2

Preliminaries

This chapter defines most of our notation, and introduces general assumptions and preliminary results required in the sequel. We present a key formula concerning the Laplace transform of a sampled signal that will play an important role in the rest of this thesis. This formula yields the well-known infinite summation expression showing that the response of the discretized plant at a given frequency depends upon that of the analog plant at infinitely many frequencies. We finish the chapter reviewing the basic conditions for closed-loop stability of sampled-data systems; i.e., a non-pathological sampling assumption, and the closed-loop stability of the discretized system.

2.1 Analog and Discrete Signals

2.1.1 Signal Spaces

We start introducing some standard signal spaces. We denote the set of complex numbers by $\mathbb{C}$. The open and closed right halves of $\mathbb{C}$ are denoted by $\mathbb{C}^+$ and $\overline{\mathbb{C}^+}$ respectively, and sometimes we shall use the acronyms ORHP and CRHP. Correspondingly, we denote by $\mathbb{C}^-$ and $\overline{\mathbb{C}^-}$ the open and closed left halves of $\mathbb{C}$, also referred as OLHP and CLHP, respectively. We denote the set of real numbers by $\mathbb{R}$, and by $\mathbb{R}_0^+$ we represent the set of non-negative real numbers, i.e., the segment $[0, \infty)$. The open and closed unit disks in $\mathbb{C}$ are denoted by $\mathbb{D} \triangleq \{z : |z| < 1\}$ and $\overline{\mathbb{D}} \triangleq \{z : |z| \leq 1\}$ respectively; we denote their complements by $\mathbb{D}^{\mathsf{C}}$ and $\overline{\mathbb{D}}^{\mathsf{C}}$.

As usual, $L_p^n(\mathbb{R}_0^+)$ denotes the space of Lebesgue measurable functions f from $\mathbb{R}_0^+$ to $\mathbb{R}^n$ that satisfy

$$\|f\|_{L_p} \triangleq \left(\int_0^\infty |f(t)|^p \, dt\right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|f\|_{L_\infty} \triangleq \operatorname*{ess\,sup}_{t \in \mathbb{R}_0^+} |f(t)| < \infty,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. We denote by $L_{pe}^n(\mathbb{R}_0^+)$ the extended space of $L_p^n(\mathbb{R}_0^+)$, i.e., the space of functions whose truncations to intervals $[0, a)$ are in $L_p^n(\mathbb{R}_0^+)$ for any finite real number a.

In a similar way, L_2^n denotes the space of functions $F(j\omega)$ defined on $j\mathbb{R}$ with values over $\mathbb{C}^n$ and satisfying

$$\|F\|_{L_2} \triangleq \left(\int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega\right)^{1/2} < \infty.$$

Here the Euclidean norm $|\cdot|$ is taken on $\mathbb{C}^n$, i.e., $|F| = \sqrt{F^*F}$, where F^* denotes the complex conjugate transpose of F. In general, we shall denote the transpose of a matrix M by M^T, and by $\overline{M}$ its conjugate.

In discrete-time we represent by ℓ_p^n the space of sequences $u \triangleq \{u_k\}_{k=-\infty}^{\infty}$ valued in $\mathbb{C}^n$ and satisfying

$$\|u\|_{\ell_p} \triangleq \left(\sum_{k=-\infty}^{\infty} |u_k|^p\right)^{1/p} < \infty \quad \text{for } 1 \le p < \infty,$$

and

$$\|u\|_{\ell_\infty} \triangleq \sup_k |u_k| < \infty.$$

We shall dispense with the superscript n in the above notations whenever the dimension of the spaces is clear from the context. We shall also omit the subindex that indicates the spaces in the notation of norms $\|\cdot\|$ when they are clear from the context.

We shall represent linear dynamic systems as input-output operators acting on L_p spaces. If $\mathcal{M}$ is a linear operator defined by

$$\begin{aligned} \mathcal{M} \quad &: \quad L_p(\mathbb{R}_0^+) \to L_p(\mathbb{R}_0^+) \\ &: \quad u \mapsto y = \mathcal{M}u, \end{aligned}$$

the L_p-induced norm of the operator $\mathcal{M}$ is defined as

$$\|\mathcal{M}\|_p \triangleq \sup \left\{ \frac{\|\mathcal{M}y\|_{L_p}}{\|u\|_{L_p}} : \text{for } u \text{ in } L_p(\mathbb{R}_0^+), \text{ and } \|u\|_{L_p} \neq 0 \right\}.$$

A quick-reference list of the above notations may be found on page 156.

