that the fundamental component of the disturbance response can potentially be reduced through use of feedback. The fundamental component of the noise response, on the other hand, is only increased by using feedback. These facts are analogous to the continuous-time case. Two other properties of (4.9)-(4.10) are unique to hybrid systems. First is the presence of harmonics at frequencies other than that of the input. The existence of these harmonics is due to the use of sampled-data feedback, and is a cost of feedback having no counterpart for analog systems. A second difference between analog and hybrid feedback systems is a limitation upon the ability of high-gain feedback to reduce the magnitude of the fundamental component of the disturbance response[4].

Lemma 4.1.3 (High Compensator Gain)
Assume that $(FPH)_d(e^{j\omega T}) \neq 0$. *Then, in the limit as* $|C_d(e^{j\omega T})| \to \infty$, *we have that* $S^0(j\omega) \to S_{HG}(j\omega)$, *where*

$$S_{HG}(s) \triangleq 1 - \frac{F(s)P(s)H(s)}{T(FPH)_d(e^{sT})} \tag{4.11}$$

and

$$S_d(e^{j\omega T}) \to 0. \tag{4.12}$$

Furthermore, the steady-state responses of the system output and the sampler input to a disturbance $d(t) = e^{j\omega t}$, $t \geq 0$, *satisfy*

$$y_{ss}^d(kT) = \left[1 - \frac{F(j\omega)\sum_{n=-\infty}^{\infty} P_n(j\omega)H_n(j\omega)}{T(FPH)_d(e^{j\omega T})}\right] e^{j\omega kT} \tag{4.13}$$

and

$$v_{ss}^d(kT) = 0. \tag{4.14}$$

Proof: The formula (4.11) follows from (2.9) and the proof of Theorem 4.2.1 (vii). The limit (4.12) is obvious. Equation (4.13) follows by setting $t = kT$ in (4.9). Finally, (4.14) follows by first showing that

$$v_{ss}^d(t) = F(j\omega)S^0(j\omega)\, e^{j\omega T} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T^0(j(\omega + k\omega_s))F(j(\omega + k\omega_s))e^{j(\omega+k\omega_s)t}$$

yielding

$$v_{ss}^d(kT) = S_d(e^{j\omega T})F(j\omega)e^{j\omega kT}. \tag{4.15}$$

Together (4.12) and (4.15) yield (4.14). □

[4]By contrast, recall that the disturbance response of an analog system can be made arbitrarily small at a given frequency provided that the plant gain is nonzero there.

It follows from (4.11) that use of high gain in the digital controller *does not* in general diminish the fundamental component of the disturbance response arbitrarily closely to zero[5]. For a disturbance lying in the Nyquist range, the obstruction to doing so is precisely the fact that the discrete frequency response depends upon the high frequency behavior of the plant, prefilter, and hold. It is true, of course, that the response of the sampler input may be reduced to zero at the sampling instants (cf. (4.14)). On the other hand, the sampled steady-state output due to a disturbance in the Nyquist range will be nonzero unless F is an ideal low pass filter, i.e., $F(j\omega) = 1$, for all $\omega \in \Omega_N$, and $F(j\omega) = 0$ otherwise.

The fundamental sensitivity and complementary sensitivity functions, together with the harmonic response functions, may be also used to describe differential sensitivity properties of a hybrid feedback system. It is well-known Bode [1945], that the sensitivity function of a continuous-time feedback system governs the relative change in the command response of the system with respect to small changes in the plant. Derivations similar to those of Lemma 4.1.2 show that the steady-state response of the system in Figure 2.4 to a command input $r(t) = e^{j\omega t}$, $t \geq 0$, is given by

$$y_{ss}^r(t) = T^0(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} T^k(j\omega)e^{j(\omega + k\omega_s)t}. \tag{4.16}$$

Since $T^0(j\omega)$ depends upon $S_d(e^{j\omega T})$, it follows from (2.8) that the fundamental component of the command response at a particular frequency is sensitive to variations in the plant response at infinitely many frequencies.

Lemma 4.1.4 (Differential Sensitivity)
At each frequency ω, the relative sensitivity of the steady state command response (4.16) *to variations in* $P(j(\omega + \ell\omega_s))$ *is given by*

(i) For $\ell = 0$,

$$\frac{P(j\omega)}{T^0(j\omega)} \frac{\partial T^0(j\omega)}{\partial P(j\omega)} = S^0(j\omega). \tag{4.17}$$

(ii) For all $\ell \neq 0$,

$$\frac{P_\ell(j\omega)}{T^0(j\omega)} \frac{\partial T^0(j\omega)}{\partial P_\ell(j\omega)} = -T^0(j(\omega + \ell\omega_s)) \tag{4.18}$$

Proof: The proof is a straightforward calculation, keeping in mind the dependence of $S_d(e^{j\omega T})$ upon $P(s + j\ell\omega_s)$. □

These results may best be interpreted by considering frequencies in the Nyquist range. Fix $\omega \in \Omega_N$. Then (4.17) states that the sensitivity of the fundamental component of the command response to small variations in the plant *at that frequency* is governed by the fundamental sensitivity function and hence may potentially

[5]However, see the remarks following Theorem 4.2.1 concerning the ZOH and use of integrators in C_d.

be reduced through use of feedback[6]. On the other hand, (4.18) states that the sensitivity of the fundamental component to *higher frequency* plant variations is governed by the fundamental complementary sensitivity function evaluated at the higher frequency, and thus *cannot* be reduced through the use of feedback. Further note that since

$$T^0(j(\omega + \ell\omega_s)) = \frac{1}{T}P(s + j\ell\omega_s)\,H(s + j\ell\omega_s)F(s + j\ell\omega_s)\,C_d(e^{j\omega T})\,S_d(e^{j\omega T}),$$

the sensitivity of the command response at a frequency in the Nyquist range to higher frequency plant variations is proportional to the gain of the hold frequency response at the *higher* frequency, thus suggesting that the hold response should not be excessively large at high frequencies[7].

4.2 Interpolation Constraints

It is well known that the sensitivity and complementary sensitivity functions of a stable, continuous-time feedback system must satisfy certain interpolation constraints at the CRHP poles and zeros of the plant and compensator. Specifically, the sensitivity function must equal zero at the CRHP poles, and the complementary sensitivity function must equal zero at the CRHP zeros.

As shown by Freudenberg and Looze [1985], these constraints may be used in conjunction with the Poisson integral to describe frequency dependent design tradeoffs. An entirely analogous set of interpolation constraints and design tradeoffs applies to the discrete-time part of a hybrid feedback system Sung and Hara [1988], Mohtadi [1990], Middleton [1991], Middleton and Goodwin [1990].

In this section we present a set of interpolation constraints that must be satisfied by the hybrid sensitivity functions defined in (4.6)-(4.8). Hybrid sensitivity responses have fixed values on $\overline{\mathbb{C}^+}$ that are determined by the open-loop zeros and poles of the plant, hold response, and digital compensator. As we shall see later, a significant difference between the hybrid case and the continuous-time only or discrete-time only cases is that the poles and zeros of the compensator yield different constraints than do those of the plant. The following theorem describe these interpolation relations for the fundamental sensitivity and complementary sensitivity functions.

Theorem 4.2.1 (Interpolation Constraints for S^0 and T^0)
Assume that P, F, H *and* C_d *satisfy all conditions stated in Chapter 2 and that the hybrid feedback system of Figure 2.4 is stable. Then the following conditions are satisfied:*

(i) S^0 *and* T^0 *have no poles in* $\overline{\mathbb{C}^+}$.

[6] See the comments following Lemma 4.1.2.
[7] See also the preliminary remarks in Subsection 3.2.2, Chapter 3.

(ii) Let $p \in \overline{\mathbb{C}^+}$ *be a pole of* P. *Then*

$$\begin{aligned} S^0(p) &= 0, \\ T^0(p) &= 1. \end{aligned} \tag{4.19}$$

(iii) Let $\zeta \in \overline{\mathbb{C}^+}$ *be a zero of* P. *Then*

$$\begin{aligned} S^0(\zeta) &= 1, \\ T^0(\zeta) &= 0. \end{aligned} \tag{4.20}$$

(iv) Let $\gamma \in \overline{\mathbb{C}^+}$ *be a zero of* H. *Then*

$$\begin{aligned} S^0(\gamma) &= 1, \\ T^0(\gamma) &= 0. \end{aligned}$$

(v) Let $a \in \mathbb{D}^c$ *be a zero of* C_d. *Define*

$$a_m \triangleq \frac{1}{T}\log(a) + jm\omega_s, \quad m = 0, \pm 1, \pm 2, \cdots$$

Then

$$\begin{aligned} S^0(a_m) &= 1, \\ T^0(a_m) &= 0. \end{aligned}$$

(vi) Let $p \in \overline{\mathbb{C}^+}$ *be a pole of* P. *Define*

$$p_m \triangleq p + jm\omega_s, \quad m = \pm 1, \pm 2, \cdots$$

Then

$$\begin{aligned} S^0(p_m) &= 1, \\ T^0(p_m) &= 0. \end{aligned} \tag{4.21}$$

(vii) T^0 *has no CRHP zeros other than those given by (iii) - (vi) above.*

(viii) Let $b \in \mathbb{D}^c$ *be a pole of* C_d. *Define*

$$b_m \triangleq \frac{1}{T}\log(b) + jm\omega_s, \quad m = 0, \pm 1, \pm 2, \cdots$$

