

# Multivariable Quadratically-stabilizing Quantizers with Finite Density

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## Abstract

This paper deals with quadratic stabilization of discrete-time multiple-input (MI) systems by means of quantized static feedback. We consider the minimization of quantization density, which is still an open problem for MI systems. Our first contribution in this regard is to partially derive the structure that a quantizer that minimizes density for a MI system should have. The second and main contribution of the paper is to develop a systematic design procedure for finite-density quadratically-stabilizing quantizers for MI systems. The resulting quantizers have a simple geometric structure and can be implemented via simple function evaluations. To the best of the authors' knowledge, no such design of finite-density quadratically-stabilizing quantizers has been previously proposed in the literature for general MI systems.

*Key words:* Quantizers, Quadratic stabilization, Quantization density.

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## 1 Introduction

Quantization is an inescapable phenomenon in engineering systems, especially when digital implementations are involved. Strategies to deal with quantization effects are advisable in the design of such systems to avoid any deleterious impact that quantization may have on their stability and performance. Numerous works have been published which explicitly deal with quantization while focusing on stabilization in a networked control setting. Within these works, we can distinguish between the ones where the quantization strategy is dynamic and time-varying (for example, Wong and Brockett, 1999; Brockett and Liberzon, 2000; Liberzon, 2003; Nair and Evans, 2004; Li and Baillieul, 2004; Tatikonda and Mitter, 2004; Tatikonda and Elia, 2004) and where it is fixed and static (for example, Delchamps, 1990; Elia and Mitter, 2001; Elia and Frazzoli, 2002; Kao and Venkatesh, 2002; Fu and Xie, 2005; Baillieul, 2002; Ishii and Francis, 2003; Goodwin et al., 2004).

Throughout this paper, we regard a quantizer as a fixed and static component of the system and deal with quadratic stabilization by means of quantized static feedback. The approach that we follow is related to the work of Elia and

Mitter (2001); Elia and Frazzoli (2002); Elia (2002); Kao and Venkatesh (2002); Fu and Xie (2005); Haimovich and Seron (2005, 2007); Haimovich (2005); Haimovich et al. (2006); Haimovich (2006). Elia and Mitter (2001) introduce a measure of density of quantization. Intuitively, the density of a quantizer is lower than that of another quantizer if the values of the former are more separated than those of the latter. Consequently, a quantizer can be regarded as being more efficient in the use of its quantization levels if its density is lower. In this context, an important question that is posed and answered in Elia and Mitter (2001) is: for a linear single-input system, what is the most efficient quantizer over all quadratically stabilizing quantizers? Elia and Mitter (2001) thus find least dense quantizers over all quantizers that quadratically stabilize a given linear single-input system and also show how such a least dense quantizer may be constructed.

Haimovich and Seron (2007) develop a geometric characterization of quadratically-stabilizing quantizers for single-input systems. This geometric characterization is therein employed to develop a state-space approach to quantization density. In this approach a least dense quantizer has its quantization *regions* as separated as possible, as opposed to the input-space-based quantization density of Elia and Mitter, where the separation of the quantization *levels* are of importance.

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The interesting results of Elia and Mitter (2001) apply only to single-input systems. Generalizing these results to MI systems is recognized as an extremely difficult task. Indeed, for MI systems, the quantization density problem introduced in Elia and Mitter (2001) still remains largely open. Elia and Frazzoli (2002) and Elia (2002) provide lower bounds on the infimum quantization density for two-input systems. Haimovich and Seron (2005) show that Theorem 1 of Elia and Frazzoli (2002) is incorrect and provide a partial replacement. It should be pointed out, however, that the main result of Elia and Frazzoli (2002) remains valid, provided it is carefully (re)interpreted (see Haimovich, 2006, §5, for further details). Kao and Venkatesh (2002) analyze different quantization schemes and their densities for linear MI systems. However, explicit design of a multivariable quantizer with finite (though not necessarily infimum) quantization density is performed only when quadratic stabilization is possible through the use of a two-dimensional subspace of the input space. Haimovich et al. (2006) extend the geometric characterization of Haimovich and Seron (2007) to MI systems in the case when the quantized control, that is, the quantizer output, is constrained to lie in a subspace of minimum dimension of the input space. Haimovich et al. (2006) also mention a possible path toward designing finite-density quadratically-stabilizing quantizers.