2.1.2 Samplers and Holds

As discussed in Chapter 1, the implementation of a controller for a continuous-time system by means of a digital device, such as a computer, implies the process of sampling and reconstruction of analog signals. By sampling, an analog signal is converted into a sequence of numbers that can then be digitally manipulated. The hold device performs the inverse operation, translating the output of the digital controller into a continuous-time signal. We shall assume throughout that nonlinearities associated with the process of discretization, such as finite memory word-length, quantization, etc., have no significant effect on the sampled-data system.

We assume also that sampling is regular, i.e., if T is the *sampling period*, sampling is performed at instants $t = kT$, with $k = 0, \pm 1, \pm 2, \ldots$. Associated with T, we define the *sampling frequency* $\omega_s = 2\pi/T$. By Ω_N we denote the *Nyquist range* of frequencies $[-\omega_s/2, \omega_s/2]$.

We consider an idealized model of the sampler. If y is an analog signal defined on the time set $\mathbb{R}_0^+$ with values over $\mathbb{C}^n$, we define the sampling operator with sampling period T, denoted by $\mathcal{S}_T$, as

$$\mathcal{S}_T\{y\} = \{y_k\}_{k=-\infty}^{\infty}, \tag{2.1}$$

where $\{y_k\}_{k=-\infty}^{\infty}$ is the sequence representing the sampled signal, and $y_k = y(kT^+)$[1]. Thus, the sampler is a linear, periodically time-varying operator. Note that the sampler operator may be unbounded in many standard signal spaces, as for example from $L_p(\mathbb{R}_0^+)$ to ℓ_p when $1 \leq p < \infty$ Chen and Francis [1991]. Therefore, we need to specify with some care the class of signals that are "sampleable".

A class of functions that guarantee that the sampling operator is well-defined is the class of functions of *bounded variation* (BV). These functions will be required to define the hold devices we shall deal with, and to assure the validity of a sampling formula that will be the starting point of our approach to sampled-data systems. The following definition is taken from Riesz and Sz.-Nagy [1990].

Definition 2.1.1 (Function of Bounded Variation)
A function f defined over a real interval (a, b) is of BV if the following sum is bounded,

$$\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| < \infty, \tag{2.2}$$

for every partition of the interval (a, b) into subintervals (t_k, t_{k-1}), where $k = 1, 2, \ldots, n$, and $t_0 = a$, $t_n = b$. The least upper bound of the sum in (2.2) is called the *total variation* of f in the interval (a, b). $\diamond$

A function of BV is not necessarily continuous, but it is differentiable almost everywhere and its derivative is a function in $L_1(a, b)$ Rudin [1987]. Moreover, the limits $f(t^+)$ and $f(t^-)$ are well defined for every t in (a, b), which means that f can have discontinuities of at most the *finite-jump* type.

The hold device that we shall consider is a GSHF *a la* Kabamba [1987], defined by the operation

$$u(t) = h(t - kT)\, u_k, \quad \text{for } kT \leq t < (k+1)T, \tag{2.3}$$

where $\{u_k\}_{k=-\infty}^{\infty}$ is a discrete sequence, and h is a bounded function with support on the interval $[0, T)$. We consider the case in which the sequence $\{u_k\}_{k=-\infty}^{\infty}$ takes values in $\mathbb{R}^p$, and so h takes values in $\mathbb{R}^{p \times p}$. We shall assume throughout that h satisfies the following technical conditions.

[1]Here, $y(kT^{\pm})$ denotes the *right (left) limit* of $y(\cdot)$ at $t = kT$, i.e.,

$$y(kT^{\pm}) \triangleq \lim_{\epsilon \downarrow 0} f(kT \pm \epsilon), \quad \text{for } \epsilon > 0,$$

whenever the limit exists.

Assumption 1
The hold function h *is a function of BV on* $[0, T)$. ◦

As discussed in Middleton and Freudenberg [1995], we can associate a *frequency response function* to this hold device, defined by

$$H(s) = \int_0^T e^{-st}\, h(t)\, dt. \tag{2.4}$$

Since h is supported on a finite interval, it follows that H is an entire function, i.e., analytic at every s in $\mathbb{C}$. For example, in the case of the ZOH we have the well-known response $H(s) = (1 - e^{-sT})/s$. Frequency responses of other types of holds will be studied in detail in Chapter 3.