Then

$$S^0(b_m) = 1 - \frac{P(b_m)\,H(b_m)\,F(b_m)}{T(FPH)_d(b)}, \tag{4.22}$$

$$T^0(b_m) = \frac{P(b_m)\,H(b_m)\,F(b_m)}{T(FPH)_d(b)}. \tag{4.23}$$

Proof: Introduce factorizations

$$P(s)\,F(s) = e^{-s\tau}\frac{N(s)}{M(s)},$$

where N and M are coprime rational functions with no poles in $\overline{\mathbb{C}^+}$, and

$$(FPH)_d(z) = \frac{N_d(z)}{M_d(z)}, \tag{4.24}$$

where N_d and M_d are coprime rational functions with no poles in $\mathbb{D}^c$. By the Youla parameterization, all controllers C_d that stabilize (4.24) have the form[8]

$$C_d = \frac{Y_d + M_d Q_d}{X_d - N_d Q_d}, \tag{4.25}$$

where Q_d, X_d, and Y_d are stable, and X_d and Y_d satisfy the Bezout identity

$$M_d X_d + N_d Y_d = 1. \tag{4.26}$$

It follows that $S_d = M_d(X_d - N_d Q_d)$ and

$$C_d S_d = M_d(Y_d + M_d Q_d). \tag{4.27}$$

Using (4.27) in (4.6) and (4.7) yields

$$S^0(s) = 1 - \frac{1}{T}e^{-s\tau}\frac{N(s)H(s)}{M(s)}M_d(e^{sT})\left[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})\right] \tag{4.28}$$

and

$$T^0(s) = \frac{1}{T}e^{-s\tau}\frac{N(s)H(s)}{M(s)}M_d(e^{sT})\left[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})\right]. \tag{4.29}$$

(i) T^0 is stable because each factor in the numerator of (4.29) is stable, and because the assumption of non-pathological sampling guarantees that any unstable pole of $1/M$ must be canceled by a corresponding zero of $M_d(e^{sT})$.

(ii) It follows from (4.26) that $Y_d(e^{pT}) = 1/N_d(e^{pT})$. Using this fact, and evaluating (4.28) in the limit as $s \to p$ yields

$$S^0(s) \longrightarrow 1 - \lim_{s\to p}\frac{F(s)P(s)H(s)}{T(FPH)_d(e^{sT})}.$$

Replace $(FPH)_d(e^{sT})$ using (2.8):

$$S^0(s) \longrightarrow 1 - \lim_{s\to p}\frac{H(s)P(s)F(s)}{\sum_{k=-\infty}^{\infty} F(s+jk\omega_s)\,P(s+jk\omega_s)\,H(s+jk\omega_s)}. \tag{4.30}$$

By the assumptions that F is stable and that sampling is non-pathological, P and F have no poles at $p + jk\omega_s$, $k \neq 0$. Using this fact, and the fact that H has no finite poles, yields that each term in the denominator of (4.30) remains finite as $s \to p$ except the term $k = 0$. The result follows.

[8] We suppress dependence on the transform variable when convenient; the meaning will always be clear from context.

To prove (iii)-(vi), observe first that (4.29) implies T^0 can have CRHP zeros only at the CRHP zeros of N, H, $M_d(e^{sT})$, or $[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})]$.

(iii)-(vii) By Assumption 3, P, F, and PF are each free of unstable hidden modes. Hence N and M can have no common CRHP zeros and (iii) follows. By the assumption of non-pathological sampling, neither can H and M. Hence (iv) follows. Note next that the zeros of $M_d(e^{sT})$ lie at $p + jk\omega_s$, $k = 0, \pm 1, \pm 2, \cdots$, where p is any CRHP pole of P and hence a zero of M. It follows from this fact that the ratio $M_d(e^{sT})/M(s)$ can have zeros only for $k = \pm 1, \pm 2, \cdots$. By the assumption of non-pathological sampling, no other cancelations occur, and all these zeros are indeed zeros of T^0. This proves (vi). By (4.25), the CRHP zeros of $[Y_d(e^{sT}) + M_d(e^{sT})Q_d(e^{sT})]$ are identical to the CRHP zeros of $C_d(e^{sT})$. By the hypotheses of Lemma 2.2.2, none of these zeros can coincide with those of $M_d(e^{sT})$, and thus with those of M. This proves (v). Statement (vii) now follows because (iii)-(vi) exhaust all possibilities for T^0 to have CRHP zeros.

(viii) It follows from (4.27) that $C_d(b)S_d(b) = 1/(FPH)_d(b)$. Substitution of this identity into (4.6)-(4.7) yields (4.22)-(4.23).

□

A summary of the interpolation constraints satisfied by S^0 and T^0 is given in Table 4.1.

$\xi \in \overline{\mathbb{C}^+}$	**Fundamental Sensitivity**	**Fundamental Complementary Sensitivity**
Plant pole $P(\xi) = \infty$	$S^0(\xi) = 0$ $S^0(\xi + jk\omega_s) = 1$ $k \neq 0$	$T^0(\xi) = 1$ $T^0(\xi + jk\omega_s) = 0$ $k \neq 0$
Plant zero $P(\xi) = 0$	$S^0(\xi) = 1$	$T^0(\xi) = 0$
Controller pole $C_d(e^{\xi T}) = \infty$	$S^0(\xi) = 1 - \frac{F(\xi)P(\xi)H(\xi)}{T(FPH)_d(e^{\xi T})}$	$T^0(\xi) = \frac{F(\xi)P(\xi)H(\xi)}{T(FPH)_d(e^{\xi T})}$
Controller zero $C_d(e^{\xi T}) = 0$	$S^0(\xi) = 1$	$T^0(\xi) = 0$
Hold zero $H(\xi) = 0$	$S^0(\xi) = 1$	$T^0(\xi) = 0$

Table 4.1: Summary of interpolation constraints on S^0 and T^0.

Harmonic response functions T^k also satisfy interpolation constraints, which are easily derived from the previous theorem.

Corollary 4.2.2 (Interpolation constraints for T^k)
Under the same hypotheses of Theorem 4.2.1 the following conditions are satisfied:

(i) T^k *has no poles in* $\overline{\mathbb{C}^+}$.

(ii) Let p *be a pole of* P *with* p *in* $\overline{\mathbb{C}^+}$. *Then*

$$T^k(p) = 0. \tag{4.31}$$

(iii) Let ζ *be a zero of* P *with* ζ *in* $\overline{\mathbb{C}^+}$. *Then*

$$T^k(\zeta - jk\omega_s) = 0. \tag{4.32}$$

(iv) Let γ *be a zero of* H *with* γ *in* $\overline{\mathbb{C}^+}$. *Then*

$$T^k(\gamma - jk\omega_s) = 0.$$

(v) Let a *be a zero of* C_d *with* a *in* $\mathbb{D}^c$, *and* a_m *as defined in Theorem 4.2.1 (v). Then*

$$T^k(a_m) = 0.$$

(vi) Let p *be a pole of* P *with* p *in* $\overline{\mathbb{C}^+}$, *and* p_m *as defined in Theorem 4.2.1 (vi). Then*

$$T^k(p_m) = \begin{cases} \dfrac{F(p_m)}{F(p)} & \text{if } m = -k, \\ 0 & \text{if } m \neq -k \end{cases} \tag{4.33}$$

(vii) T^k *has no CRHP zeros other than those given by (iii) - (vi) above.*

(viii) Let b *be a pole of* C_d *with* b *in* $\mathbb{D}^c$, *and* b_m *as defined in Theorem 4.2.1 (viii). Then*

$$T^k(b_{m-k}) = \frac{P(b_m)\,H(b_m)\,F(b_{m-k})}{T(FPH)_d(b)}. \tag{4.34}$$

Proof: Note that the harmonic functions T^k can be expressed as

$$T^k(s - jk\omega_s) = \frac{F(s - jk\omega_s)}{F(s)} T^0(s). \tag{4.35}$$

By Assumption 3, F is minimum phase and stable, so F_{-k}/F is bistable. The result then follows straightforward from Theorem 4.2.1. □

There are a number of differences between the interpolation constraints for the hybrid and the continuous-time cases; we now describe these in detail.