As mentioned above, the solution to the infimum-quantization-density problem is still unknown for general MI systems. For these systems, in addition, none of the available results even give insight into the structure of infimum-density quantizers or a finite-density quantizer design procedure. Note that a first step towards the obtention of infimum-density quantizers may be the construction of finite-density quantizers, as opposed to the infinite-density ones employed in, for example, Ishii and Francis (2003) and Fu and Xie (2005) (see comments after Definition 5). These results consider valid and interesting approaches to the design of quantizers for MI systems, but do not address the MI infimum quantization density problem. In this context, the current paper provides two related but different contributions. The first contribution consists in the partial derivation of the structure that an optimal quantizer for a MI system should have. The second and main contribution of the paper is to develop a systematic finite-density quadratically-stabilizing quantizer design procedure for MI systems. The results of the current paper build on previous work by the authors (Haimovich et al., 2006) and formed part of H. Haimovich's Ph.D. work (Haimovich, 2006).

## 2 Quadratic Stabilization and Quantization Density

We consider a discrete-time linear time-invariant system:

$$x(k+1) = Ax(k) + Bu(k), \quad (1)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $u(k) \in \mathbb{R}^m$  is the current control, and  $x(k) \in \mathbb{R}^n$  is the current state. We assume that the matrix  $A$  has at least one eigenvalue outside or on the

unit circle,  $B$  has full column rank and the pair  $(A, B)$  is stabilizable. Consider a positive definite quadratic function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{+,0}$ , of the form

$$V(x) = x^T P x, \quad \text{where } P = P^T > 0. \quad (2)$$

Given a function  $V$  of the form (2), we will analyze feedback laws that render  $V$  a Lyapunov function for the closed-loop system. However, not every function  $V$  of the form (2) will allow such a feedback law to exist. We therefore employ the following definition (adapted from Sontag, 1998).

**Definition 1 (CLF)** A positive definite quadratic function of the form (2) is said to be a control Lyapunov function (CLF) for system (1) if a static feedback  $u = q(x)$  exists such that the closed-loop system  $x(k+1) = Ax(k) + Bq(x(k))$  admits  $V$  as a Lyapunov function.

Let  $\Delta V(x, u)$  denote the increment of a function  $V$  of the form (2) along the trajectories of system (1):

$$\begin{aligned} \Delta V(x, u) &\triangleq V(Ax + Bu) - V(x) \\ &= x^T L x + 2x^T M u + u^T B^T P B u, \end{aligned} \quad (3)$$

$$\text{where } L \triangleq A^T P A - P, \quad M \triangleq A^T P B. \quad (4)$$

Since, by assumption,  $P = P^T > 0$  and  $B$  has full column rank, then  $B^T P B > 0$  and hence  $B^T P B$  is invertible. We then define the matrices

$$Q \triangleq M(B^T P B)^{-1} M^T - L, \quad K_{GD} \triangleq -(B^T P B)^{-1} M^T. \quad (5)$$

We have the following well-known result (see, for example, Haimovich, 2006, §2.2, for a proof).

**Lemma 2** A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{+,0}$  of the form (2) is a CLF for system (1) if and only if  $Q > 0$ , where  $Q$  was defined in (5) with  $L$  and  $M$  as in (4).

We consider quadratic stabilization of system (1) when the control is a *quantized* static state feedback.

**Definition 3 (Quantizer)** A quantizer  $q$  is a discrete-range function  $q : \mathbb{R}^r \rightarrow \mathbb{R}^s$  of the form

$$q(x) = u_i \text{ if and only if } x \in \mathcal{R}_i, \quad \text{for } i \in \mathbb{Z}. \quad (6)$$

The sets  $\mathcal{R}_i$  are called the quantization regions of  $q$  and  $u_i$  is called the value or level of  $q$  corresponding to  $\mathcal{R}_i$ . The sets  $\mathcal{R}_i$ ,  $i \in \mathbb{Z}$ , satisfy

$$\bigcup_{i \in \mathbb{Z}} \mathcal{R}_i = \mathbb{R}^r, \text{ and } \mathcal{R}_i \cap \mathcal{R}_j = \emptyset \text{ whenever } i \neq j. \quad (7)$$

If  $r = n$  in Definition 3, we say that  $q$  is a state quantizer. If  $r = s = 1$ , we say that  $q$  is a scalar quantizer.