We shall be particularly interested in the *zeros* of the response function H. These have transmission blocking properties, and may affect the stabilizability of the discretized system Middleton and Freudenberg [1995]. Furthermore, as we shall see in Chapter 4, they are an important factor in analysis of the achievable performance of the sampled-data system.

Definition 2.1.2 (Zeros of the Hold Middleton and Freudenberg [1995])
Consider a response function defined by (2.4) and suppose that $\det(H)$ is not identically zero. Then the *zeros* of H are those values s in $\mathbb{C}$ for which $H(s)$ has less than full rank. ◇

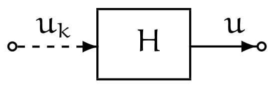

Figure 2.1: Response of a GSHF.

The frequency response of the hold defined in (2.4) is useful to compute the Laplace transform of the output of the hold device (see Figure 2.1). As described in Middleton and Freudenberg [1995], the i-th column of the frequency response function (2.4) represents the Laplace transform of the output of the hold to an unitary pulse in the ith input. More generally, if U_d is the $\mathcal{Z}$-transform of the input sequence $\{u_k\}_{k=-\infty}^{\infty}$, then we have the following Åström and Wittenmark [1990].

Lemma 2.1.1
Consider the hold defined by (2.3) *and its associated frequency response* (2.4). *Then*

$$U(s) = H(s)\, U_d(e^{sT}).$$

◦

GSHFs have been proposed as a more versatile alternative to the traditional ZOH [see for example Kabamba, 1987], and indeed, recent studies have shown that if a solution to the sampled-data H_∞ control problem exists, then it may be realized by a LTI discrete controller and a GSHF Sun et al. [1993]. Nevertheless, these devices certainly are much more complex to be implemented and — as some authors have suggested and we shall expand on — they may bring in serious intersample difficulties.

2.1.3 A Key Sampling Formula

Our approach to sampled-data systems is in the frequency-domain. We now present a result that is essential to the understanding of the frequency-domain properties of sampled-data systems and will play a central role throughout the following chapters. Unfortunately, despite the fact that the result is well-known and appears in many textbooks [e.g., Åström and Wittenmark, 1990, Franklin et al., 1990, Kuo, 1992, Ogata, 1987], it is difficult to find in the literature a proof that is rigorous and self-contained, and which clearly delineates the classes of signals to which it is applicable. Indeed, this fact has stimulated discussion in the past [cf. Pierre and Kolb, 1964, Carroll and W.L. McDaniel, 1966, Phillips et al., 1966, 1968].

Let g be a function of BV in every finite interval of $\mathbb{R}_0^+$, and let G be its Laplace transform,

$$G(s) = \int_0^\infty e^{-st}\, g(t)\, dt.$$

If σ_G is the abscissa of absolute — and uniform — convergence of G, we denote by $\mathcal{D}_G$ the strip

$$\mathcal{D}_G \triangleq \{s = x + jy, \text{with } x > \sigma_G \text{ and } y \text{ in } \Omega_N\}.$$

Given a sequence $\{g_k\}_{k=0}^\infty$, we introduce the $\mathcal{Z}$-transform, $G_d = \mathcal{Z}\{\{g_k\}\}$, defined by

$$G_d(z) = \sum_{k=0}^\infty g_k z^{-k}. \tag{2.5}$$

For a continuous-time signal g defined on $\mathbb{R}_0^+$, and $g(t) = 0$ for $t < 0$, we define the $\mathcal{Z}$-transform as the transformation of its sampled version,

$$\begin{aligned} G_d(z) &= \mathcal{Z}\{\mathcal{S}_T\{g\}\} \\ &= \sum_{k=0}^\infty g(kT^+)\, z^{-k}. \end{aligned}$$

Then we have the following lemma.