Remark 4.2.1 (CRHP Plant Poles) Each CRHP plant pole yields constraints (4.19) directly analogous to the continuous-time case. Furthermore, each of these poles yields the additional constraints (4.21), which arise from the periodically spaced zeros of $S_d(e^{sT})$ and the fact that non-pathological sampling precludes all but one of these zeros from being canceled by a pole of P. ⋄

Remark 4.2.2 (CRHP Plant Zeros) Each CRHP plant zero yields constraints (4.20) directly analogous to the continuous-time case. Note in particular that these constraints are present *independently* of the choice of the hold function. The zeros of the discretized plant lying in $\mathbb{D}^C$, on the other hand, do not impose any inherent constraints on S^0. Indeed, suppose that $\nu \in \mathbb{D}^C$ is a zero of $(FPH)_d$. Then for each $\nu_k \triangleq \frac{1}{T}\log(\nu) + jk\omega_s$, $k = 0, \pm 1, \pm 2, \cdots$, it follows that

$$S^0(\nu_k) = 1 - \frac{1}{T}P(\nu_k)H(\nu_k)C_d(\nu)F(\nu_k), \tag{4.36}$$

and thus the size of $S^0(\nu_k)$ is *not* independent of the choice of compensator. $\diamond$

Remark 4.2.3 (Unstable Compensator Poles) For analog systems, unstable plant and compensator poles yield identical constraints on the sensitivity and complementary sensitivity functions; namely, when evaluated at such a pole, sensitivity must equal zero and complementary sensitivity must equal one. Comparing (ii) and (vii) in Theorem 4.2.1, we see that in a hybrid system unstable plant and compensator poles will generally yield different constraints on sensitivity and complementary sensitivity. In particular, unstable compensator poles will yield corresponding zeros of S^0 only in special cases. $\diamond$

Remark 4.2.4 (Zeros of C_d) Each zero of the compensator lying in $\mathbb{D}^C$ imposes infinitely many interpolation constraints upon the continuous-time system because there are infinitely many points in the s-plane that map to the location of the zero in the z-plane. These constraints are due to the fact that a pole at *any* of these points will lead to an unstable discrete pole-zero cancelation. $\diamond$

Remark 4.2.5 (Zeros of Hold Response) By Theorem 4.2.1 (iv), zeros of H lying in the CRHP impose constraints on the sensitivity function identical to those imposed by CRHP zeros of the plant. A ZOH has CRHP zeros only on the $j\omega$-axis. As discussed in Chapter 3, GSHF response functions may have zeros in the *open* right half plane. $\diamond$

Remark 4.2.6 (Zeros of S^0) Our list of CRHP zeros for T^0 and T^k is exhaustive; however, our list for S^0 is not. It is interesting to contrast this situation with the analog case. For analog systems, the CRHP zeros of the sensitivity function consist precisely of the union of the CRHP poles of the plant and compensator. On the other hand, by Theorem 4.2.1 (ii) and (vii), unstable plant poles yield zeros of S^0 while unstable compensator poles generally do not. Furthermore, as the following example shows, S^0 may have CRHP zeros even if both plant and compensator are stable.

Example 4.2.1 Consider the plant $P(s) = 1/(s+1)$. Discretizing with a ZOH, sample period $T = 1$, and no anti-aliasing filter (i.e., $F(s) = 1$) yields $(FPH)_d(z) = .6321/(z - .3670)$. A stabilizing discrete controller for this plant is

$$C_d(z) = \frac{(4.8158)(z^2 + .1z + 0.3988)}{(z^2 - 1.02657z + 0.9025)}.$$

Both plant and compensator are stable; yet it may be verified that S^0 has zeros at $s = 0.2 \pm j$ (see Figure 4.1).

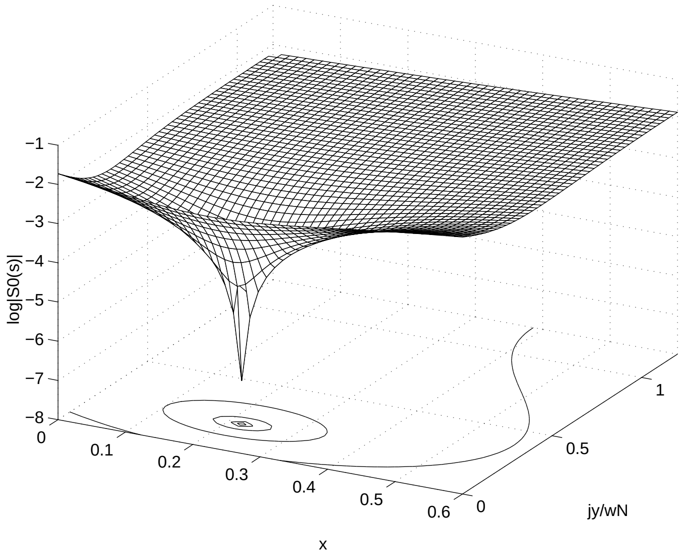


Figure 4.1: Fundamental sensitivity for Example 4.2.1

$\diamond$

4.3 Hybrid Disturbance Rejection Properties

In the last section we derived a set of interpolation constraints that must be satisfied by the hybrid sensitivity responses S^0, T^0, and T^k in the CRHP. As we shall see in this section, these constraints have interpretations in terms of steady-state disturbance rejection properties of the system. In particular, we analize periodic disturbances of frequencies within and outside the Nyquist range, and the effect that corresponding unstable poles of the plant and compensator have on these rejection properties. For analog systems, it is well-known that, by the internal model principle, an input disturbance can be asymptotically rejected if the system includes the dynamics of the disturbance [e.g., Wonham, 1985]. As we shall see in this section, this is not generally the case for sampled-data systems.

We start analyzing those properties that are associated with unstable poles of the plant.

Corollary 4.3.1
Assume that P *has a pole at* $s = j\omega$, $\omega \in \Omega_N$. *Then the steady-state response to a disturbance* $d(t) = e^{j(\omega + \ell\omega_s)t}$, $t \geq 0, \omega \in \Omega_N$, $\ell = 0, \pm 1, \pm 2, \ldots$, *is given as follows.*

(i) If $\ell = 0$*, then*

$$y_{ss}^{d}(t) = 0\,. \tag{4.37}$$

(ii) If $\ell \neq 0$*, then*

$$y_{ss}^{d}(t) = e^{j(\omega+\ell\omega_s)t} - \frac{F(j(\omega+\ell\omega_s))}{F(j\omega)} e^{j\omega t}. \tag{4.38}$$

Proof: By (4.9) the steady-state response is given by

$$y_{ss}^{d}(t) = S^{0}(j(\omega+\ell\omega_s))e^{j(\omega+\ell\omega s)t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T^{k}(j(\omega+\ell\omega_s))e^{j(\omega+(k+\ell)\omega_s)t}$$

(i) By (4.19) and (4.31), $S^0(j\omega) = 0$ and $T^k(j\omega) = 0$.

(ii) By (4.21), $S^0(j(\omega+\ell\omega_s)) = 1$. From (4.33) follows that $T^k(j(\omega+\ell\omega_s)) = 0$ if $k \neq -\ell$, and $T_{-\ell}(j(\omega+\ell\omega_s)) = F(j(\omega+\ell\omega_s))/F(j\omega)$ if $k = -\ell$.

□

It follows from (4.37) that a disturbance of frequency $\omega \in \Omega_N$ will be asymptotically rejected if the plant has a pole at $j\omega$ (if necessary, the pole may be augmented to the plant via an *analog* precompensator). However, any high frequency disturbance of frequency $\omega+\ell\omega_s$, $\omega \in \Omega_N$, $\ell \neq 0$ will be passed directly through to the output along with an alias of frequency ω whose amplitude is determined by the ratio of the gains of the anti-aliasing filter evaluated at the two frequencies.

As pointed out in Remark 4.2.4, unstable compensator poles do not yield the same type of constraints on S^0 as the unstable poles of the plant. Moreover, unstable compensator poles do not in general yield asymptotic disturbance rejection, as the following result shows.

Corollary 4.3.2
Assume that C_d *has a pole at* $z = e^{j\omega T}$*, and that* P *has no poles at* $s = j(\omega+k\omega_s)$*,* $k = 0, \pm1, \pm2, \ldots$*. Then the steady-state response to a disturbance input* $d(t) = e^{j(\omega+\ell\omega_s)t}, t \geq 0, \omega \in \Omega_N, \ell = 0, \pm1, \pm2, \ldots$*, satisfies*

$$y_{ss}^{d}(t) = S_{\ell}^{0}(j\omega)e^{j(\omega+\ell\omega s)t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_{\ell}^{k}(j\omega)e^{j(\omega+(k+\ell)\omega_s)t}, \tag{4.39}$$

where

$$S_{\ell}^{0}(j\omega) = 1 - \frac{P(j(\omega+\ell\omega_s))\,H(j(\omega+\ell\omega_s))\,F(j(\omega+\ell\omega_s))}{T(FPH)_d(e^{j\omega T})} \tag{4.40}$$

and

$$T_{\ell}^{k}(j\omega) = \frac{P(j(\omega+(k+\ell)\omega_s))\,H(j(\omega+(k+\ell)\omega_s))\,F(j(\omega+\ell\omega_s))}{T(FPH)_d(e^{j\omega T})}. \tag{4.41}$$

◦

Note that even for $\ell = 0$ the steady-state disturbance response is in general *nonzero*. For continuous-time systems, it is well known that a periodic disturbance may be asymptotically rejected by incorporating the dynamics of the disturbance into the system. For a hybrid system, Corollaries 4.3.1 and 4.3.2 show that for asymptotic disturbance rejection to be present, the dynamics should be augmented to the plant using an *analog precompensator*. Including a discretized version of these dynamics in the digital compensator will not, in general, achieve the desired result. Exceptions to this statement may be obtained by imposing additional structure on the hold response function.

