Variations of Classical Extremal Graph Theoretical Problems: Moore Bound and Connectivity

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Certificate of Originality

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(Signed) ______________________________

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3. J. Tang, Y. Lin, C. Balbuena and M. Miller, Calculating the extremal number $\mathrm{ex}(n; \{C_3, C_4, \ldots, C_t\})$, *Discrete Applied Mathematics*, Volume 157, Issue 9, May 2009, pp. 2198 – 2206.

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Abstract

Interconnection networks form an important research area which has received much attention, both in theoretical research and in practice. Design of interconnection networks is much concerned with the topology of networks. The topology of a network is usually studied in terms of extremal graph theory. Consequently, from the extremal graph theory point of view, designing the topology of a network involves various extremal graph problems. One of these problems is the well-known fundamental problem called the *degree/diameter problem*, which is to determine the largest (in terms of the number of vertices) graphs or digraphs of given maximum degree and given diameter. General upper bounds, called Moore bounds, exist for the largest possible order of such graphs and digraphs of given maximum degree Δ (respectively, out-degree d) and diameter D. However, quite a number of open problems regarding the degree/diameter problem do still exist. Some of these problems, such as constructing a Moore graph of degree $\Delta = 57$ and diameter $D = 2$, have been open for over 50 years.

Another extremal graph problem regarding the design of the topology of a network is called the *construction of EX graphs*, which is to obtain graphs of the largest size (in terms of the number of edges), given order and forbidden cycle lengths. In this thesis, we obtain large graphs whose sizes either improve the lower bound of the size of EX graphs, or even reach the optimal value.

We deal with designing the topology of a network, but we are also interested in the issue of fault tolerance of interconnection networks. This leads us to another extremal graph problem, that is, *connectivity*. In this thesis, we provide an overview of the current state of research in connectivity of graphs and digraphs. We also present our contributions to the connectivity of general regular graphs with small diameter, and the connectivity of EX graphs.

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Chapter 1

Introduction

The design of large interconnection networks has become of growing interest due to recent advances in very large scale integrated technology. Our study has been focusing on designing static networks, that is, networks that are established and grow according to a topological design. For this static type of network, research concentrates mainly on discovering optimal designs for network topology and then developing algorithms that take advantage of the topology.

It is well known that an interconnection network can be modelled by a graph, or a directed graph. In this case, the vertices of the graph, or directed graph, represent the nodes of the interconnection network, and the edges of the graph, or the arcs of the directed graph, represent the connections between the nodes in the network. The degree of a vertex is the number of vertices connected to it. This corresponds to a constraint on the number of connections from any one node. The diameter of the graph, or directed graph, measures the maximum data communication delay.

Since the purpose of having a communication network is to exchange information efficiently, so the performance is one of the important aspects that are taken into consideration when designing a network. In order to have a good performance when we design the topology of an interconnection network, many requirements should be considered. In this thesis we restrict ourselves to focus on two main requirements. Firstly, the number of nodes, such as computers and switching devices, should be as large as possible; secondly, the connections between the nodes should be as many as possible.

Consequently, from the point of view of extremal graph theory, which is a branch of graph theory, these two main requirements correspond to the two well-known fundamental extremal graph problems:

- *Degree/diameter Problem*:

  For given integer numbers Δ and D, construct graphs of maximum degree Δ and diameter $\leq D$ with the largest possible number of vertices $n_{\Delta,D}$.

- *EX Graph Problem*:

  For given integer numbers n and t, construct graphs of given order n and girth $\geq t+1$ with the largest possible number of edges $ex(n;t)$.

The directed version of the degree/diameter problem differs only in that 'degree' is replaced by 'out-degree' in the statement of the problem:

- *Degree/diameter Problem*:

  For given integer numbers d and D, construct digraphs of maximum out-degree d and diameter $\leq D$ with the largest possible number of vertices $n_{d,D}$.

The directed version of the EX graph problem could also be formulated but we do not include it here. There are many other related open problems concerning the degree/diameter problem. There are large gaps between the best current lower and upper bounds on the largest possible order in graphs and digraphs. Furthermore, due to the fact that the extremal Moore graphs and Moore directed graphs (Moore digraphs) do not exist, with a few exceptions, we are interested in finding graphs and digraphs that are close to being Moore. The development of optimization algorithms, such as Simulated Annealing and Genetic Algorithms, to handle designing network topologies has been receiving much attention, especially in the last decade. For instance, the well known travelling salesman problem is NP-hard. However, solutions that are very close to optimal can be obtained by using Simulated Annealing and Genetic Algorithms. We believe that there is a great scope to gain improved solutions or even optimal results in the degree/diameter problem using clever optimization algorithms.

Apart from performance, reliability is another important aspect to be considered in designing a network. Network nodes and communication links sometimes fail and must be removed from service. When components fail, the network should continue to function, even if with reduced capacity. Many parameters, such as connectivity, have been introduced to measure the resilience of a network and its ability to continue operation despite disabled components. Among these parameters, a very important parameter is the connectivity of the network. Connectivity of graphs is defined as the minimum number of nodes or links that must fail in order to partition the network into two or more disjoint networks. Connectivity of digraphs means the minimum number of nodes or arcs such that these nodes can be partitioned into two parts V_1, V_2 in such a way that there are no arcs from V_1 to V_2, but arcs from V_2 to V_1 can exist. Naturally, it is desirable to have high network connectivity.

It is well known that Moore digraphs exist only for regular degree $d = 1$ or diameter $D = 1$. In Chapter 4, we present our new results in constructing large digraphs, called 'nearly Moore digraphs', which are in some way 'close' to Moore digraphs, by relaxing the maximum out-degree d. Since no Moore graphs exist for $\Delta \geq 3$ and $D \geq 3$, we list some open problems in constructing large graphs, called nearly Moore graphs, which are in some way 'close' to Moore graphs, by relaxing the maximum degree Δ.

Apart from constructing nearly Moore graphs and digraphs, we have also tried to construct 'EX graphs', that is, graphs having as many edges as possible, for given order and given forbidden cycle lengths. Obtaining the value of the size of EX graphs is known as a difficult task. In Chapter 6, we present our contribution to the construction of EX graphs when $5 \leq t \leq 7$, and some new upper bounds on the size of EX graphs and in some cases the exact values of size of EX graphs is provided.

In order to construct reliable and fault-tolerant networks we are interested in finding sufficient conditions for graphs and digraphs to satisfy large connectivity, such as the cardinality of a minimum vertex-cut of a graph G being not less than the minimum degree δ of G. Apart from the construction of EX graphs, we have been interested in the connectivity of EX graphs. In Chapter 7, we show our contribution to the connectivity of EX graphs. Furthermore, we present new general results on the connectivity of regular graphs with a small diameter D.

To summarize, this thesis is organized as follows.

Chapter 1 (this chapter) gives an introduction of the thesis.

In Chapter 2 we introduce basic concepts of Graph Theory which will be used throughout this thesis.