Lemma 2.1.2
If g is a function of BV in every finite interval of $\mathbb{R}_0^+$, then for every s in $\mathcal{D}_G$

$$G_d(e^{sT}) = \frac{g(0^+)}{2} + \sum_{k=1}^\infty \frac{g(kT^+) - g(kT^-)}{2}\, e^{-skT} + \frac{1}{T} \sum_{n=-\infty}^\infty G(s + jn\omega_s). \tag{2.6}$$

Proof: See Appendix A, §A.1. □

Lemma 2.1.2 shows the well-known fact that the frequency response of a sampled signal is built upon the superposition of infinitely many copies of the original frequency response of the signal. If the signal has finite discontinuities at the sampling instants, then correction terms of half of the jumps at the corresponding sampling instants have to be included — cf. the property of the Laplace and Fourier inverse transforms which converge to the average of the limits of the function from left and right at a jump discontinuity. In particular, (2.6) is closely related to an important identity in Fourier analysis known as the *Poisson Summation Formula*[2]. See further remarks in Appendix A, §A.1.

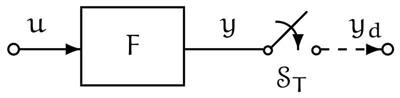

Figure 2.2: Filtered sampling.

Moreover, Lemma 2.1.2 clearly delineates two important classes of signals and systems to which the formula is applicable, as we shall see in the following two corollaries. The first one is concerned with sampling the response of a strictly proper finite dimensional (FD), LTI system (see Figure 2.2). This represents a common practice in digital control engineering, i.e., to precede the sampler by an anti-aliasing filter, and is also required for the the sampling operation to be well-defined [e.g., Chen and Francis, 1991].

Corollary 2.1.3
Let u *be a signal in* $L_{1e}(\mathbb{R}_0^+)$*, and let* F *be a strictly proper rational function. Then for every* s *in* $\mathcal{D}_{FU}$

$$(FU)_d(e^{sT}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(s + jn\omega_s)U(s + jn\omega_s).$$

Proof: Immediate from Lemma 2.1.2 by noting that the response of a FDLTI strictly proper system to an input in $L_{1e}(\mathbb{R}_0^+)$ is continuous [e.g., Desoer and Vidyasagar, 1975], so $y(t^+) = y(t^-)$ for every t. In particular, since $y(t) = 0$ for $t < 0$, this also implies that $y(0) = 0$, and the result then follows. □

The second corollary deals with sampling the pulse response of a hold function followed by a FDLTI strictly proper system, and displays the relation between the discrete equivalent of this cascade and the corresponding continuous-time Laplace transforms (see Figure 2.3).

Corollary 2.1.4
Let H *be a hold frequency-response function as described in Subsection 2.1.2 and* P *a strictly proper rational function. Let* $(PH)_d$ *denote the discrete equivalent of the cascade connection* PH *defined as*

$$(PH)_d(z) = \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{P(s)H(s)\}\}\}.$$

[2]This is the following Rudin [1987]. If G is the Fourier transform of g, then

$$\sum_{k=-\infty}^{\infty} g(k\alpha) = \beta \sum_{k=-\infty}^{\infty} G(jk\beta),$$

where $\alpha > 0$, $\beta > 0$, and $\alpha\beta = 2\pi$. Although named after S.D. Poisson, this formula seems to have been first discovered by A.L. Cauchy in 1817 [Grattan-Guinness, 1990, p. 793].

Then for every s *in* $\mathcal{D}_P$*,*

$$(PH)_d(e^{sT}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} P(s + jn\omega_s)\, H(s + jn\omega_s). \qquad (2.7)$$

Proof: Since the pulse response of H is of BV by assumption, we then have that the output of P is continuous Desoer and Vidyasagar [1975]. The result then follows from Lemma 2.1.2. □

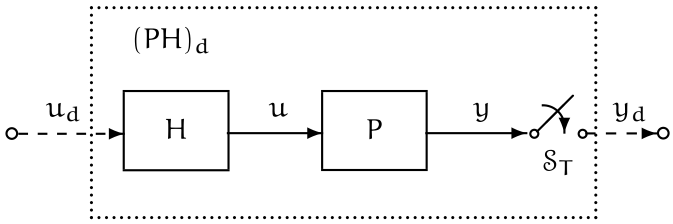

Figure 2.3: Discrete equivalent of the cascade of a hold and a FDLTI system.

Note that the domains of validity of these results can be further extended by analyticity of Laplace transforms.

Equation (2.7) appears in many digital control textbooks [e.g., Åström and Wittenmark, 1990, Franklin et al., 1990], and it has been the starting point of a number of recent frequency-domain approaches to sampled-data systems Goodwin and Salgado [1994], Araki and Ito [1993], Araki et al. [1993], Freudenberg et al. [1995]. Some authors refer to (2.7) as the *impulse modulation formula* [e.g., Araki and Ito, 1993, Araki et al., 1993].