**Definition 4 (QS Quantizer)** Consider a CLF  $V$  of the form (2) and its increment along the trajectories of system (1),  $\Delta V(x, u)$  in (3). A quantizer  $q : \mathbb{R}^n \rightarrow \mathbb{R}^m$  that satisfies  $q(0) = 0$  and

$$\Delta V(x, q(x)) < 0, \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \quad (8)$$

is called quadratically stabilizing (QS) with respect to  $V$ . We say that a quantizer  $q$  is just ‘QS’ instead of ‘QS with respect to  $V$ ’ when the CLF  $V$  is clear from the context.

It is well-known (Delchamps, 1990) that any quantizer that quadratically stabilizes a given open-loop-unstable discrete-time LTI system necessarily has an infinite number of levels, which become increasingly closer near the origin. Therefore, the density of quantization has been defined to yield a finite value for quantizers having these features (for example, for logarithmic quantizers in the scalar case). The quantization density concept introduced in Elia and Mitter (2001) applies to symmetric quantizers with scalar levels, that is, quantizers  $q : \mathbb{R}^r \rightarrow \mathbb{R}$  that satisfy  $q(x) = -q(-x)$  for all  $x \in \mathbb{R}^r$ . A generalization to quantizers with two-dimensional levels ( $q : \mathbb{R}^r \rightarrow \mathbb{R}^2$ ) appears in Elia and Frazzoli (2002) and Elia (2002). We next provide a straightforward generalization to quantizers with levels of arbitrary dimension.

**Definition 5 (Quantization Density)** Given a quantizer  $q : \mathbb{R}^r \rightarrow \mathbb{R}^s$ , let  $\mathcal{U}(q)$  denote the range of  $q$ , that is,

$$\mathcal{U}(q) \triangleq \{u \in \mathbb{R}^s : u = q(x) \text{ for some } x \in \mathbb{R}^r\}. \quad (9)$$

For  $\epsilon \in (0, 1]$ , let  $C^s(\epsilon)$  be the following region in  $\mathbb{R}^s$ :

$$C^s(\epsilon) \triangleq \{u \in \mathbb{R}^s : \epsilon \leq \|u\|_2 \leq 1/\epsilon\}. \quad (10)$$

The density of  $q$ , denoted  $\eta(q)$ , is defined as follows, where  $\#[\cdot]$  denotes the number of elements (cardinality) of a set:

$$\eta(q) \triangleq \limsup_{\epsilon \rightarrow 0} \frac{\#[\mathcal{U}(q) \cap C^s(\epsilon)]}{-2 \ln \epsilon}. \quad (11)$$

It can readily be verified that the measure of density in Definition 5 coincides with the one given in Elia and Mitter (2001) when the output of the quantizer  $q$  is a scalar ( $s = 1$ ) and  $q$  satisfies  $q(x) = -q(-x)$ . According to (11), the density of a quantizer with a finite number of levels is zero and the density of a quantizer with radially uniformly spaced values is infinite. The density of a quantizer  $q$  also is infinite if, for some  $0 < \epsilon \leq 1$ ,  $q$  has an infinite number of levels in the set  $C^s(\epsilon)$ . This is the case if the range of  $q$  is the Cartesian product of the ranges of  $s$  scalar logarithmic quantizers, as in, for example, Ishii and Francis (2003) and Fu and Xie (2005). The quantization density in Definition 5 is finite for quantizers with radially logarithmically spaced values, as follows from the next result (see Haimovich, 2006, §2.4 for a proof).

**Theorem 6** Let  $0 < \rho < 1$  and let  $u_i \in \mathbb{R}^s$ , for  $i = 1, \dots, N$ , be any nonzero vectors such that the sets  $\mathcal{U}_i \triangleq \{\rho^j u_i : j \in \mathbb{Z}\}$  are disjoint. Suppose that the range of a quantizer  $q : \mathbb{R}^r \rightarrow \mathbb{R}^s$ , namely  $\mathcal{U}(q)$ , satisfies

$$\mathcal{U}(q) = \biguplus_{i=1}^N \mathcal{U}_i \cup \{0\}, \quad (12)$$

where  $\biguplus$  denotes disjoint union. Then,

$$\eta(q) = \frac{N}{-\ln \rho}. \quad (13)$$

From expression (13), it follows that the lower  $\rho$  is, the lower the density of the quantizer  $q$  (recall that  $0 < \rho < 1$ ). Note that the lower  $\rho$  is, the more radially separated the values of  $q$  are. The following result, whose proof can be found in Haimovich (2006, §2.4) shows that the density of a quantizer is preserved under a linear one-to-one transformation.