In Chapter 3 we present an overview of several known extremal graph problems concerning Moore bounds, girth, and connectivity. Furthermore, in Chapter 3, we show the details of some extremal graph problems based on these three parameters. The degree/diameter problem will be discussed in Section 3.2. Constructing EX graphs based on forbidden smallest cycles will be shown in Section 3.3. The current knowledge of minimal vertex-cut and minimal edge-cut of graphs in terms of connectivity is given in Section 3.4.

In Chapter 4, we discuss the problem of finding nearly Moore graphs and nearly Moore digraphs, which are in some way 'close' to Moore graphs and digraphs, and we present new results on nearly Moore digraphs.

In Chapter 5 we give a summary of the current knowledge of the construction of cages. In the second part of this chapter, two main conjectures are discussed for directed cages.

In Chapter 6 we construct some large graphs with given order and girth, whose sizes increase the best currently known lower bounds on the size of EX graphs. Moreover, when order $n \leq 40$, we improve some upper bounds on the size of EX graphs. In addition, some exact value of the size of EX graphs are obtained.

In Chapter 7 we prove that regular graphs of small diameter D are superconnected. In addition, we show that a graph $G \in EX(n;t)$ is edge-superconnected, and when

t is even, G is at least δ-connected, where δ is the smallest degree in G. When t is odd, we prove that G is at least 4-connected.

In Chapter 8, we present conclusions and some open problems arising from this thesis.

Apart from research in the topology of networks, during my PhD candidature I also conducted research in optimization algorithms. In Appendix A, the performance of algorithms is discussed, including simulated annealing (SA), genetic algorithms (GA) and a hybrid of simulated annealing and genetic algorithms (HSAGA).

All original results, mainly to be found in Chapters 4, 6 and 7, are marked with $*$.

Chapter 2

Basic Concepts

2.1 Basic concepts

In this chapter, we introduce basic concepts, definitions and notations in graph theory which will be used throughout this thesis. Those notations and concepts which are only used in a particular chapter will be defined thereof. For other concepts, definitions and notations not covered in this chapter, see [46]. We give the basic definitions for undirected graphs and directed graphs in separate sections.

2.2 Undirected graphs

A *graph* G is defined to be a pair of sets $(V(G), E(G))$, where $V(G)$ is a finite nonempty set of elements, called *vertices*, and $E(G)$ is a set (possibly empty) of unordered pairs $\{u, v\}$ of vertices $u, v \in V(G)$, called *edges*. The set $V(G)$ is called the *vertex-set* of G and $E(G)$ is called the *edge-set* of G. For the sake of brevity, an edge between u and v is often denoted by uv. In this thesis we only focus on simple graphs, so we will not consider *loops*, that is, edges of the form $\{v, v\}$, or *multiple edges*, that is, edges which occur more than once between a particular pair of vertices. The *order* of a graph G is the number of vertices in G. Figure 2.1 shows an example of a graph of order 8 with vertex set $\{v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3\}$ and edge set $\{v_1u_1, v_2u_1, v_3u_1, v_4u_1, v_2u_2, v_3u_2, v_5u_2\}$.

Let u and v be vertices of a graph G. We say that u is *adjacent* to v if there is an edge e between u and v, that is, $e = uv$. Then we call v a *neighbour* of u. The set of all neighbours of u is called the *neighbourhood* of u and is denoted by $N(u)$. We then also say that both vertices u and v are *incident with* e. The set of all edges incident with vertex u is denoted by $E(u)$. For example, in Figure 2.1, vertex v_1 is adjacent to vertex u_1, vertex u_1 is incident with edges v_2u_1, v_2u_1, v_3u_1 and v_4u_1, then $E(u_1) = \{u_1v_1, u_1v_2, u_1v_3, u_1v_4\}$.

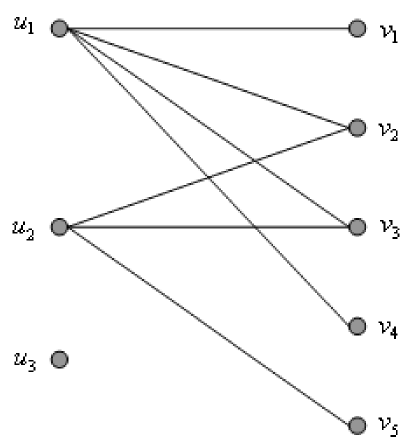


Figure 2.1: Example of a graph.

The *degree* of a vertex v of G, denoted by $d(v)$, is the number of vertices adjacent to v, that is, the number of all the neighbours of v. If a vertex v has degree 0, which means that v is not adjacent to any other vertex, then v is called an *isolated vertex*, or an *isolate*. A vertex of degree 1 is called an *end vertex*. In Figure 2.1, $d(u_1) = 4$, u_3 is an isolated vertex, and v_5 is an end vertex. We also have $d(u_2) = 3$, $d(v_1) = 1$, $d(v_2) = 2$, $d(v_3) = 2$ and $d(v_4) = 1$. Given a graph G, a *degree sequence*, denoted by $\mathcal{D} = \mathcal{D}(G)$, is a monotonic non-increasing sequence of the degrees of all the vertices in G. In Figure 2.1, the degree sequence $\mathcal{D} = (4, 3, 2, 2, 1, 1, 1, 0)$. If a graph contains many vertices, the degree sequence of this graph can be written in the superscript notation. In Figure 2.1, the degree sequence $\mathcal{D} = (4, 3, 2^2, 1^3, 0)$. Furthermore, if every vertex of a graph G has the same degree then G is called *regular*. In the graph of Figure 2.2, we have $d(v_1) = d(v_2) = d(v_3) = d(v_4) = d(u_1) = d(u_2) = d(u_3) = 4$. Therefore, the graph in Figure 2.2 is regular of degree 4. The *minimum degree* of a graph G is denoted by $\delta(G) = \delta$. Similarly, the maximum degree of a graph G is denoted by $\Delta(G) = \Delta$.

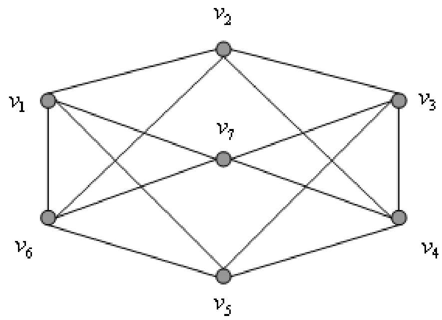


Figure 2.2: Example of a regular graph.

A $v_0 - v_l$ *walk* in a graph G is a finite alternating sequence $v_0, e_1, v_1, e_2, ..., e_l, v_l$ of vertices and edges in G, where $e_i = v_{i-1}v_i$, for each i, $1 \leq i \leq l$. Such a walk may also be denoted by $v_0, v_1...v_l$. The *length* of a walk is the number of edges in the walk. A $v_0 - v_l$ walk is *closed* if $v_0 = v_l$. If all the vertices of a $v_0 - v_l$ walk are distinct, then the walk is called a *path*. A $v_0 - v_l$ walk is called a *cycle* if $v_0 = v_l$. In Figure 2.2, $v_6v_5v_3v_7v_6v_2$ is a walk of length 5 which is not a path, $v_6v_5v_3v_7$ is a path of length 3, and $v_6v_5v_3v_7v_6$ is a cycle of length 4, this cycle is denoted by C_4. The smallest length of cycles is called *girth*.