Corollary 4.3.3
Let the hypotheses of Corollary 4.3.2 be satisfied. Choose $\omega \in \Omega_N$. *Assume that*

$$\begin{aligned} P(j\omega) &\neq 0 & (4.42) \\ H(j\omega) &\neq 0 & (4.43) \end{aligned}$$

and

$$H(j(\omega + k\omega_s)) = 0, \text{ for all } k = \pm 1, \pm 2, \cdots \tag{4.44}$$

Then

(i) the steady-state response to an input $d(t) = e^{j\omega t}$, $t \geq 0$ *satisfies* $y_{ss}^d(t) = 0$,

(ii) the steady-state response to an input $d(t) = e^{j(\omega+\ell\omega_s)t}$, $t \geq 0$, $\ell = \pm 1, \pm 2, \cdots$ *satisfies*

$$y_{ss}^d(t) = e^{j(\omega+\ell\omega_s)t} - \frac{F(j(\omega + \ell\omega_s))}{F(j\omega)} e^{j\omega t} \tag{4.45}$$

Proof:

(i) The steady-state response to $e^{j\omega t}$ is given by (4.9). Hypotheses (4.42)-(4.43) imply that $(FPH)_d(e^{j\omega t}) = \frac{1}{T}F(j\omega)P(j\omega)H(j\omega) \neq 0$, and it follows from Theorem 4.2.1 (iv) that the coefficient of $e^{j\omega t}$ in (4.9) equals one. Using (4.44) in (4.41) shows that the coefficients of the higher frequency terms in (4.9) all equal zero.

(ii) Hypotheses (4.42)-(4.44) imply that (4.40) equals one, (4.41) equals zero for $k + \ell \neq 0$, and that

$$T_{-\ell}(j(\omega + \ell\omega_s)) = \frac{F(j(\omega + \ell\omega_s))}{F(j\omega)}.$$

These facts, together with (4.39) yield (4.45).

□

A consequence of Corollary 4.3.3 is the well known fact that a discrete integrator may be used in conjunction with a ZOH to achieve asymptotic rejection of constant disturbances. Related results for hybrid systems with a ZOH are found in Franklin and Emami-Naeini [1986] and Urikura and Nagata [1987]. A recent and more general study of tracking problems in hybrid systems is given in Yamamoto [1994].

The role played by the hold frequency response function in the disturbance rejection properties of the system may be further explored by considering plant *input* disturbances. We shall now see that, in conjunction with a pole of the analog plant at the frequency of the disturbance, input disturbance rejection is essentially determined by the shape of the frequency response of the hold.

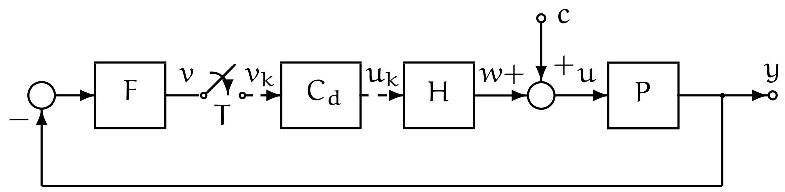


Figure 4.2: System with input disturbance.

Consider the SISO sampled-data system of Figure 4.2. In a similar way to the cases of output disturbance and noise, we can derive expressions describing the steady-state response of the plant input u to a periodic disturbance c. The following lemma, analogous to Lemma 4.1.2, shows this.

Lemma 4.3.4 (Steady-state Frequency Response to Input Disturbance)
Suppose the hypothesis of Lemma 2.2.2 are satisfied, and assume that $c(t) = e^{j\omega}$, $t \geq 0$. *Then as* $t \to \infty$, $u(t) \to u_{ss}(t)$, *where*

$$u_{ss}(t) = S^0(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} R^k(j\omega)e^{j(\omega+k\omega_s)t}, \tag{4.46}$$

and

$$R^k(s) \triangleq \frac{1}{T}H_k(s)S_d(e^{sT})C_d(e^{sT})F(s)P(s). \tag{4.47}$$

Proof: The proof is similar to that of Lemma 4.1.2. □

Note that, like the response to plant output disturbances, the fundamental component of u_{ss} is governed by S^0, and so may also be potentially reduced by feedback. The main difference in this case lies on the harmonics, which are given by the responses R^k. From (4.47) we could foresee that the behavior of the hold response function at high frequencies will have a significant role in the relative magnitude of these harmonics. This is perhaps further clarified by the following result, which describes the steady-state disturbance rejection properties of the system in Figure 4.2 when the plant has an unstable pole at the frequency of the disturbance[9].

[9] Compare with Corollary 4.3.1.

Lemma 4.3.5
Assume that P *has a pole at* $s = j\omega$, ω *in* Ω_N. *Then the steady-state response to an input disturbance* $c(t) = e^{j(\omega+\ell\omega_s)t}$, $t \geq 0$, $\ell = 0, \pm 1, \pm 2, \ldots$, *is given as follows:*

(i) *if* $\ell = 0$

$$u_{ss}(t) = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{H_k(j\omega)}{H(j\omega)} e^{j(\omega+k\omega_s)t}, \tag{4.48}$$

(ii) *if* $\ell \neq 0$

$$u_{ss}(t) = e^{j(\omega+\ell\omega_s)t}. \tag{4.49}$$

Proof: From Lemma 4.3.4 we have that the steady-state response is given by

$$u_{ss}(t) = S^0(j(\omega+\ell\omega))e^{j(\omega+\ell\omega_s)t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} R^k(j(\omega+\ell\omega_s))e^{j(\omega+(k+\ell)\omega_s)t}. \tag{4.50}$$

(i) By the assumption of non-pathological sampling $H(j\omega) \neq 0$ holds, and so comparing (4.47) and (4.7) we may alternatively write

$$R^k(j\omega) = \frac{H_k(j\omega)}{H(j\omega)} T^0(j\omega). \tag{4.51}$$

Replacing (4.51) in (4.50), and applying Theorem 4.2.1 (ii) gives $S^0(j\omega) = 0$ and $T^0(j\omega) = 1$, from which the result follows.

(ii) Theorem 4.2.1 (vi) yields $S^0(j(\omega+\ell\omega_s)) = 1$. Since $S_d(e^{j\omega T}) = 0$ and $s = j(\omega+\ell\omega_s)$ could not be a pole of P by the non-pathological sampling assumption, we have that

$$\begin{aligned} R^k(j(\omega+\ell\omega_s)) &= \frac{1}{T} H_{k+\ell}(j\omega) S_d(e^{j\omega T}) C_d(e^{j\omega T}) F_\ell(j\omega) P_\ell(j\omega) \\ &= 0. \end{aligned}$$

Equation (4.49) follows.

□

Notice from (4.49) that harmonics are asymptotically rejected if the frequency of the disturbance is higher than the Nyquist frequency, but the fundamental component is passed directly to the input of the plant (cf. Corollary 4.3.1).

In the case of a disturbance of frequency ω within Ω_N, it follows from (4.48) that the fundamental component of the steady-state response will be asymptotically rejected if the plant has a pole at $s = j\omega$. However, harmonics of this frequency will pass with amplitudes proportional to the ratios $|H_k(j\omega)/H(j\omega)|$, with $k = \pm 1, \pm 2, \ldots$

In practice, the system will have acceptable asymptotic rejection properties if the hold frequency response rolls off at frequencies higher than the Nyquist frequency, as is the case of a ZOH[10]. If on the other hand the hold response is large at high frequencies, as from Chapter 3 we know it may happen with GSHFs, harmonics at those frequencies will be amplified, thereby degrading the input disturbance rejection properties of the hybrid system. In addition, the presence of these harmonics in conjunction with plant input saturation phenomena may compromise the system's overall performance[11].

The connection between the hold response and the "size" of the steady-state signals generated by a disturbance in case (i) of Lemma 4.3.5 is further illustrated by the following straightforward corollary. Let w denote the output of the hold device, i.e.,

$$w(t) = u_{ss}(t) - c(t). \tag{4.52}$$

From Lemma 4.3.5 follows that if $c(t) = e^{j\omega T}$, and the plant has a pole at $s = j\omega$, then w is given by

$$w(t) = \sum_{k=-\infty}^{\infty} \frac{H_k(j\omega)}{H(j\omega)} e^{j(\omega + k\omega_s t)}. \tag{4.53}$$

Notice that w is not necessarily periodic. However, its amplitude does correspond to that of a periodic function, since

$$|w(t)| = \left| \sum_{k=-\infty}^{\infty} \frac{H_k(j\omega)}{H(j\omega)} e^{jk\omega_s t} \right|. \tag{4.54}$$

We measure the size of the steady-state value of w by its 2-norm, over an interval of length T. From (4.54) it follows that this is the same as

$$\|w\|_2 = \left(\int_0^T |w(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

We then have the following result.

Corollary 4.3.6
Assume the conditions of Lemma 4.3.5 are satisfied. Then for a disturbance $c(t) = e^{j\omega t}$*, with* ω *in* Ω_N*,*

$$\frac{1}{T}\|w\|_2^2 = \frac{\|h\|_2^2}{|H(j\omega)|^2}. \tag{4.55}$$

Proof: From (4.53) we have that $w(t)e^{-j\omega T}$ is periodic with Fourier Series representation

$$w(t)e^{-j\omega T} = \sum_{k=-\infty}^{\infty} \frac{H_k(j\omega)}{H(j\omega)} e^{jk\omega_s t}. \tag{4.56}$$

[10]Notice then that the hold should have similar roll-off properties as those of the anti-aliasing filter. Further related comments are given in Chapter 5, Remark 5.2.3.