**Lemma 7** Let  $\bar{q} : \mathbb{R}^r \rightarrow \mathbb{R}^p$  be a quantizer, let  $W \in \mathbb{R}^{s \times p}$  be a matrix having linearly independent columns and let  $q : \mathbb{R}^r \rightarrow \mathbb{R}^s$  be defined by  $q(x) = W\bar{q}(x)$ , for all  $x \in \mathbb{R}^r$ . Then,  $\eta(q) = \eta(\bar{q})$ .

### 3 Infimum Quantization Density

In this section, we present the first contribution of the paper. Specifically, Theorem 8 below partially reveals the structure of a quantizer that optimizes density over all quantizers that are QS with respect to a given CLF, for a MI system.

**Theorem 8** Let  $q : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a QS quantizer for system (1) with respect to a CLF  $V$  of the form (2). Consider the matrices  $L$  and  $M$  defined in (4) and the matrix  $Q$  defined in (5). Define

$$Z \triangleq (B^T P B)^{1/2}, \quad K \triangleq Z^{-1} M^T, \quad (14)$$

and find the following singular value decomposition<sup>2</sup> of  $KQ^{-1/2}$ :

$$KQ^{-1/2} = S_1 \Sigma S_2^T \quad (15)$$

where  $S_1 \in \mathbb{R}^{m \times m}$ ,  $S_2 \in \mathbb{R}^{n \times m}$ ,  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_m)$ , and  $S_1^T S_1 = I_m = S_2^T S_2$ . Define the quantizer  $\bar{q} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by

$$\bar{q}(x) = q \left( Q^{-1/2} S_2 \Sigma^T Q^{1/2} x \right), \quad \text{for all } x \in \mathbb{R}^n. \quad (16)$$

Then,  $\bar{q}$  is QS and  $\eta(\bar{q}) \leq \eta(q)$ .

<sup>2</sup> This decomposition was proposed in Kao and Venkatesh (2002).

**PROOF.** We begin by showing that  $\bar{q}$  is QS. Since  $q$  is QS by assumption, then  $q(0) = 0$ . Then, from (16) it follows that  $\bar{q}(0) = 0$ . Consider the increment of  $V$ , as defined in (3). Using (3)–(5) and (14), we can write  $\Delta V(x, u)$  as

$$\Delta V(x, u) = [Kx + Zu]^T [Kx + Zu] - x^T Qx. \quad (17)$$

Using (15), (17), and simplifying, yields

$$\begin{aligned} \Delta V(Q^{-1/2} S_2 S_2^T Q^{1/2} x, u) = \\ [Kx + Zu]^T [Kx + Zu] - x^T Q^{1/2} S_2 S_2^T Q^{1/2} x. \end{aligned} \quad (18)$$

It can readily be verified that  $Q \geq Q^{1/2} S_2 S_2^T Q^{1/2}$ . It then follows from (17) and (18) that for all  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ :

$$\Delta V(x, u) \leq \Delta V(Q^{-1/2} S_2 S_2^T Q^{1/2} x, u). \quad (19)$$

From (19), with  $u = \bar{q}(x)$ , then

$$\Delta V(x, \bar{q}(x)) \leq \Delta V(Q^{-1/2} S_2 S_2^T Q^{1/2} x, \bar{q}(x)). \quad (20)$$

Since  $q$  is QS by assumption, then  $\Delta V(x, q(x)) < 0$  for all nonzero  $x \in \mathbb{R}^n$ . In particular, using (16),

$$\Delta V(Q^{-1/2} S_2 S_2^T Q^{1/2} x, \bar{q}(x)) < 0, \quad (21)$$

for all  $x \in \mathbb{R}^n$  satisfying  $Q^{-1/2} S_2 S_2^T Q^{1/2} x \neq 0$ . Note that  $Q^{-1/2} S_2 S_2^T Q^{1/2} x \neq 0$  if and only if  $S_2^T Q^{1/2} x \neq 0$ . Combining (19)–(21), it follows that  $\Delta V(x, \bar{q}(x)) < 0$  for all  $x \in \mathbb{R}^n$  such that  $S_2^T Q^{1/2} x \neq 0$ . If  $S_2^T Q^{1/2} x = 0$ , then from (16) and since  $q$  is QS, we have  $\bar{q}(x) = q(0) = 0$ . From (17) and (15), it follows that  $\Delta V(x, 0) = x^T K^T Kx - x^T Qx = -x^T Qx$  if  $S_2^T Q^{1/2} x = 0$ . We have thus established that  $\Delta V(x, \bar{q}(x)) < 0$  for all nonzero  $x \in \mathbb{R}^n$ . Hence,  $\bar{q}$  is QS.