The *distance* from vertex u to vertex v in G, denoted by $d_G(u, v)$, is the length of a shortest path from u to v. For example, the distance from v_1 to v_6 of the graph in Figure 2.3 is 2. Similarly, the *distance* from vertex u to a set of vertices X in G, denoted by $d(u, X) = d_G(u, X)$, is the length of a shortest path from u to the set X. The set $N_r(v) = \{w \in V : d(w, v) = r\}$ denotes the *neighborhood of v at distance r*. For $S \subset V$, the *neighborhood of S at distance r* is denoted by $N_r(S) = \{w \in V : d(w, S) = r\}$. Observe that $N_0(v) = v$ and $N_0(S) = S$. When $r = 1$, we write $N(v)$ and $N(S)$, instead of $N_1(v)$ and $N_1(S)$.

The *eccentricity* $e(u)$ of a given vertex u of a graph G is $e(u) = max_{v \in V(G)} d(u, v)$, that is, the distance between u and a vertex furthest from u. For instance, the eccentricity of vertex v_1 in Figure 2.3 is $e(v_1) = 3$, while $e(v_2) = 3$, $e(v_3) = 4$,

$e(v_4) = 2$, $e(v_5) = 2$, $e(v_6) = 3$, $e(v_7) = 4$ and $e(v_8) = 3$. Given a graph G, the *eccentricity sequence*, denoted by $\mathcal{E} = \mathcal{E}(G)$, is a monotonic non-increasing sequence of the eccentricities of the vertices of G. In Figure 2.3, the eccentricity sequence $\mathcal{E} = \mathcal{E}(G) = (4, 4, 3, 3, 3, 3, 2, 2)$. If a graph contains many vertices, the eccentricity sequence of the graph can be written in the superscript notation. For example, in Figure 2.3, the eccentricity sequence $\mathcal{E} = (4^2, 3^4, 2^2)$. The *radius* of G is the minimum eccentricity among all the vertices of G. A vertex is *central* if its greatest distance from any other vertex is equal to the radius of G. For example, the radius of the graph in Figure 2.3 is 2, and vertices v_4 and v_5 are central vertices. The *diameter* $D = D(G)$ of a graph G is the maximum eccentricity among all the vertices of G. In other words, the longest distance between any two vertices in G is the diameter of G. For instance, the graph in Figure 2.3 has diameter 4.

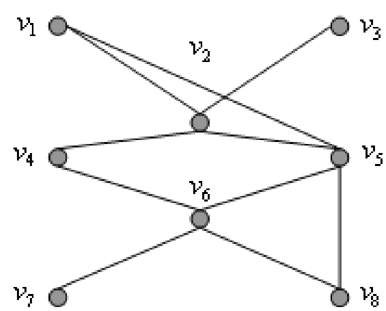


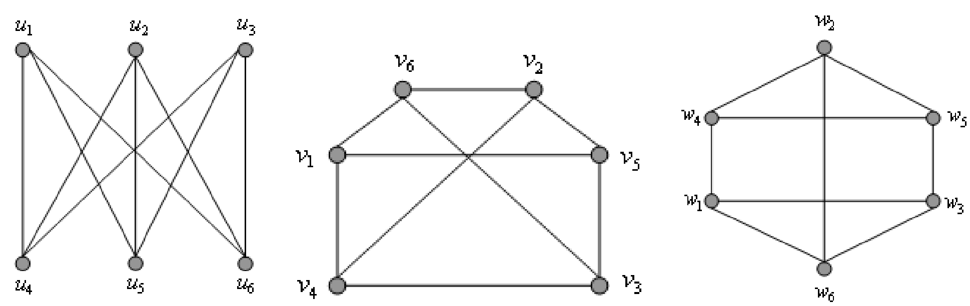
Figure 2.3: Example of a walk and a path in a graph.

A graph G is *connected* if, for any two distinct vertices u and v of G, there is a path between u and v. Otherwise, G is *disconnected*. A graph H is a *subgraph* of G if its edges and vertices form subsets of the vertex and edge sets of G. A maximal connected subgraph of G is called a *component* of G. Thus a disconnected graph contains at least two components. For example, the graph in Figure 2.2 is connected, but the graph in Figure 2.1 is disconnected (because there is no path between u_3 and any other vertex).

A *bipartite* graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If G contains every possible edge joining V_1 and V_2 then G is a *complete bipartite graph*; if V_1 and V_2

have m and n vertices, respectively, then we write $G = K_{m,n}$.

Two graphs G_1 and G_2, each with n vertices, are said to be *isomorphic* if there exists a one-to-one mapping $f : V(G_1) \to V(G_2)$ which preserves all the adjacencies, that is, $f(u)$ and $f(v)$ are adjacent in G_2 if and only if u and v are adjacent in G_1. In Figure 2.4, graphs G_1 and G_2 are isomorphic under the mapping $f(u_i) = v_i$, for every $i = 1, 2, \ldots, 6$. However, graphs G_1 and G_3 are not isomorphic because graph G_1 is bipartite, and graph G_3 is not bipartite. Consequently, there cannot be any one-to-one mapping preserving adjacencies.



(a) G_1 (b) G_2 (c) G_3

Figure 2.4: Isomorphism and non-isomorphism in graphs.

An *automorphism* of a graph G is an isomorphism where $G_1 = G_2 = G$, that is, a one-to-one mapping $f : V(G) \to V(G)$ which preserves all the adjacencies, that is, $f(u)$ and $f(v)$ are adjacent if and only if u and v are. For example, consider the graph G_2 in Figure 2.4 under the mapping f, defined by $f(v_1) = v_5, f(v_4) = v_3, f(v_6) = v_2$. Then f is an automorphism of the graph G_2.

A complete graph on n vertices, denoted K_n, is a graph in which every vertex is adjacent to every other vertex. Thus K_n has $\frac{n(n-1)}{2}$ edges. Figure 2.5 shows an example of a complete graph, K_6. A *clique* is a proper subgraph of a graph G such that every vertex is connected to every other vertex in the subgraph.

The *adjacency matrix* of a graph G with vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the

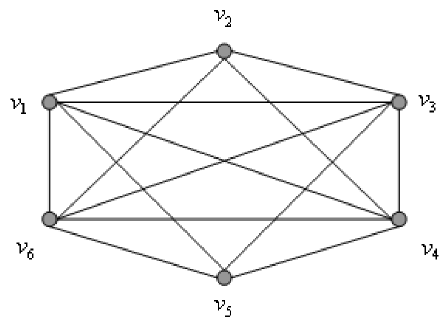
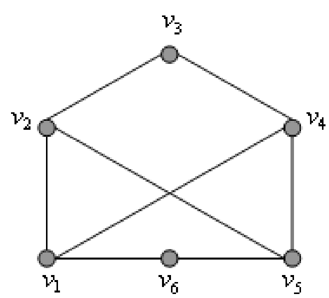


Figure 2.5: Complete graph K_6.