[11]See remarks following Lemma 3.1.2 in Chapter 3.

Application of Parseval's Identity [e.g., Rudin, 1987] to the series (4.56) yields

$$\begin{aligned}\int_0^T |w(t)|^2\,dt &= \sum_{k=-\infty}^{\infty}\left|\frac{H_k(j\omega)}{H(j\omega)}\right|^2\\ &= \frac{T\|h\|_2^2}{|H(j\omega)|^2}\end{aligned}$$

where in the last equality we have used Lemma 3.1.1. The result follows. □

Corollary 4.3.6 shows that the "average power" of the signal generated at the output of the hold device by a periodic disturbance in the Nyquist range Ω_N is proportional to the 2-norm of the hold pulse response h. Noting that

$$\begin{aligned}|H(j\omega)| &\le \int_0^T |h(t)|\,dt\\ &= \|h\|_1,\end{aligned}$$

we obtain the lower bound

$$\|w\|_2 \ge \sqrt{T}\frac{\|h\|_2}{\|h\|_1}. \tag{4.57}$$

Since $\|h\|_1 \ge \sqrt{T}\|h\|_2$, we get that $\|w\|_2 \ge 1$ always. It is interesting to note that for a ZOH $\|h\|_1 = \sqrt{T}\|h\|_2$, and so it achieves the lowest bound for the size of the signal w.

We illustrate these results with a numerical example.

Example 4.3.1 (GSHF control of a harmonic oscillator) We consider Example 1 in Kabamba [1987], where a GSHF is designed to stabilize the plant

$$P(s) = \frac{1}{s^2+1}$$

by output feedback. The setup corresponds to the system in Figure 4.2 with $F(s) = 1$. This system cannot be made asymptotically stable by continuous-time direct output feedback. However, it can be asymptotically stabilized by a digital compensator and a ZOH, although the closed-loop eigenvalues cannot be arbitrarily assigned. The technique proposed by Kabamba allows the stabilization with just a GSHF (i.e., $C_d = 1$), and arbitrary closed-loop eigenvalues. The hold suggested is a FDLTI GSHF (Definition 3.1.1) given by the matrices

$$K = \begin{bmatrix}0 & 1\end{bmatrix} \quad L = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \quad M = \begin{bmatrix}-13.1682\\ 7.0898\end{bmatrix}.$$

The sampling period selected was $T = 1$. This GSHF sets the closed-loop discrete eigenvalues to $z = 0$, so the system is stabilized in two sampling periods. For comparison, we alternatively computed a stabilizing solution using a ZOH and a discrete compensator. A constant compensator of gain $k = -0.9348$ renders a double real pole of the discretized system at $z = 0.7552$.

We computed the frequency response of this GSHF using the formula given by Lemma 3.1.5 in Chapter 3. This is plotted in Figure 4.3, together with the response of the ZOH for reference. We have indicated with dotted lines the abscissas corresponding to the frequencies ω, $\omega + \omega_s$, and $\omega + 2\omega_s$, where $\omega = 1$ is the frequency of the complex poles of the plant.

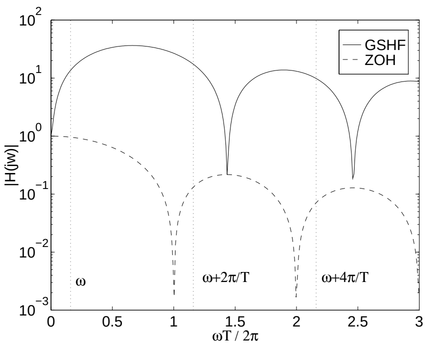


Figure 4.3: Frequency response of GSHF and ZOH.

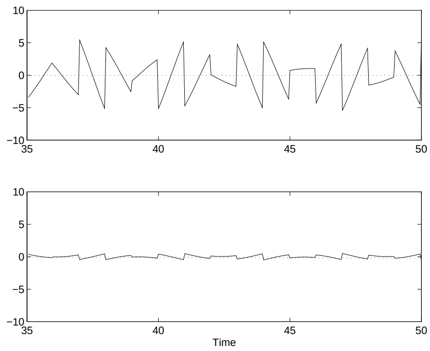


Figure 4.4: Response to input disturbance.

In reference to the summation of (4.48), we can see in this plots that the GSHF will have a larger number of terms with significant contribution than the ZOH. Therefore, we should expect from the GSHF solution a larger steady-state response to a sinusoidal input disturbance of frequency $\omega = 1$. This is illustrated in Figure 4.4, where we plotted the corresponding steady-state responses u for the GSHF system (above) and for the ZOH system (below). The amplitude of the signal produced in the GSHF case is approximately 10 times larger than that of the ZOH case. ◇

4.4 Integral Relations

The interpolation constraints derived in the preceding section fix the values of the hybrid response functions at points of the CRHP. For continuous-time systems, the Poisson integral may be used to translate the interpolation constraints into equivalent integral relations that the sensitivity and complementary sensitivity functions must satisfy along the $j\omega$ - axis Freudenberg and Looze [1985]. In this section, we show that similar integral relations must be satisfied by the hybrid sensitivity functions. We also show that these functions must satisfy generalizations of the Bode sensitivity integral Bode [1945] and its dual for complementary sensitivity Middleton and Goodwin [1990], Middleton [1991].

4.4.1 Notation

Denote the non-minimum phase zeros of P by

$$\{\zeta_i; i = 1, \dots, N_\zeta\}, \tag{4.58}$$

the non-minimum phase zeros of H by

$$\{\gamma_i; i = 1, \dots, N_\gamma\}, \tag{4.59}$$

the non-minimum phase zeros of C_d by

$$\{a_i; i = 1, \dots, N_a\} \tag{4.60}$$

and the ORHP poles of P by

$$\{p_i; i = 1, \dots, N_p\}, \tag{4.61}$$

including multiplicities in each case. To each a_i and p_i, denote the associated NMP zeros of T^0 by

$$\{a_{ik} = \frac{1}{T}\log(a_i) + jk\omega_s, k = 0, \pm 1, \pm 2, \dots\} \tag{4.62}$$

and

$$\{p_{ik} = p_i + jk\omega_s, k = \pm 1, \pm 2, \dots\}, \tag{4.63}$$

respectively. As discussed in Chapter 3, it is possible that the hold function has a countable infinity of NMP zeros, and thus that N_γ in (4.59) equals infinity.

Denote the Blaschke products of NMP zeros of P and H by

$$B_\zeta(s) = \prod_{i=1}^{N_\zeta} \left(\frac{\zeta_i - s}{\bar{\zeta}_i + s}\right) \tag{4.64}$$

and[12]

$$B_\gamma(s) = \prod_{i=1}^{N_\gamma} \left(\frac{\gamma_i - s}{\bar{\gamma}_i + s}\right). \tag{4.65}$$

Denote the Blaschke product of ORHP plant poles by

$$B_p(s) = \prod_{i=1}^{N_p} \left(\frac{p_i - s}{\bar{p}_i + s}\right) \tag{4.66}$$

For each NMP zero of C_d and for each ORHP pole of P, denote the Blaschke products of associated NMP zeros of T^0 by[12]

$$B_{a_i}(s) = \prod_{k=-\infty}^{\infty} \left(\frac{a_{ik} - s}{\bar{a}_{ik} + s}\right) \tag{4.67}$$

[12]That the Blaschke product (4.65) converges even if N_γ is infinite follows from Hoffman [1962]. The same is valid for (4.67)-(4.68).

and

$$B_{p_i}(s) = \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{p_{ik} - s}{\bar{p}_{ik} + s} \right). \tag{4.68}$$

4.4.2 Poisson Sensitivity Integral

We now derive an integral *inequality* that must be satisfied by $\log|S^0(j\omega)|$.

Theorem 4.4.1 (Poisson integral for S^0)
Assume that the hypotheses of Lemma 2.2.2 are satisfied. Let $\xi = x + jy$ equal one of (4.58)-(4.59) *or* (4.62)-(4.63). *Then*

$$\int_0^\infty \log|S^0(j\omega)|\,\Psi(\xi,\omega)d\omega \geq \pi \log|B_p^{-1}(\xi)|, \tag{4.69}$$

where $\Psi(s,\omega)$ is the Poisson kernel for the half plane defined in (3.30).

Proof: Denote the NMP zeros of S^0 by $\mu_1, \mu_2, \cdots$ and define the Blaschke product

$$B_\mu(s) = \prod_i \frac{\mu_i - s}{\bar{\mu}_i + s}.$$

Then $S^0 = \check{S}B_\mu$ where $\check{S}$ has no poles or zeros in the ORHP. The Poisson integral [Levinson and Redheffer, 1970, p. 225] implies that (4.69) holds with *equality* if $B_p(\xi)$ is replaced by $B_\mu(\xi)$. Since the set of NMP zeros of S^0 due to the ORHP poles of P is generally a *proper* subset of all such zeros (cf. Remark 4.2.6) inequality (4.69) follows. □

Theorem 4.4.1 has several design implications, which we describe in a series of remarks.