We next show that  $\eta(\bar{q}) \leq \eta(q)$ . Let  $\mathcal{U}(\cdot)$  denote the range of a quantizer [recall (9)]. From (16), note that  $\mathcal{U}(\bar{q}) \subseteq \mathcal{U}(q)$ . Therefore, it follows that

$$\#\mathcal{U}(\bar{q}) \cap C^m(\epsilon) \leq \#\mathcal{U}(q) \cap C^m(\epsilon), \quad (22)$$

for all  $\epsilon \in (0, 1]$ , where  $C^m(\epsilon)$  is the set defined in (10). From (11), then  $\eta(\bar{q}) \leq \eta(q)$ . This concludes the proof.  $\square$

Theorem 8 shows that, given any QS quantizer  $q$ , we can construct a QS quantizer  $\bar{q}$  with a specific structure that is also QS and whose density is not greater than that of  $q$ . The key structural difference between an arbitrary QS quantizer  $q$  and a quantizer  $\bar{q}$  constructed from  $q$  according to (16) is that the matrix  $Q^{-1/2} S_2 S_2^T Q^{1/2}$  has rank  $m$  where  $m \leq n$  (this inequality follows because the system input matrix  $B$  has full column rank). Note that the quantizer  $\bar{q}$  can be written as

$$\bar{q}(x) = \tilde{q}(S_2^T Q^{1/2} x), \quad (23)$$

where  $\tilde{q} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ . The result of Theorem 8 then implies that the search for an infimum-density QS quantizer can be performed exclusively over quantizers  $\tilde{q}$  having the specific structure (23). Since the ranges of  $\bar{q}$  and  $\tilde{q}$  coincide, then  $\eta(\bar{q}) = \eta(\tilde{q})$  and thus the search for the infimum density, which has to be performed over quantizers  $q : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is reduced to a search over quantizers  $\tilde{q} : \mathbb{R}^m \rightarrow \mathbb{R}^m$  ( $m \leq n$ ).

**Remark 9** In the single-input case,  $S_2^T Q^{1/2} = \alpha K_{GD}$ , with  $\alpha \in \mathbb{R}$  and  $K_{GD}$  as in (5). Then, Theorem 8 shows that the search for an infimum density quantizer can be performed exclusively over quantizers  $\tilde{q} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  of the form  $\tilde{q}(x) = q_s(K_{GD}x)$ , where  $q_s$  is a scalar quantizer. This structure is precisely the one derived in the first part of the proof of Theorem 2.1 of Elia and Mitter (2001).

## 4 Finite-density Multivariable QS Quantizers

In this section, we present the main contribution of the paper. Specifically, given system (1) and a CLF  $V$  of the form (2), we design a QS quantizer  $q : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that  $\eta(q) < \infty$ . Our quantizer design procedure utilizes several results of Haimovich et al. (2006). Following the latter reference, we construct quantizers having the structure:

$$q(x) = W\hat{q}(D^T x), \quad \text{for all } x \in \mathbb{R}^n, \quad (24)$$

where  $\hat{q} : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$ ,  $W \in \mathbb{R}^{m \times \ell}$  and  $D \in \mathbb{R}^{n \times \ell}$  have linearly independent columns, and  $\ell$  is the number of positive eigenvalues of the matrix  $L$  in (4). This number is the lowest possible dimension for  $\hat{q}$  in (24), for the given system and CLF. The results of Haimovich et al. (2006) require that the matrix  $L$  in (4) be invertible. We therefore make this assumption in the sequel. Relating the quantizer structure (24) with the result of Theorem 8 in §3, it follows that as far as quantization density is concerned, it is advantageous that the matrix  $D^T$  have the form

$$D^T = \Gamma^T S_2^T Q^{1/2}, \quad (25)$$

with  $Q$  as in (5),  $S_2$  as in Theorem 8, and  $\Gamma \in \mathbb{R}^{m \times \ell}$ , since in this case  $q$  in (24) satisfies (23), with  $\tilde{q}$  defined by  $\tilde{q}(\tilde{x}) = W\hat{q}(\Gamma^T \tilde{x})$ . The following theorem provides the main ingredient for our design procedure.