$n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Figure 2.6 shows a graph of order 6 with its adjacency matrix.



(a)

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b)

Figure 2.6: Graph G and its adjacency matrix A.

2.3 Directed graphs

A *directed graph* or a *digraph* G is a pair of sets $G = (V(G), A(G))$, where $V(G)$ is a finite nonempty set of distinct elements called *vertices* and $A(G)$ is a set of ordered pairs u, v of vertices $(u, v) \in V(G)$, called *arcs*. The set $V(G)$ is called the *vertex-set* of G and $A(G)$ is called the *arc-set* of G. For brevity, an arc (u, v) is often denoted by uv. Similarly to the undirected case, the number of vertices in G is called the *order* of the digraph G. The set of all the arcs incident from vertex u is denoted by $A^+(u)$ and the set of the arcs incident to vertex u is denoted by $A^-(u)$. Figure 2.7 shows an example of a digraph G of order 8 with vertex-set $V(G) = \{v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3\}$ and arc-set $A(G) = \{(v_1, u_1), (u_1, v_2), (u_1, v_3), (u_1, v_4), (v_2, u_2), (v_3, u_2), (u_2, v_5)\} = \{v_1u_1, u_1v_2, u_1v_3, u_1v_4, v_2u_2, v_3u_2, u_2v_5\}$; $A^+(u_1) = \{u_1v_2, u_1v_3, u_1v_4\}$ and $A^-(u_1) = \{v_1u_1\}$.

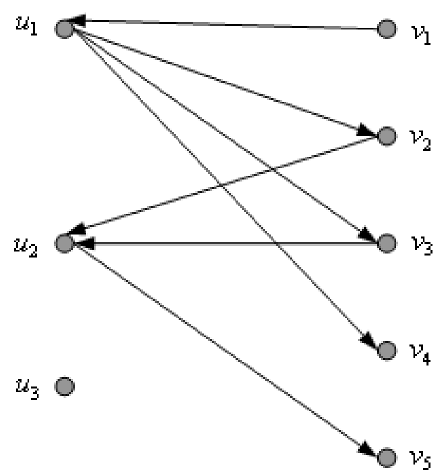
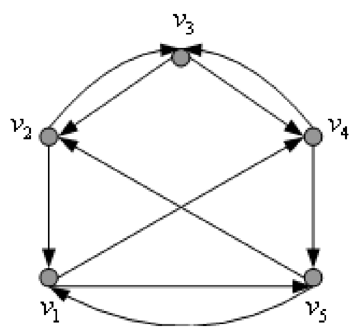
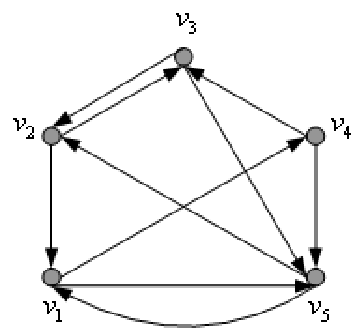


Figure 2.7: Example of a digraph.

An *in-neighbour* (respectively, *out-neighbour*) of a vertex v in G is a vertex u (respectively, w) such that $(u, v) \in A(G)$ (respectively, $(v, w) \in A(G)$). The set of all the in-neighbours (respectively, out-neighbours) of a vertex v is called the *in-neighbourhood* (respectively, *out-neighbourhood*) of v, denoted by $N^-(v)$ (respectively, $N^+(v)$). The *in-degree* (respectively, *out-degree*) of a vertex v, denoted by $d^-(v)$ (respectively, $d^+(v)$), is the number of all its in-neighbours (respectively, out-neighbours). If every vertex of a digraph G has the same in-degree (respectively, out-degree) then G is said to be *in-regular* (respectively, *out-regular*). If a digraph G is in-regular of in-degree d and out-regular of out-degree d, then G is called a *diregular* digraph of degree d (or *d-regular*). For example, the digraph G_1 in Figure 2.8 is diregular of degree 2 but the digraph G_2 is not diregular (G_2 is out-regular but not in-regular). The *minimum in-degree* of digraph G is represented by $\delta^- = \delta^-(G)$ (respectively, the *minimum out-degree* of G is denoted by $\delta^+ = \delta^+(G)$. Similarly, the *maximum in-degree* of digraph G is represented by $\Delta^- = \Delta^-(G)$ (respectively, the *maximum out-degree* of G is denoted by $\Delta^+ = \Delta^+(G)$. Furthermore, the *minimum degree* of a digraph G is defined as $\delta(G) = min\{\delta^-, \delta^+\}$. Similarly, the *maximum degree* of a digraph G is defined as $\Delta(G) = max\{\Delta^-, \Delta^+\}$.



(a) G_1



(b) G_2

Figure 2.8: Diregular digraph and non-diregular digraph.

For digraph G_2 in Figure 2.8, we have $N^+(v_1) = \{v_4, v_5\}$, $N^+(v_2) = \{v_1, v_3\}$, $N^+(v_3) = \{v_2, v_5\}$, $N^+(v_4) = \{v_3, v_5\}$, $N^+(v_5) = \{v_1\}$. Furthermore, $d^+(v_1) = 2$, $d^+(v_2) = 2$, $d^+(v_3) = 2$, $d^+(v_4) = 2$ and $d^+(v_5) = 1$. Given a digraph G, its *out-degree sequence*, denoted by $\mathcal{D}^+ = \mathcal{D}^+(G)$, is a monotonic non-increasing sequence of the out-degrees of its vertices. In Figure 2.7, the out-degree sequence $\mathcal{D}^+ = (3, 1, 1, 1, 1, 0, 0, 0)$. If a digraph contains many vertices, the out-degree sequence of this digraph can be written in the superscript notation. For example, in Figure 2.7, the out-degree sequence $\mathcal{D}^+ = (3, 1^4, 0^3)$. Given a digraph G, an *in-degree sequence*, denoted by $\mathcal{D}^- = \mathcal{D}^-(G)$, is a monotonic non-increasing sequence of the in-degrees of the vertices in G. In Figure 2.7, we have $N^-(u_1) = \{v_1\}$, $N^-(u_2) = \{v_2, v_4\}$, $N^-(u_3) = \emptyset$, $N^-(v_1) = \emptyset$, $N^-(v_2) = \{u_1\}$, $N^-(v_3) = \{u_1\}$, $N^-(v_4) = \{u_1\}$ and $N^-(v_5) = \{u_2\}$, and the in-degree sequence of G is $\mathcal{D}^- = (2, 1, 1, 1, 1, 1, 0, 0)$. The in-degree sequence of a digraph can be written in the superscript expression. For example, in Figure 2.7, the in-degree sequence is $\mathcal{D}^- = (2, 1^5, 0^2)$.