Remark 4.4.1 (NMP Plant Zeros) As in the continuous time case, if the plant is non-minimum phase, then requiring that $|S^0(j\omega)| < 1$ over a frequency range Ω implies that, necessarily, $|S^0(j\omega)| > 1$ at other frequencies. The severity of this tradeoff depends upon the relative location of the NMP zero and the frequency range Ω. We now discuss this in more detail.

We recall the definition of the weighted length of an interval by the Poisson kernel for the half plane, introduced in Chapter 3, (3.35). Let $\xi = x + jy$ be a point lying in $\mathbb{C}^+$, and consider the frequency interval $\Omega = [0, \omega_0)$. Then, we had that

$$\Theta(\xi, \Omega) \triangleq \int_0^{\omega_0} \Psi(\xi, \omega)\, d\omega.$$

We have seen in Subsection 3.3.1 that $\Theta(\xi, \Omega)$ equals the negative of the phase lag contributed by a Blaschke product of ξ at the upper end point of the interval Ω. With this notation, the following result is an immediate consequence of (4.69).

Corollary 4.4.2
Suppose that $\zeta = x + jy$ *is a NMP zero of the plant, and suppose that*

$$|S^0(j\omega)| \leq \alpha, \quad \textit{for all } \omega \textit{ in } \Omega.$$

Then

$$\sup_{\omega > \omega_0} |S^0(j\omega)| \geq (1/\alpha)^{\frac{\Theta(\zeta,\Omega)}{\pi - \Theta(\zeta,\Omega)}} \left|B_p^{-1}(\zeta)\right|^{\frac{\pi}{\pi - \Theta(\zeta,\Omega)}} \tag{4.70}$$

○

The bound (4.70) shows that if disturbance attenuation is required throughout a frequency interval in which the NMP zero contributes significant phase lag, then disturbances will be greatly amplified at some higher frequency. The term due to the Blaschke product in (4.69) shows that plants with approximate ORHP pole-zero cancelations yield particularly sensitive feedback systems. ◇

Remark 4.4.2 (NMP Hold Zeros) A non-minimum phase zero of the hold response imposes precisely the same tradeoff as does a zero of the plant in the same location. This tradeoff is exacerbated if the NMP hold zero is near an unstable plant pole. Poor sensitivity in this case is to be expected, as an exact pole-zero cancelation yields an unstable hidden mode in the discretized plant[13]. ◇

Remark 4.4.3 (Unstable Plant Poles) Using an analog controller, the sensitivity function of a system with an unstable, but minimum phase, plant can be made arbitrarily small over an arbitrarily wide frequency range Zames and Bensoussan [1983] while maintaining sensitivity bounded outside this range. This is no longer true for digital controllers and the fundamental sensitivity function. The following result is an immediate consequence of (4.69). ◇

Corollary 4.4.3
(i) *Assume that the plant has a real ORHP pole,* $p = x$. *Then*

$$\|S^0\|_\infty \geq \sqrt{1 + \left(\frac{x}{\omega_N}\right)^2} \tag{4.71}$$

(ii) *Assume that the plant has an ORHP complex conjugate pole pair,* $p = x + jy, \bar{p} = x - jy$. *Then for* $k = \pm 1, \pm 2 \ldots$

$$\|S^0\|_\infty \geq \sqrt{1 + \left(\frac{x}{k\omega_N}\right)^2} \sqrt{1 + \left(\frac{x}{y - k\omega_N}\right)^2} \tag{4.72}$$

○

[13] See the conditions for non-pathological sampling in Lemma 2.2.1. An example of a poorly conditioned discretized system is given at the end of §7.1 in Chapter 7.

Proof: We show only (i); (ii) is similar. Evaluate (4.69) with ξ equal one of (4.63), i.e., $\xi = x + jk\omega_s$, with $k = \pm 1, \pm 2, \ldots$ Then

$$\begin{aligned} \pi \log \|S^0\|_\infty &\geq \int_0^\infty \log |S^0(j\omega)| \Psi(\xi, \omega)\, d\omega \\ &\geq \pi \log \left| \frac{2x + jk\omega_s}{-jk\omega_s} \right| . \end{aligned} \tag{4.73}$$

From (4.73) follows

$$\begin{aligned} |S^0\|_\infty &\geq \sqrt{1 + \left(\frac{x}{k\omega_N}\right)^2} \\ &\geq \sqrt{1 + \left(\frac{x}{\omega_N}\right)^2} . \end{aligned}$$

□

In either case of Corollary 4.4.3, the fundamental sensitivity function necessarily has a peak strictly greater than one.

For a real pole, achieving good sensitivity requires that the sampling rate be sufficiently fast with respect to the time constant of the pole; e.g., achieving $\|S^0\|_\infty < 2$ requires that $\omega_N > x/\sqrt{3}$. This condition is also necessary for a complex pole pair. Furthermore, sensitivity will be poor if $y \approx k\omega_N$ for some $k \neq 0$. The reason for poor sensitivity in this case is clear; if $y = k\omega_N$, then the complex pole pair violates the non-pathological sampling condition (2.12), and the discretized plant will have an unstable hidden mode.

More generally, we have

Corollary 4.4.4
Assume that the plant has unstable poles p_i *and* p_ℓ *with* $p_i \neq \bar{p}_\ell$. *Then*

$$\|S^0\|_\infty \geq \max_{k \neq 0} \left| \frac{\bar{p}_i + p_\ell + jk\omega_s}{p_i - p_\ell - jk\omega_s} \right| \tag{4.74}$$

and

$$\|S^0\|_\infty \geq \max_{k \neq 0} \left| \frac{p_i + p_\ell + jk\omega_s}{\bar{p}_i - p_\ell - jk\omega_s} \right| \tag{4.75}$$

○

It follows that if sampling is "almost pathological", in that $p_i - p_\ell \approx jk\omega_s$, or $\bar{p}_i - p_\ell \approx jk\omega_s$, then sensitivity will be large.

Remark 4.4.4 (Approximate Discrete Pole Zero Cancelations) Suppose that the discrete compensator has an NMP zero a_i. Then (4.69) holds with ξ equal to one of the points a_{ik} (4.62) in the s-plane that map to a_i in the z-plane. If the plant has an unstable pole near one of these points, then the right hand side of (4.69) will be large, and S^0 will have a large peak. Poor sensitivity is plausible, because this situation corresponds to an approximate pole-zero cancelation between a NMP zero of the compensator and a pole of the discretized plant. ◇

4.4.3 Poisson Complementary Sensitivity Integral

We now derive a result for T^0 dual to that for S^0 obtained in the previous section. An important difference is that we can characterize *all* NMP zeros of T^0, and thus obtain integral *equalities*.

First, we note an additional property of the hold response function.

Lemma 4.4.5
The hold response function (2.4) *may be factored as*

$$H(s) = \check{H}(s)e^{-s\tau_H} B_\gamma(s), \tag{4.76}$$

where $\tau_H \geq 0$, B_γ *is given by* (4.65), *and* $\log|\check{H}|$ *satisfies the Poisson integral relation.*

Proof: Follows from Hoffman [1962, pp. 132-133]. □

As we discussed in Chapter 3, page 27, for a FDLTI GSHF, $\tau_H = 0$. For a PC GSHF, defined by (3.11), if $\bar{k}$ denotes the smallest value of k for which $a_k \neq 0$, then is easily seen from (3.13)-(3.14) that $\tau_H = \bar{k}T/N$.

We have seen in Chapter 3 explicit expressions for the zeros of a piecewise constant hold with $\alpha_0 \neq 0$ and approximations to the zeros of a FDLTI hold. We remark that, in each case, H possesses infinitely many zeros which approach infinity along well defined paths which may lie in the ORHP (See §3.2).

Theorem 4.4.6 (Poisson integral for T^0)
Assume that the hypotheses of Lemma 2.2.2 are satisfied. Let $p_\ell = x + jy$ *be an ORHP pole of* P. *Then*

$$\begin{aligned}\int_0^\infty \log|T^0(j\omega)|\,\Psi(p_\ell,\omega)\,d\omega &= \pi x\tau_P + \pi x\tau_H + \pi x N_C T \\ &\quad + \pi\log|B_\zeta^{-1}(p_\ell)| + \pi\log|B_\gamma^{-1}(p_\ell)| \\ &\quad + \pi\sum_{i=1}^{N_p}\log|B_{p_i}^{-1}(p_\ell)| + \pi\sum_{i=1}^{N_a}\log|B_{a_i}^{-1}(p_\ell)|\end{aligned} \tag{4.77}$$

where $\Psi(s,\omega)$ *is the Poisson kernel for the half plane defined in* (3.30).

Proof: Note that T^0 has an inner-outer factorization

$$T^0(s) = \check{T}(s)\,e^{-s\tau_P}\,e^{-s\tau_H}\,e^{-s\tau N_C T}\,B_\zeta(s)\,B_\gamma(s)\prod_{i=1}^{N_p}B_{p_i}(s)\prod_{i=1}^{N_a}B_{a_i}(s)$$

where $\log|\check{T}|$ satisfies the Poisson integral Levinson and Redheffer [1970]. Since $\log|\check{T}(j\omega)| = \log|T^0(j\omega)|$, the result follows. □

We comment on the design implications of Theorem 4.4.6 in a series of remarks.