2.2 Hybrid Systems

2.2.1 Basic Feedback Configuration

The basic feedback system of study is shown in Figure 2.4. The analog plant is a linear time-invariant system represented by the transfer matrix P, and the controller is given by the discrete transfer matrix C_d. The digital controller interfaces with the analog parts of the system by a sampler $\mathcal{S}_T$ and a hold function H as described in Subsection 2.1.2. The transfer matrix F represents the anti-aliasing filter.

Signals in Figure 2.4 are as follows,

r reference input,	u_k discrete control sequence,
y plant output,	u analog control signal,
d output disturbance,	v analog output of the filter,
n sensor noise,	v_k sampled output of the filter.

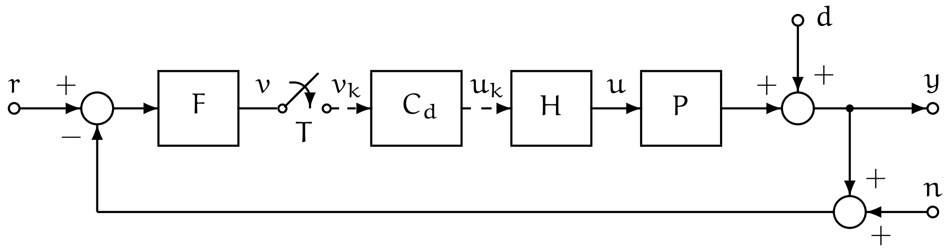

Figure 2.4: Sampled-data control system.

Analog signals are given as functions defined over t in $\mathbb{R}_0^+$, while discrete signals are sequences defined at entire multiples k of the sampling time T. We shall assume that the input signals satisfy the following condition.

Assumption 2
The reference signal r*, disturbance* d*, and noise* n *are functions in* $L_{1e}(\mathbb{R}_0^+)$. ○

It is straightforward to verify that this assumption is satisfied by signals that are steps, ramps, sinusoids or exponentials, and signals in $L_p(\mathbb{R}_0^+)$ for $1 \le p \le \infty$ Chen and Francis [1991]. Signals that contain impulses are excluded.

We shall assume throughout that the following conditions are satisfied by the plant, filter, and compensator.

Assumption 3
The plant, filter, and compensator are full column rank rational transfer matrices, each free of unstable hidden modes, and they satisfy the following additional hypotheses,

(i) $P(s) = P_0(s)\, e^{-s\tau}$, *where* P_0 *is proper and* $\tau \ge 0$,

(ii) F *is strictly proper, stable and minimum-phase, and*

(iii) C_d *is proper.* ○

The assumption that the filter F is strictly proper is standard and guarantees that the sampling operation is well-defined [e.g., Chen and Francis, 1991, Sivashankar and Khargonekar, 1993]. The assumptions that F has no poles or zeros in $\overline{\mathbb{C}^+}$ may be removed, and are only invoked to facilitate discussion. In practice anti-aliasing filters will satisfy these assumptions.

We define the *discretized plant* as the discrete transfer function of the series connection of hold, plant, filter, and sampler,

$$(FPH)_d(z) \triangleq \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{F(s)P(s)H(s)\}\}\}. \tag{2.8}$$

It follows from Assumptions 1 and 3, and Corollary 2.1.4 that the discretized plant satisfies the well-known relation

$$(FPH)_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(s)\,P_k(s)\,H_k(s), \tag{2.9}$$

where the notation $F_k(s)$ represents $F(s + jk\omega_s)$, i.e., the shift of $F(\cdot)$ by an entire number of times the sampling frequency in the direction of the imaginary axis. We shall use this notation throughout this thesis.

Suppose now that in the loop of Figure 2.4 we assume $r = 0$ and consider a disturbance x at the input of the plant. Introduce a ficticious hold at x, and shift the filter and sampler to the inputs at the summation point of n, as shown in Figure 2.5. From this diagram we obtain the discrete loop of Figure 2.6.