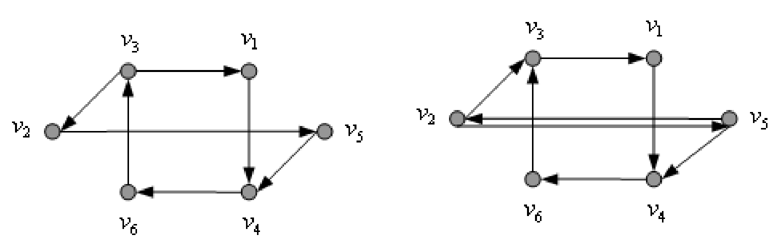
The directed versions of the terms *walk* and *path* are defined in the expected manner. A $v_0 - v_l$ directed path is called a *directed cycle* denoted by $\overrightarrow{C_l}$ if $v_0 = v_l$. The smallest length of directed cycle in G is called the *directed girth* of G. A concept that is unique to digraphs is the *semi-walk*, which is a finite, alternating sequence $v_0v_1 \ldots v_l$ beginning with vertex v_0 and ending with vertex v_l, such that either $v_{i-1}v_i$ or v_iv_{i-1} is an arc in $A(G)$, for each i, $0 \leq i \leq l$. For instance, in Figure 2.8(a), $v_4v_5v_1v_5v_2v_1$ is a semi-walk of length 5. If all vertices are distinct then we call the semi-walk a *semi-path*. For example, in Figure 2.7, $v_1u_1v_2u_2v_5$ is a semi-path of length 4.

The *distance* from vertex u to vertex v, denoted by $d(u,v)$, is the length of the shortest directed path from u to v, if any; otherwise $d(u,v) = \infty$. Note that $d(u,v)$ is not necessarily equal to $d(v,u)$. The set $N_r^-(v) = \{w \in V : d(w,v) = r\}$ and $N_r^+(v) = \{w \in V : d(v,w) = r\}$ are the *in-neighborhood* and the *out-neighborhood*, respectively, of v at distance r. The *distance from vertex w to a set of vertices S* in G, denoted by $d(w,S) = d_G(w,S)$, is the length of a shortest path from w to any vertex in S. Similarly, the *distance from a set of vertices S to a vertex w* in G, denoted by $d(S,w) = d_G(S,w)$, is the length of a shortest path from any vertex in S to w. For $S \subset V$, the *in-neighborhood* and *out-neighborhood of S at distance* r is denoted by $N_r^-(S) = \{w \in V : d(w,S) = r\}$ and $N_r^+(S) = \{w \in V : d(S,w) = r\}$. Observe that $N_0^-(v) = N_0^+(v) = \{v\}$ and $N_0^-(S) = N_0^+(S) = S$. When $r = 1$, we write $N^-(v)$, $N^+(v)$, $N^-(S)$ and $N^+(S)$, instead of $N_1^-(v)$, $N_1^+(v)$, $N_1^-(S)$ and $N_1^+(S)$.

The *in-eccentricity* $e^-(v)$ of a vertex v in a digraph G is $e^-(v) = max_{u \in G}\ d(u,v)$, that is, the distance from a vertex u furthest to v. For instance, the in-eccentricity of vertex v_1 in the digraph G_2 in Figure 2.8 is $e^-(v_1) = 2$, also we have $e^-(v_2) = 2$, $e^-(v_3) = 3$, $e^-(v_4) = 2$ and $e^-(v_1) = 2$. Similarly, The *out-eccentricity* $e^+(v)$ of a vertex v in a digraph G is $e^+(v) = max_{u \in G}\ d(v,u)$, that is, the distance from the a vertex v furthest to a vertex u. For example, the out-eccentricity of vertex v_1 in the digraph G_2 in Figure 2.8 is $e^+(v_1) = 2$. Also we have $e^+(v_2) = 2$, $e^+(v_3) = 3$, $e^+(v_4) = 2$ and $e^+(v_5) = 2$. Given a digraph G, an *in-eccentricity sequence* (respectively, *out-eccentricity sequence*) of G, denoted by $\mathcal{E}^- = \mathcal{E}^-(G)$ (respectively, denoted by $\mathcal{E}^+ = \mathcal{E}^+(G)$), is a monotonic non-increasing sequence of the in-eccentricities (respectively, out-eccentricities) of the vertices in G. In Figure 2.8, the in-eccentricity sequence $\mathcal{E}^- = (3, 2, 2, 2, 2) = (3, 2^4)$, and the out-eccentricity sequence $\mathcal{E}^+ = (3, 2, 2, 2, 2) = (3, 2^4)$). The *in-radius* (respectively, *out-radius*) is the minimum in-eccentricity (respectively, *out-eccentricity*). The *radius* of a digraph is the minimum value between in-radius and out-radius among all vertices in G. The *in-diameter* (respectively, *out-diameter*) of a digraph G is the maximum value of in-eccentricity (respectively, out-eccentricity). The *diameter* of a digraph G is the maximum value of in-eccentricity and out-eccentricity among all the vertices in G. In other words, the diameter is the longest distance between any two vertices in G. For example, the digraph G_1 in Figure 2.8 has diameter 2.

From now on, we denote by $\mathcal{G}(n, d, k)$ the set of all digraphs G, not necessarily diregular, of order n, maximum out-degree d, and diameter k.

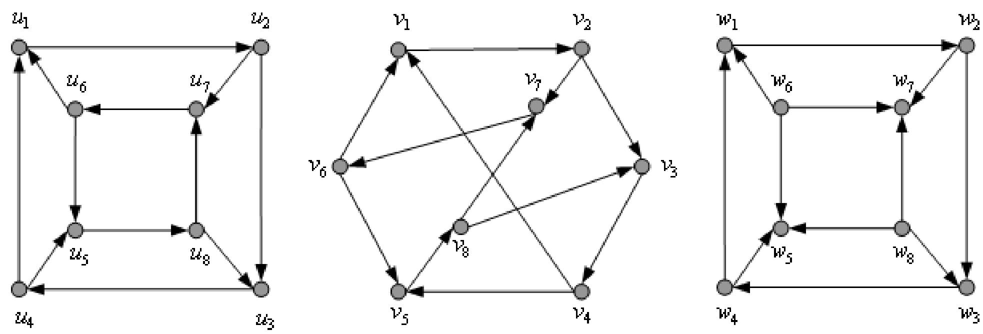
We say that a vertex v is *reachable* from a vertex u in a digraph G if there is a directed path from u to v. *The underlying graph* of a digraph G is obtained from replacing each directed arc with an undirected edge. A digraph G is *connected* if there is a path between any pair of vertices in underlying graph of G. A digraph G is called *strongly connected* if, for any two distinct vertices of G, each vertex is reachable from the other. For example, the digraph G_1 in Figure 2.9 is strongly connected but the digraph G_2 in Figure 2.9 is connected but not strongly connected because the underlying graph is connected but v_5 is not reachable from vertices v_1, v_3, v_4 and v_6.



(a) G_1 (b) G_2

Figure 2.9: A strongly connected digraph G_1 and a non-strongly connected digraph G_2.