Remark 4.4.5 The first three terms on the right hand side of (4.77) show that $|T^0(j\omega)|$ will display a large peak if there is a long time delay in the plant, digital controller, or hold function. $\diamond$

Remark 4.4.6 The fourth and fifth terms on the right hand side of (4.77) show that $|T^0(j\omega)|$ will display a large peak if there is an approximate unstable pole-zero cancelation in the plant, or between the plant and the hold function. By the non-pathological sampling condition (ii) in Lemma 2.2.1, the latter peak corresponds to an approximate unstable pole-zero cancelation in the *discretized* plant. $\diamond$

The following result is analogous to Corollary 4.4.4 for T^0.

Corollary 4.4.7
(i) Assume that $p_\ell = x$*, a real pole. Then*

$$\|T^0\|_\infty \geq \frac{\sinh\left(\dfrac{\pi x}{\omega_N}\right)}{\left(\dfrac{\pi x}{\omega_N}\right)}. \tag{4.78}$$

(ii) Assume that $p_\ell = x + jy$*, a complex pole. Then*

$$\|T^0\|_\infty \geq \frac{\sinh\left(\dfrac{\pi x}{\omega_N}\right)}{\left(\dfrac{\pi x}{\omega_N}\right)} \left|\frac{\sinh\left(\dfrac{\pi p_\ell}{\omega_N}\right)}{\left(\dfrac{\pi p_\ell}{\omega_N}\right)}\right| \left|\frac{\left(\dfrac{\pi y}{\omega_N}\right)}{\sin\left(\dfrac{\pi y}{\omega_N}\right)}\right|. \tag{4.79}$$

Proof: By rearranging definition (4.68), we have

$$B_{p_i}(s) = \prod_{k=1}^{\infty} \frac{1 - \left(\dfrac{p_i - s}{jk\omega_s}\right)^2}{1 - \left(\dfrac{\bar{p}_i + s}{jk\omega_s}\right)^2}$$

Using the identities [Levinson and Redheffer, 1970, p. 387],

$$\frac{\sin \pi\alpha}{\pi\alpha} = \prod_{k=1}^{\infty}\left(1 - \frac{\alpha^2}{k^2}\right)$$

and $\sin j\alpha = j \sinh \alpha$ yields

$$B_{p_i}(s) = \frac{\sinh \pi\left(\dfrac{p_i - s}{\omega_s}\right)}{\pi\left(\dfrac{p_i - s}{\omega_s}\right)} \frac{\pi\left(\dfrac{\bar{p}_i + s}{\omega_s}\right)}{\sinh \pi\left(\dfrac{\bar{p}_i + s}{\omega_s}\right)} \tag{4.80}$$

Note that the first factor on the right hand side of (4.80) converges to one as $s \to p_i$. It follows that

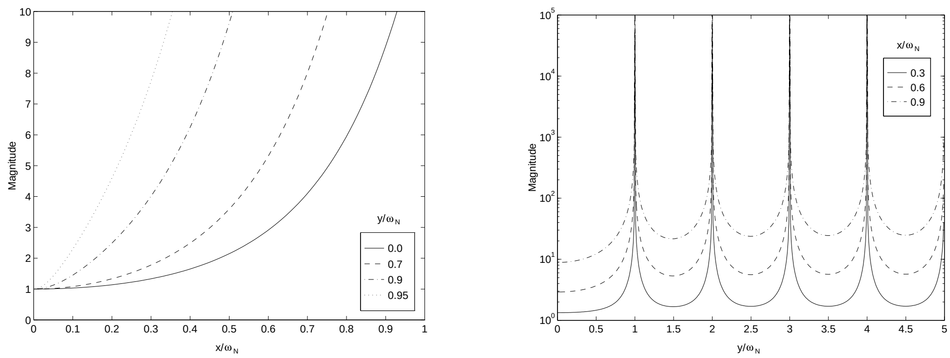
$$B_{p_\ell}(p_\ell) = \frac{\left(\dfrac{\pi x}{\omega_N}\right)}{\sinh\left(\dfrac{\pi x}{\omega_N}\right)} \tag{4.81}$$

Inverting yields (4.78). Furthermore

$$B_{\bar{p}_\ell}(p_\ell) = \frac{\sin\left(\dfrac{\pi y}{\omega_N}\right)}{\left(\dfrac{\pi y}{\omega_N}\right)} \frac{\left(\dfrac{\pi p_\ell}{\omega_N}\right)}{\sinh\left(\dfrac{\pi p_\ell}{\omega_N}\right)} \tag{4.82}$$

Together (4.81)-(4.82) yield (4.79). $\square$

Figure 4.5(a) give plots of the bound (4.79) versus $\mathbb{R}e\{p_\ell\}$ for various values of $\mathbb{I}m\{p_\ell\}$, and Figure 4.5(b) give plots of the bound (4.79) versus $\mathbb{I}m\{p_\ell\}$ for various values of $\mathbb{R}e\{p_\ell\}$. The pole location has been normalized by the Nyquist frequency. Note in Figure 4.5(b) that for a complex pole $\|T^0\|_\infty$ will become arbitrarily large as $y \to k\omega_N, k = \pm 1, \pm 2 \ldots$, because sampling becomes pathological at such frequencies. It follows from these plots that to achieve good robustness the Nyquist frequency should be chosen several times larger than the radius of any unstable pole.



(a) (b)

Figure 4.5: Lower bounds on $\|T^0\|_\infty$.

4.4.4 Poisson Harmonic Response Integral

Harmonic response functions (4.8) also satisfy Poisson integral relations. As for the case of T^0, in this case we also obtain an integral *equality*, since *all* the zeros of

T^k in $\overline{\mathbb{C}^+}$ are characterized. The result follows as a straightforward corollary to Theorem 4.4.6.

Corollary 4.4.8 (Poisson Integral for T^k)
Assume that the hypotheses of Lemma 2.2.2 are satisfied. Let $p_\ell = x + jy$ *be an ORHP pole of* P. *Then*

$$\begin{aligned}\int_0^\infty \log|T^k(j\omega)|\,\Psi(p_\ell - jk\omega_s, \omega)\,d\omega = \pi \log\left|\frac{F_{-k}(p_\ell)}{F(p_\ell)}\right| &+ \pi x\tau_P + \pi x\tau_H + \pi x N_C T \\ &+ \pi\log|B_\zeta^{-1}(p_\ell)| + \pi\log|B_\gamma^{-1}(p_\ell)| \\ &+ \pi\sum_{i=1}^{N_p}\log|B_{p_i}^{-1}(p_\ell)| \\ &+ \pi\sum_{i=1}^{N_a}\log|B_{a_i}^{-1}(p_\ell)|\end{aligned} \tag{4.83}$$

where $\Psi(s,\omega)$ *is the Poisson kernel for the half plane defined in* (3.30).

Proof: Immediately from relation (4.35),

$$\begin{aligned}\int_0^\infty \log|T^k(j\omega)|\,\Psi(p_\ell - jk\omega_s, \omega)\,d\omega &= \int_0^\infty \log\left|\frac{F(j\omega)}{F_k(j\omega)}\right|\,\Psi(p_\ell - jk\omega_s, \omega)\,d\omega \\ &+ \int_0^\infty \log|T^0(j(\omega + k\omega_s))|\,\Psi(p_\ell - jk\omega_s, \omega)\,d\omega.\end{aligned}$$

The first integral on the RHS of the equation above gives the first term on the RHS of (4.83), since by our assumptions on F, $\log(F(s)/F(s - jk\omega_s))$ satisfies a Poisson integral relation. The second integral is the Poisson Complementary Sensitivity Integral of (4.77). The result follows. □

The implications of this integral constraint are similar to those for T^0, since — except for the first — all terms on the RHS of (4.83) are the same on the RHS of (4.77). Hence $|T^k(j\omega)|$ will display a large peak if there are long time delays in the plant, digital controller, or hold function. There will be also large peaks if there are approximate unstable pole-zero cancelations in the plant or between the plant and the hold function. Differently in this case, these constraints are relaxed by the presence of the first term on the RHS of (4.83), which will be generally negative since the anti-aliasing filter is normally designed to roll off at high frequencies. The following corollary, corresponding with Corollary 4.4.7, shows this.

Corollary 4.4.9
(i) Assume that $p_\ell = x$, *a real pole. Then*

$$\|T^k\|_\infty \geq \frac{\sinh\left(\dfrac{\pi x}{\omega_N}\right)}{\left(\dfrac{\pi x}{\omega_N}\right)}\left|\frac{F_{-k}(x)}{F(x)}\right|.$$

(ii) Assume that $p_\ell = x + jy$*, a complex pole. Then*

$$\|T^k\|_\infty \geq \frac{\sinh\left(\dfrac{\pi x}{\omega_N}\right)}{\left(\dfrac{\pi x}{\omega_N}\right)} \left|\frac{\sinh\left(\dfrac{\pi p_\ell}{\omega_N}\right)}{\left(\dfrac{\pi p_\ell}{\omega_N}\right)}\right| \left|\frac{\left(\dfrac{\pi y}{\omega_N}\right)}{\sin\left(\dfrac{\pi y}{\omega_N}\right)}\right| \left|\frac{F_{-k}(p_\ell)}{F(p_\ell)}\right|.$$

○

4.4.5 Bode Sensitivity Integral

The following is a generalization of the classical sensitivity integral theorem of Bode [1945]. As in the case of the Poisson sensitivity integral, we only obtain an *inequality*, because S^0 may possess NMP zeros in addition to those associated with ORHP plant poles.