**Theorem 10** Let  $D \in \mathbb{R}^{n \times \ell}$  have linearly independent columns and satisfy  $D^T L^{-1} D = I_\ell$ , where  $L$  was defined in (4) and  $\ell$  is the number of positive eigenvalues of  $L$ . Define

$$H \triangleq B^T P B - M^T L^{-1} M. \quad (26)$$

Let  $W \in \mathbb{R}^{m \times \ell}$  be such that  $S \triangleq D^T L^{-1} M W = -I_\ell$ , with  $M$  as defined in (4), and such that  $J \triangleq -W^T H W > 0$ . Let  $\lambda$  denote the smallest eigenvalue of the matrix  $J$ , and let  $c$  be an odd integer satisfying  $c \geq 3$  and  $c > 1 + \sqrt{\ell/\lambda}$ . Let  $\rho \triangleq (c-2)/c$  and consider the quantizer  $q$  defined by (24)

with  $\hat{q} : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  satisfying

$$\hat{q}(a) \triangleq \begin{cases} 0 & \text{if } a = 0, (27a) \\ \rho^{j(\|a\|_\infty)} \mathcal{I}(a \rho^{-j(\|a\|_\infty)}) & \text{if } a \neq 0, (27b) \end{cases}$$

the function  $\mathcal{I} : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  is defined by

$$\mathcal{I}(a) \triangleq [\mathcal{I}(a_1) \dots \mathcal{I}(a_\ell)]^T, \quad (28)$$

where  $\mathcal{I} : \mathbb{R} \rightarrow \frac{2}{c-1}\mathbb{Z}$  and  $j : \mathbb{R}_+ \rightarrow \mathbb{Z}$  are defined by

$$\mathcal{I}(b) \triangleq \frac{2}{c-1} \left\lceil \frac{c-1}{2} |b| - \frac{1}{2} \right\rceil \text{sgn}(b), \quad (29)$$

$$j(b) \triangleq \left\lceil \left( \ln \frac{c}{(c-1)b} \right) / \left( \ln \frac{1}{\rho} \right) \right\rceil, \quad (30)$$

$\lceil b \rceil$  denotes the least integer not less than  $b$  and  $\lfloor b \rfloor$  denotes the greatest integer not greater than  $b$ . Then,  $q$  is QS and

$$\eta(q) = \frac{c^\ell - (c-2)^\ell}{\ln c - \ln(c-2)}. \quad (31)$$

**PROOF.** The proof that  $q$  is QS is omitted for space reasons. This proof employs the necessary and sufficient conditions derived in Haimovich et al. (2006) and can be consulted in Haimovich (2006, §3.5).

We next establish (31). Let  $\bar{q} : \mathbb{R}^n \rightarrow \mathbb{R}^\ell$  be the quantizer defined by  $\bar{q}(x) = \hat{q}(D^T x)$ . Since  $D$  has linearly independent columns, then  $\mathcal{U}(\bar{q}) \triangleq \{\bar{q}(x) : x \in \mathbb{R}^n\}$  and  $\mathcal{U}(\hat{q}) \triangleq \{\hat{q}(a) : a \in \mathbb{R}^\ell\}$  are equal. Therefore,  $\eta(\bar{q}) = \eta(\hat{q})$  because quantization density depends only on the range of a quantizer (recall Definition 5). From (24) and since  $W$  has linearly independent columns, then Lemma 7 establishes that  $\eta(q) = \eta(\bar{q})$ . We next note that the range of  $\hat{q}$  has the form (12) as in Theorem 6, with  $N = c^\ell - (c-2)^\ell$  (see Figure 1 below). Therefore, Theorem 6 yields  $\eta(q) = \frac{c^\ell - (c-2)^\ell}{-\ln \rho}$ . Eq. (31) then follows by substituting  $\rho = (c-2)/c$  into the former equation.  $\square$

**Remark 11** Haimovich et al. (2006) give necessary and sufficient conditions on the matrices  $W$  and  $D$  so that a QS quantizer of the form (24) exists. It can easily be shown that the additional assumptions required by Theorem 10 incur no loss of generality [see steps (vi) and (vii) below].

We are now ready to present our quantizer design procedure.

#### Finite-density QS Quantizer Design

- (i) Given the system and CLF matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $P \in \mathbb{R}^{n \times n}$  such that  $A$  is unstable,  $(A, B)$  is stabilizable and  $P = P^T > 0$ .