Two digraphs G_1 and G_2, each with n vertices, are said to be *isomorphic*, if there exists a one-to-one mapping $f : V(G_1) \to V(G_2)$ which preserves all the adjacencies, that is, $f(u)$ is adjacent to $f(v)$ if and only if u is adjacent to v. Otherwise, G_1 is said to be *non-isomorphic* to G_2. In Figure 2.10, digraphs G_1 and G_2 are isomorphic under the mapping $f(u_i) = v_i$, for every $i = 1, 2, \ldots, 8$. However, digraphs G_1 and G_3 are not isomorphic because G_3 contains two vertices of in-degree 3 while G_1 does not and consequently there is no one-to-one mapping preserving adjacencies.

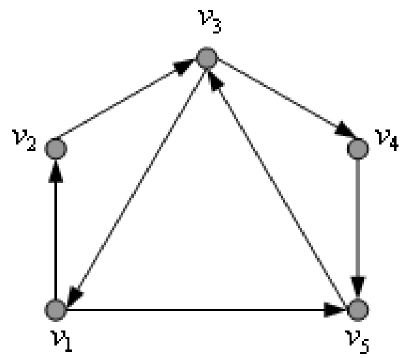


(a) G_1 (b) G_2 (c) G_3

Figure 2.10: Isomorphism and non-isomorphism in digraphs.

A complete digraph on n vertices, denoted $\overrightarrow{K_n}$, is a digraph that has every pair of distinct vertices adjacent to each other. Thus $\overrightarrow{K_n}$ has $n(n-1)$ arcs and $\overrightarrow{K_n}$ is diregular of degree $n-1$. A *directed clique* is a subdigraph of a digraph G such that every vertex is connected to and from every other vertex in the subdigraph. A *bipartite* digraph G is a digraph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every arc of G joins a vertex of V_1 with a vertex of V_2. If G contains every possible arc joining vertices in V_1 and V_2, then G is a *complete bipartite digraph*. If V_1 and V_2 have m and n vertices, we write $G = \overrightarrow{K_{m,n}}$ for the complete bipartite graph. The *adjacency matrix* of a digraph G with vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in A(G), \\ 0 & \text{otherwise.} \end{cases}$$



(a) G_1

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(b) G_2

Figure 2.11: Digraph G and its adjacency matrix A.

Figure 2.11 shows a digraph G of order 5 and its adjacency matrix $A(G)$.

Chapter 3

Extremal Graph Theory

3.1 Introduction

Extremal graph theory is a branch of graph theory concerned with inequalities among functions of graph invariants, such as order, size, connectivity, minimum or maximum degree, girth, etc., and the structures that demonstrate that these inequalities are best possible. Since the first major result by Tur*á*n in 1941 [84], numerous mathematicians have contributed to make extremal graph theory a vibrant and deep subject.

There are many interesting problems in extremal graph theory. In this chapter, we list some classic extremal graph problems as follows.

(i) *Moore graphs and digraphs.* One of the interesting questions we are asking when designing a network is the following: how to design a network which reaches its maximum capacity under limited budget and satisfies a desired performance? One way of stating this question in extremal graph theory terminology could be as follows: what is the maximum possible order $n_{\Delta,D}$ of a graph with given maximum degree Δ and diameter at most D? This problem is known as the 'degree/diameter problem'. The directed version of 'degree/diameter problem' is defined as: what is the maximum possible order $n_{d,D}$ of a digraph with given maximum out-degree d and diameter at most D?

(ii) *Girth.* In connection with cycle lengths, we are especially interested in the length of a shortest cycle, that is, the *girth* of the graph. Concerning the girth of graphs, two important unsolved problems will be discussed. One problem is the construction of graphs with the smallest possible order n, given regular degree d and girth g. Such graphs are called *cages.* The directed version of cages problem is defined as: find digraphs with the smallest possible order n, given regular degree d and directed girth g. Another famous problem about girth is constructing graphs with the largest possible number of edges, given order n and given the lengths of forbidden cycles, called EX graphs problem. In this thesis, the directed version of EX graphs problem has not been considered. Possibly, this problem could be interesting as a future research direction.

(iii) *Connectivity.* Reliability, such as fault tolerance, is a very important aspect to be considered when designing a network. Network nodes and connection links sometimes fail but the rest of the network should continue to function. From the standpoint of extremal graph theory, we investigate what is the minimum cardinality of a 'cut set' if G is a connected graph or digraph, where *cut set* is defined as the number of vertices whose removal disconnects the graph or digraph.

In this thesis, we focus on studying three problems of extremal graph theory, namely, Moore graphs and digraphs, girth and connectivity. Next, we will discuss these three problems in detail.

3.2 Moore bounds

The two problems below are extremal graph problems in terms of the maximum order of graphs and digraphs. As mentioned before, these problems are called the degree/diameter problems for undirected and for directed graphs.

Problem 3.2.1 *For given numbers Δ and D, construct graphs of maximum degree Δ and diameter $\leq D$, having the largest possible number of vertices $n_{\Delta,D}$.*

Problem 3.2.2 *For given numbers d and D, construct digraphs of maximum out-degree d and diameter $\leq D$, with the largest possible number of vertices $n_{d,D}$.*

Research activities related to the degree/diameter problem fall into two main areas. The first is search for proofs of non-existence of graphs or digraphs of orders close to the general upper bounds, known as the Moore bounds. The other area is the construction of large graphs or digraphs, in order to improve the current lower bounds on $n_{\Delta,D}$ (respectively, $n_{d,D}$). In this section, we present our current knowledge of the degree/diameter problem.

3.2.1 Moore graphs

There is a natural upper bound on the largest possible order $n_{\Delta,D}$ of a graph G of maximum degree Δ and diameter at most D. If $\Delta = 1$ then $D = 1$ and $n_{1,1} = 2$. Hence, we assume that $\Delta \geq 2$ in the rest of this section. Let x be a vertex of the graph G and let n_i, for $0 \leq i \leq D$, be the number of vertices at distance i from x. Then $n_i \leq \Delta(\Delta - 1)^{i-1}$, for $1 \leq i \leq D$ (see Figure 3.1).