Theorem 4.4.10
Assume that the hypotheses of Lemma 2.2.2 are satisfied.

$$\int_0^\infty \log|S^0(j\omega)|d\omega \geq \pi \sum_{i=1}^{N_p} \mathbb{R}e\{p_i\} \tag{4.84}$$

Proof: Assumption 1 implies that $|sH(s)|$ is bounded on $\overline{\mathbb{C}^+}$ (see (A.12) in Chapter A). This fact, together with the assumption that F is strictly proper, imply that

$$\lim_{\substack{s\to\infty \\ \mathbb{R}e\{s\}\geq 0}} |s\,T^0(s)| = 0.$$

Hence the technique used in Freudenberg and Looze [1985] to derive the continuous-time version of (4.84) may be applied. □

The sensitivity integral states that if $|S^0(j\omega)| < 1$ over some frequency range, then necessarily $|S^0(j\omega)| > 1$ at other frequencies. Hence there is a tradeoff between reducing and amplifying the fundamental component of the response to disturbances at different frequencies. This tradeoff is exacerbated if the plant has ORHP poles. As in the analog cases, (4.84) does not impose a meaningful design limitation unless an additional bandwidth constraint is imposed [e.g., Freudenberg and Looze, 1985]. The need to prevent aliasing in hybrid systems implies that bandwidth constraints are potentially more severe than in the analog case. Design implications remain to be worked out, but it should be noted that the frequency response of the hold function as well as that of the anti-aliasing filter will need to be considered.

4.4.6 Middleton Complementary Sensitivity Integral

We here derive an integral relation for T^0 that is dual to the Bode sensitivity integral obtained for S^0 in the preceding section. This result is a generalization to hybrid systems of the complementary sensitivity integral introduced in Middleton and Goodwin [1990], and Middleton [1991]. As in the case of the Poisson complementary sensitivity integral, exhaustive knowledge of the zeros of T^0 yields an integral equality.

Theorem 4.4.11
Assume that the hypotheses of Lemma 2.2.2 are satisfied. Suppose also that $T^0(0) \neq 0$. *Define* $\acute{T}^0(0) = dT^0/ds|_{s=0}$. *Then*

$$\int_0^\infty \log\left|\frac{T^0(j\omega)}{T^0(0)}\right| \frac{d\omega}{\omega^2} = \frac{\pi}{2}(\tau_P + \tau_H + N_C T) + \pi \sum_{k=1}^{N_\zeta} \frac{1}{\zeta_k} + \pi \sum_{k=1}^{N_\gamma} \frac{1}{\gamma_k} - \pi \sum_{k=1}^{N_p} \frac{1}{p_k}$$
$$+ \frac{\pi T}{2} \sum_{k=1}^{N_p} \coth\left(\frac{p_k T}{2}\right) + \frac{\pi T}{2} \sum_{k=1}^{N_a} \coth\left(\frac{a_k T}{2}\right)$$
$$+ \frac{\pi}{2} \frac{\acute{T}^0(0)}{T^0(0)}. \tag{4.85}$$

Proof: See §A.3 in Appendix A. □

This result states that if the ratio $|T^0(j\omega)/T^0(0)| < 1$ over some frequency range, then necessarily this ratio must exceed one at other frequencies. Hence, there is a tradeoff between reducing and amplifying the fundamental component of the response to noise in different frequency ranges. Further comments are found in the following series of remarks.

Remark 4.4.7 The first term on the right hand side of (4.85) show that the tradeoff worsens if the plant, hold, or compensator has a time delay. The second three terms show that the tradeoff worsens if the plant, hold, or compensator has NMP zeros. It is easy to verify that the sum of the fifth and sixth terms is positive, and thus the tradeoff also worsens if the plant has ORHP poles. For an interpretation of the seventh term, see Remark 4.4.9 below. ◇

Remark 4.4.8 One difference between (4.85) and the analogous results in Middleton and Goodwin [1990], Middleton [1991] is that the latter references assume the presence of an integrator in the system. We avoid this requirement by instead assuming that $T^0(0) \neq 0$ and normalizing T^0 by its DC value. This approach could also have been taken in Middleton and Goodwin [1990], Middleton [1991]; we have adopted it here to obtain a more general result. If there is indeed an integrator in the system, then the following corollary, which follows from Corollaries 4.3.1 and 4.3.2, shows that the normalization factor is unnecessary. ◇

Corollary 4.4.12
Assume that

(i) P *contains at least one integrator, and/or*

(ii) C_d *contains at least one integrator* and H *is a ZOH.*

Then $T^0(0) = 1$. ◦

Remark 4.4.9 We now provide an interpretation for the ninth term in (4.85). To do this, we consider the hybrid system depicted in Figure 4.6.

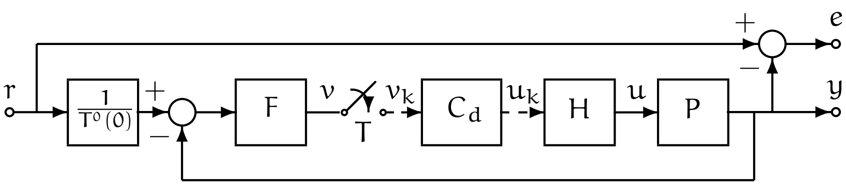


Figure 4.6: Hybrid system.

The proof of the following result is straightforward, and hence omitted.

Lemma 4.4.13
Suppose that the hypotheses of Lemma 2.2.2 are satisfied.

(i) Consider the response of the system to a unit step input. Then, as $t \to \infty$, $e \to e_{ss}$, *where*

$$e_{ss}(t) = \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} -\frac{P(j\ell\omega_s)H(j\ell\omega_s)}{P(0)H(0)} e^{j\ell\omega_s t} \tag{4.86}$$

Furthermore, if H *is a ZOH and/or if* P *contains an integrator, then* $e_{ss} = 0$.

(ii) Consider the response of the system to a unit ramp input. Then as $t \to \infty$, $e \to e_{ss}$, *where*

$$e_{ss}(t) = -\frac{\acute{T}^0(0)}{T^0(0)} - \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} [\alpha_\ell + \beta_\ell t]\, e^{j\ell\omega_s t}, \tag{4.87}$$

with

$$\alpha_\ell = \frac{d}{ds}\left[\frac{T^0(s)}{T^0(0)} \frac{F(s + j\ell\omega_s)}{F(s)}\right]\Bigg|_{s=-j\ell\omega_s} \tag{4.88}$$

and

$$\beta_\ell = \frac{P(-j\ell\omega_s)H(-j\ell\omega_s)}{P(0)H(0)}. \tag{4.89}$$

Furthermore, if H *is a ZOH and/or if* P *contains an integrator, then* $\beta_\ell = 0$ *and* e_{ss} *is bounded. If both these conditions are satisfied, then* $\alpha_\ell = 0$ *and*

$$e_{ss}(t) = -\frac{\acute{T}^0(0)}{T^0(0)}. \tag{4.90}$$

○

It follows from part (ii) of Lemma 4.4.13 that the constant $\acute{T}^0(0)/T^0(0)$ plays a role similar to that played by the reciprocal of the velocity constant in a Type-1 analog feedback system[14]. Hence the ninth term on the right hand side of (4.85) can ameliorate the severity of the design tradeoff *only* if the steady-state error to a ramp input is large and positive, so that the output lags the reference input significantly. ◇

4.5 Summary

We conclude this chapter with a brief summary of inherent design limitations for hybrid feedback systems.

Perhaps most important is the fact that those plant properties such as non-minimum phase zeros, unstable poles, and time delays that pose design difficulty for analog feedback systems continue to pose difficulty when the controller is implemented digitally. Furthermore, the existence of such a difficulty is independent of the type of hold function used. It is important, however, that the intersample behavior be examined if the problems are to be detected. Examining system response only at the sampling instants may be misleading.

There are also a number of design limitations unique to digital controller implementations. First, there are limits upon the ability of high compensator gain to achieve disturbance rejection unless the hold function satisfies additional constraints. Second, there are design limitations due to potential non-minimum phase zeros of the hold function. Perhaps most interesting are the design limitations due to unstable plant poles. If the sample rate is "almost pathological" and/or is slow with respect to the time constant of the pole, then sensitivity, robustness, and response to exogenous inputs will all be poor.

Furthermore, as it is apparent from the results in Araki and Ito [1993], Leung et al. [1991], and Thompson et al. [1983], the fundamental and harmonic response functions introduced here have connections with the L_2-induced norm of the system, and therefore with its robustness properties against linear time-varying perturbations. We shall deal with these issues in depth in the forecoming chapter.

Perhaps most interesting is the observation that the hold response function plays a role identical to that of the anti-aliasing filter in mapping high frequency plant behavior, including uncertainty, into the response of the discretized plant. This will be the main subject of Chapter 7.

[14]For an analog version of (4.90), see Truxal [1955, p. 286]