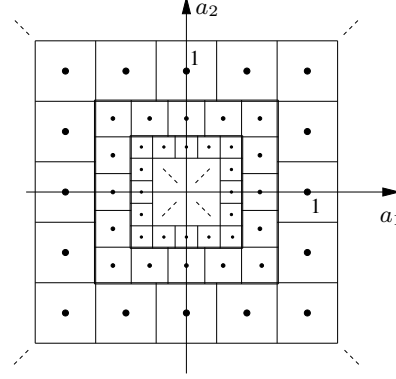


Fig. 1. Quantization regions and values of  $\hat{q} : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  for  $\ell = 2$  and  $c = 5$ . Each square is a quantization region of  $\hat{q}$  whose centre is its corresponding value.

- (ii) Compute  $L$  and  $M$  from (4), and  $Q$  from (5).
- (iii) Verify that  $L$  is nonsingular.
- (iv) Compute  $\ell$ , the number of positive eigenvalues of  $L$ .
- (v) Compute  $H$  from (26).
- (vi) Choose  $\tilde{D} \in \mathbb{R}^{n \times \ell}$  and  $\tilde{W} \in \mathbb{R}^{m \times \ell}$  satisfying the necessary conditions (Haimovich et al., 2006)  $\tilde{D}^T L^{-1} \tilde{D} > 0$  and  $\tilde{W}^T H \tilde{W} < 0$ .
- (vii) Compute  $D = \tilde{D}(\tilde{D}^T L^{-1} \tilde{D})^{-1/2}$  and  $W = -\tilde{W} \tilde{S}^{-1}(\tilde{D}^T L^{-1} \tilde{D})^{1/2}$ , where  $\tilde{S} = \tilde{D}^T L^{-1} M \tilde{W}$ . Note that we now have  $D^T L^{-1} D = I_\ell$  and  $S = D^T L^{-1} M W = -I_\ell$ .
- (viii) Compute  $J = -W^T H W$  and smallest eigenvalue,  $\lambda$ .
- (ix) Choose an odd integer  $c \geq 3$  satisfying  $c > 1 + \sqrt{\ell/\lambda}$ .
- (x) Consider  $\hat{q} : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$  defined in (27)–(30).
- (xi) The required QS quantizer  $q : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is defined by  $q(x) = W \hat{q}(D^T x)$ . Its density is given by (31).

**Remark 12** Given a system and a corresponding CLF, the above quantizer design procedure yields a finite-density QS quantizer provided that the matrix  $L$  is invertible. This latter requirement is the only additional assumption in our development, that is, there are no other requirements that may prevent our procedure from yielding the desired quantizer.

**Remark 13** Our quantizer design procedure may yield a quantizer with lower density if the matrix  $\tilde{D}$  chosen at step (vi) satisfies, in addition,  $\tilde{D}^T = \tilde{\Gamma}^T S_2^T Q^{1/2}$ , for some  $\tilde{\Gamma} \in \mathbb{R}^{m \times \ell}$ . This follows since, in such case, the corresponding  $D$  of step (vii) satisfies (25) (see the example in §5 below). A systematic procedure for selecting  $\tilde{D}$  and  $\tilde{W}$  satisfying all requirements is suggested in Haimovich (2006, §3.5).

<sup>3</sup> The matrix  $\tilde{S}$  is always invertible provided  $\tilde{D}$  and  $\tilde{W}$  are chosen as required in step (vi) (see Haimovich, 2006, Remark 3.16).

## 5 Example

We next apply our quantizer design procedure in a numerical example. Consider system (1) with matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (32)$$

and the quadratic function  $V(x) = x^T P x$  with

$$P = \begin{bmatrix} 16.6561 & -7.2172 & -15.3227 & -0.1282 & -0.08183 \\ -7.2172 & 4.4423 & 7.2172 & 0.04159 & 0.02651 \\ -15.3227 & 7.2172 & 16.6561 & 0.1282 & 0.08183 \\ -0.1282 & 0.04159 & 0.1282 & 21.5906 & 10.7047 \\ -0.08183 & 0.02651 & 0.08183 & 10.7047 & 6.6030 \end{bmatrix}. \quad (33)$$