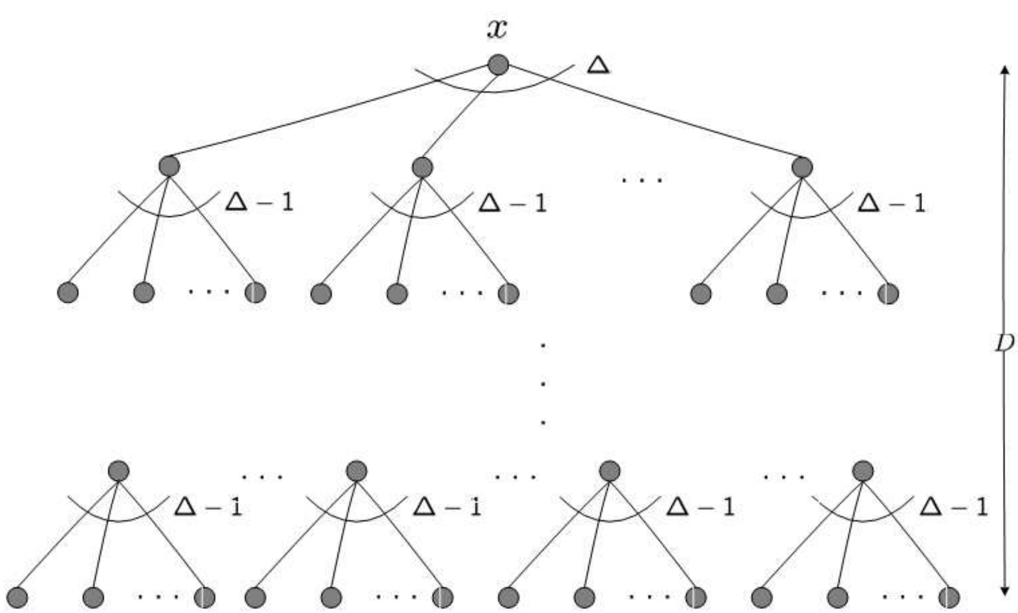


Figure 3.1: Illustration of the Moore bound for an undirected graph.

Therefore,

$$
\begin{aligned}
n_{\Delta,D} = \sum_{i=0}^{D} |N_i(x)| &\leq 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1} \\
&= 1 + \Delta(1 + (\Delta-1) + \cdots + (\Delta-1)^{D-1}) \\
&= \begin{cases} 2D+1 & \text{if } \Delta = 2 \\ 1 + \Delta\,\frac{(\Delta-1)^D - 1}{\Delta-2} & \text{if } \Delta > 2 \end{cases}
\end{aligned} \tag{3.1}
$$

The value of the right-hand side of (3.1) is called the *Moore bound* and is denoted by $M_{\Delta,D}$. The Moore bound was first mentioned by Hoffman and Singleton [48] and named after E. F. Moore. A graph of maximum degree Δ and diameter at most D whose order is equal to the Moore bound $M_{\Delta,D}$ is called a *Moore graph*; such a graph is necessarily regular of degree Δ.

Moore graphs exist only in a few cases:

- There are Moore graphs with degree $\Delta \geq 1$ when diameter $D = 1$. These Moore graphs are the complete graphs $K_{\Delta+1}$.

- There are Moore graphs, namely, the cycle C_5 (see Figure 3.2), Petersen graph (see Figure 3.3), and Hoffman-Singleton graph when diameter $D = 2$ and degree $\Delta = 2, 3, 7$. The existence of these Moore graphs was proved by Hoffman and Singleton [48]. Furthermore, it is not known if there exists a Moore graph with $D = 2$ and $\Delta = 57$.

- The only other Moore graphs, when $D \geq 3$, are the cycles on $2D + 1$ vertices C_{2D+1}. This was proved independently by Damerell [28] and by Bannai and Ito [10].

3.2.2 Moore digraphs

Similarly to the case of undirected graphs, there is a natural upper bound $n_{d,D}$ on the order of digraphs, maximum out-degree d and given diameter D. By considering a directed spanning tree of the digraph with maximum out-degree d and diameter D from a vertex in the digraph (see Figure 3.4), it is easy to derive the following bound.

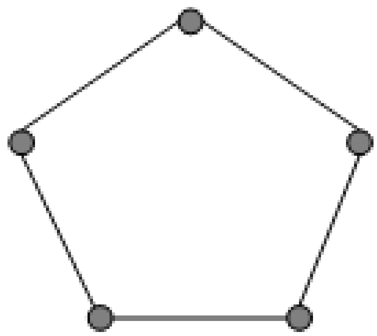


Figure 3.2: Graph of order $n = M_{2,2}$: C_5.

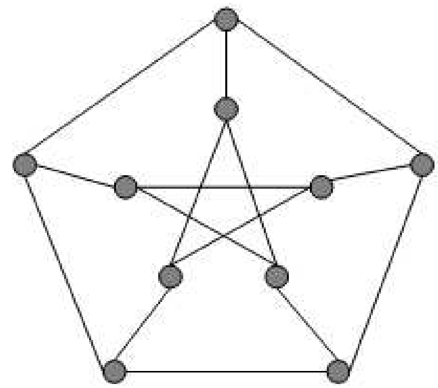


Figure 3.3: Graph of order $n = M_{3,2}$: the Petersen graph.

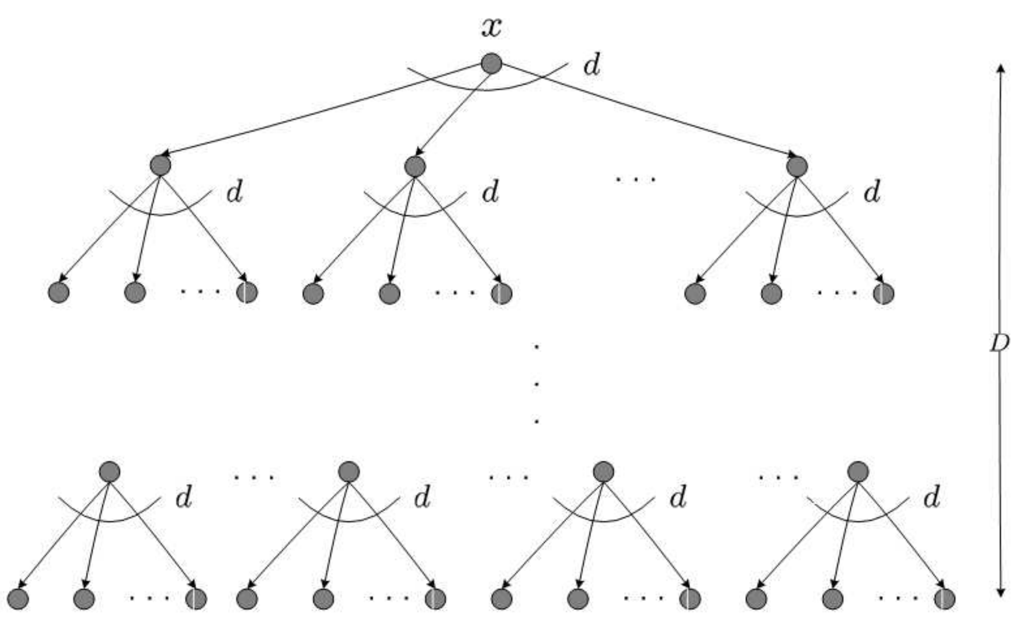


Figure 3.4: Illustration of the Moore bound for a directed graph.

$$n_{d,D} = \sum_{i=0}^{D} |N_i^+(x)| \leq 1 + d + d^2 + \cdots + d^D$$

$$= \begin{cases} D+1 & \text{if } d = 1 \\ \frac{d^{D+1}-1}{d-1} & \text{if } d > 1 \end{cases} \tag{3.2}$$

This upper bound is called the *Moore bound* for digraphs, denoted by $\overrightarrow{M_{d,D}}$. A digraph of out-degree d and diameter at most D whose order reaches the Moore bound is called a *Moore digraph*. Moore digraphs exist in the following cases:

- when $d = 1$ (directed cycles of length $D+1$, $\overrightarrow{C_{D+1}}$, for any $D \geq 1$),
- when $D = 1$ (complete digraphs of order $d+1$, $\overrightarrow{K_{d+1}}$, for any $d \geq 1$).