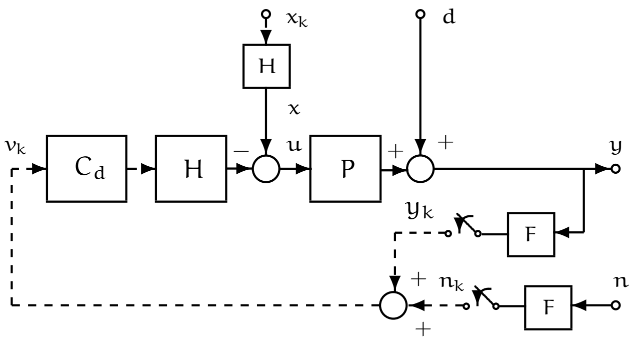

Figure 2.5: Sampled-data system with input disturbances.

We now define the discrete sensitivity and discrete complementary sensitivity functions. Since the setup is multiple-input multiple output, there are two pairs of functions corresponding to the scalar ones, depending where the loop is opened Freudenberg and Looze [1988]. We shall require only the following *input discrete sensitivity function*,

$$S_d(z) \triangleq [I + C_d(z)(FPH)_d(z))]^{-1}, \tag{2.10}$$

and *output discrete complementary sensitivity function*,

$$T_d(z) \triangleq (FPH)_d(z)S_d(z)C_d(z). \tag{2.11}$$

These functions map signals in the discrete loop of Figure 2.6 as

$$\tilde{U}_d(z) = S_d(z)X_d(z) \quad \text{and} \quad Y_d(z) = T_d(z)N_d(z),$$

where $\tilde{U}_d, X_d, Y_d$, and N_d correspond to the $\mathcal{Z}$-transforms of the signals $\tilde{u}_x, x_k, y_k$, and n_k, respectively.

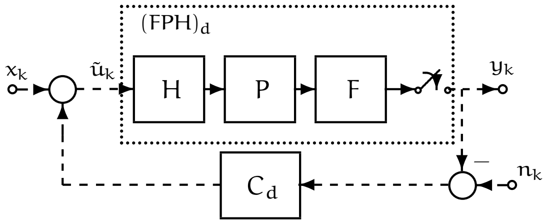

Figure 2.6: Discrete sensitivity functions.

2.2.2 Non-pathological Sampling and Internal Stability

As with the case of a ZOH, closed-loop stability is guaranteed by the assumptions that sampling is non-pathological and that the discretized system is closed-loop stable. The next result is a generalization of the well-known result of Kalman et al. [1963] to the case of GSHFs.

Lemma 2.2.1 (Non-pathological Sampling, Middleton and Freudenberg [1995])
Suppose that P *and* F *are as defined in Subsection 2.2.1 and assume further that*

(i) if λ_i *and* λ_k *are CRHP poles of* P*, then*

$$\lambda_i \neq \lambda_k + jn\omega_s, \quad n = \pm 1, \pm 2, \cdots \tag{2.12}$$

(ii) if λ_i *is a CRHP pole of* P*, then* $H(s)$ *has no zeros at* $s = \lambda_i$*.*

Then the discretized plant (2.8) *is free of unstable hidden modes.* ∘

In particular, Lemma 2.2.1 says that since the response of a GSHF may have zeros in $\overline{\mathbb{C}^+}$, it may be necessary to require that none of these coincides with an unstable pole of the analog plant (note that this *is* necessary in the SISO case). Under the non-pathological sampling hypothesis, it is straightforward to extend the exponential and L_2 input-output stability results of Francis and Georgiou [1988] and Chen and Francis [1991] to the case of GSHF.

Lemma 2.2.2
Suppose that P*,* F*,* C_d*, and* H *are as defined in Subsections 2.1.2 and 2.2.1, that the nonpathological sampling conditions (i) - (ii) are satisfied, that the product* $(FPH)_d\, C_d$ *has no pole-zero cancelations in* $\mathbb{D}^C$*, and that all poles of* S_d *lie in* $\mathbb{D}$*. Then the feedback system in Figure 2.4 is exponentially stable and* L_2 *input-output stable.*

Proof: The proof may be obtained by simple modification of the proofs of Francis and Georgiou [1988, Theorem 4] and Chen and Francis [1991, Theorem 6]. □

Lemma 2.2.2 establishes the conditions for the nominal stability of the sampled-data system of Figure 2.4, and will be required in most of the remaining chapters. In particular, this result guarantees that the operators mapping disturbances and noise to the output are bounded on L_2. This will be the starting point for the analysis developed in Chapter 5.

2.3 Summary

This chapter has introduced the main notation, definitions, and preliminary results that will be required in the rest of this thesis.