We would like to design a quantizer  $q : \mathbb{R}^5 \rightarrow \mathbb{R}^4$  with finite density that renders  $V$  a Lyapunov function for the closed-loop system  $x(k+1) = Ax(k) + Bq(x(k))$ . We hence follow steps (i)–(xi) of our quantizer design procedure above. We readily check that the matrices  $A$ ,  $B$  and  $P$  given above are such that  $A$  is unstable,  $(A, B)$  is stabilizable and  $P = P^T > 0$  [step (i)]. We compute  $L$ ,  $M$  and  $Q$  [step (ii)] and check that  $Q > 0$ , hence verifying that the given  $V$  is a CLF. We next check that  $L$  is nonsingular [step (iii)] and compute the number of positive eigenvalues of  $L$  [step (iv)],  $\ell = 3$ . We next compute  $H$  [step (v)]. For step (vi), Haimovich (2006, §3.5) outlines a systematic procedure for selecting  $\tilde{D}$  and  $\tilde{W}$  satisfying the required inequalities. Instead of applying this procedure, we here will focus on showing how, given  $\tilde{D}_1$  satisfying  $\tilde{D}_1^T L^{-1} \tilde{D}_1 > 0$ , Theorem 8 can be used to generate a matrix  $\tilde{D}_2$  that, in addition to satisfying  $\tilde{D}_2^T L^{-1} \tilde{D}_2 > 0$  also satisfies (25). To this aim, we choose the following matrices:

$$\tilde{D}_1 = \begin{bmatrix} 0.040 & -1.430 & -0.289 \\ 0.071 & -0.196 & -0.303 \\ -0.069 & 1.307 & -1.309 \\ 1.144 & 0.124 & -0.249 \\ 1.752 & 0.254 & -0.126 \end{bmatrix}; \quad \tilde{W} = \begin{bmatrix} -0.294 & 2.093 & -0.094 \\ 0.142 & -0.623 & 1.289 \\ -0.010 & -0.110 & 0.063 \\ -2.602 & 0.034 & -0.094 \end{bmatrix},$$

which satisfy the conditions of step (vi) but  $\tilde{D}_1$  does not satisfy (25), for any  $\Gamma \in \mathbb{R}^{m \times \ell}$ . We proceed with step (vii) and compute  $W$  and  $D_1$ . These matrices now satisfy the assumptions of Theorem 10. Computing  $J = -W^T H W$  and calculating its smallest eigenvalue yields  $\lambda \approx 0.0117$  [step (viii)]. The least odd integer  $C$  that satisfies  $C \geq 3$  and  $C > 1 + \sqrt{\ell/\lambda}$  is  $C = 19$  [step (ix)]. The required finite-density QS quantizer satisfies  $q(x) = W \hat{q}(\tilde{D}_1^T x)$ , with  $\hat{q}$  as described in (27)–(30) [steps (x) and (xi)]. The density of  $q$  is  $\eta(q) \approx 17496$ . This concludes the design of a QS quantizer with finite density.

We next show how Theorem 8 may be employed to design a quantizer with lower density. Theorem 8 states that the quantizer  $\bar{q}(x) = W \hat{q}(\tilde{D}_2^T x)$ , where  $\tilde{D}_2^T = D_1^T Q^{-1/2} S_2 S_2^T Q^{1/2}$ , is QS. Hence, the matrix  $\tilde{D}_2$  must also satisfy the necessary condition  $\tilde{D}_2^T L^{-1} \tilde{D}_2 > 0$ , and, in addition, has the form (25) by construction, with

$\Gamma^T = D_1^T Q^{-1/2} S_2$ . Therefore, we apply again our quantizer design procedure from step (vi) on with  $\tilde{D}$  replaced by  $\tilde{D}_2$ . At step (viii), we now obtain  $\lambda \approx 0.0167$  and at step (ix) we can choose  $C = 15$ . This yields a quantizer  $q$  with density  $\eta(q) \approx 8232$ , which is less than half the density previously obtained using  $\tilde{D}_1$ .

## 6 Conclusions

We have considered the minimization of quantization density in the context of quadratic stabilization of discrete-time MI systems by means of quantized static feedback. Our first contribution in this regard was to partially derive the structure of a quantizer that minimizes density for a MI system. Our results reveal that the search for minimum density quantizers that map the state space into the input space is reduced to a search over quantizers that map the input space into the input space. Our second and main contribution was to develop a systematic design procedure to construct finite-density quadratically-stabilizing quantizers for MI systems. The quantizers constructed according to this procedure can be implemented via simple function evaluations. To illustrate the systematic design procedure, a finite-density quadratically-stabilizing quantizer was designed for a four-input open-loop unstable system.

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