For $d \geq 2$ and $D \geq 2$, there do not exist any Moore digraphs of degree d and diameter D. This was first proved by Plesník and Znám [72], and in a simpler way by Bridges and Toueg [21] in 1980. Since there are no Moore digraphs with maximum out-degree $d \geq 2$ and diameter $D \geq 2$, the research in large digraphs focuses on digraphs whose order is close to the Moore bounds, that is, digraphs of order $n = \overrightarrow{M_{d,D}} - s$, where the *defect* s is as small as possible.

3.3 Girth

3.3.1 EX graphs

The *extremal number*, denoted by $ex(n;t) = ex(n;\{C_3, C_4, \ldots, C_t\})$, is the maximum number of edges in a graph of order n and girth at least $g \geq t+1$, and by $EX(n;t) = EX(n;\{C_3, C_4, C_5, \ldots, C_t\})$ we denote the set of graphs of order n, girth at least $t+1$, having the number of edges equal to $ex(n;t)$. Such graphs are called *EX graphs*.

Basically, we fix the length of the largest forbidden cycle and order of these EX graphs, but an EX graph need not be regular.

It is well known that $ex(n;3) = \lfloor n^2/4 \rfloor$, and the extremal graph is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. In 1975, Erdős [30] introduced the problem of determining the values of $ex(n;4)$, the maximum number of edges in a graph of order n with girth at least 5. He also conjectured that $ex(n;4) = (1/2 + o(1))^{3/2} n^{3/2}$. The current best known result [41] regarding this problem is

$$\frac{1}{2\sqrt{2}} n^{3/2} \leq ex(n;4) \leq \frac{1}{2} n^{3/2}.$$

It is known (see page 158 of the book by Bollobás [18]) that if $e > 90sn^{1+1/s}$ then the graph contains a cycle of length $2s$. Therefore, $ex(n;2s) \leq 90sn^{1+1/s}$.

A result proved by Erdős [76] gives the lower bound of $ex(n;t) \geq c_t n^{1+1/(t-1)}$, for some positive constant c_t. Lazebnik et al. [57] improved this lower bound, constructing a family of graphs which shows that for an infinite sequence of values of n the extremal number is lower bounded, $ex(n;2s+1) \geq d_s n^{1+2/(3s-3+\epsilon)}$, where $\epsilon = 0$ if $s \geq 3$ is odd and $\epsilon = 1$ if $s \geq 2$ is even. To our knowledge, this is the best asymptotic lower bound for the greatest number of edges in a graph of order n and girth g, $g \geq 5$, $g \neq 11, 12$. For $g = 11, 12$, a better bound is given by the regular 'generalized hexagon', which is defined in Chapter 5.

In a graph of girth g and an average degree $\bar{d}$, Alon et al. [2] proved the Moore bound for irregular graphs, denoted by $\overline{M}(\bar{d}, g)$:

Theorem 3.3.1 *[2] Let $g \geq 3$ and $\bar{d} \geq 2$. Then*

$$\overline{M}(\bar{d}, g) = \begin{cases} 1 + \bar{d} \displaystyle\sum_{i=0}^{\frac{g-3}{2}} (\bar{d}-1)^i & \text{if } g \text{ is odd.} \\ 2\bar{d} \displaystyle\sum_{i=0}^{\frac{g-2}{2}} (\bar{d}-1)^i & \text{if } g \text{ is even.} \end{cases}$$

In a graph $G \in EX(n;t)$, we know that G has girth $g \geq t+1$ and order n. Then using Theorem 3.3.1, we can obtain the maximum possible value of the average degree $\bar{d}$ if $g = t + 1$ and the order of an EX graph n is given. According to the definition of EX graphs, we know $t \geq 3$. For small values of t, such as $t = 3, 4$, we have:

$$n = n(\overline{d}, t) = \begin{cases} 2\bar{d}^2 & \textit{if } t \textit{ is } 3. \\ 1 + \bar{d}^2 & \textit{if } t \textit{ is } 4. \end{cases}$$

When $t \geq 5$, we obtain:

$$n = n(\overline{d}, t) = \begin{cases} 1 + \frac{\bar{d}}{\bar{d}-2}((\bar{d} - 1)^s - 1) & \textit{if } t \textit{ is even and } s = \frac{t}{2}. \\ \frac{2\bar{d}}{\bar{d}-2}((\bar{d} - 1)^s - 1) & \textit{if } t \textit{ is odd and } s = \frac{t+1}{2}. \end{cases}$$

Note that, an upper bound of an extremal number in terms of the maximum possible value of average degree $\overline{d}$ is obtained as $\lfloor \bar{d}n/2 \rfloor$.

The lower bound of $ex(n; t)$ is denoted by $ex_l(n; t)$. In [40], Garnick et al. obtained the following lower bounds of $ex(n; 4)$.

Theorem 3.3.2 *[40] Let $G \in EX(n; 4)$, and let q be the largest prime power such that $2n_q \leq n$, and $n_q = q^2 + q + 1$. Then $ex_l(n; 4) \geq 2n + (q - 3)n_q$.*

However, when $t \geq 5$ we do not have a general lower bound of $ex(n; t)$.

Constructing EX graphs is not easy because most values of the extremal number are not known. It would be helpful to know some structural properties of EX graphs. Regarding the girth of extremal graphs, several authors have obtained some results. In 1993, Garnick et al. [41] and Kwong et al. [56] independently proved that for $n \geq 5$, the girth of $G \in EX(n; 3)$ is 4 and, for $n \geq 9$, the girth of $G \in EX(n; 4)$ is 5. In 1997, Lazebnik and Wang [58] proved several results regarding the girth of EX graphs; these results are summarised below.

Theorem 3.3.3 *[58] Let $G \in EX(n; t)$, $t \geq 3$ and $n \geq t + 1$. Then*

(i) For $n \geq 8$, the girth of $G \in EX(n; 5)$ is 6.

(ii) There exists an extremal graph G of girth $t + 1$; and if $n \neq t + 2$, there exists an extremal graph G with minimum degree $\delta \geq 2$ and girth $t + 1$.

(iii) *For* $t \geq 12$, $ex(2t+2;t) = 2t+4$, *and there exists an extremal graph* G *with girth* $t+2$.

(iv) *If* $\Delta(G) \geq t$ *then the girth of* G *is girth* $t+1$.

Recently, in 2007, Balbuena and her collaborators obtained more results about the girth of extremal graphs, these results are summarised below.

Theorem 3.3.4 *Let* $G \in EX(n;t)$, $t \geq 3$ *and* $n \geq t+1$. *Then*

(i) *[6] For* $n \geq 12$, $n \notin \{15, 30, 80, 170\}$, *the girth of* $G \in EX(n;6)$ *is* 7, *and there exists an extremal graph* G *of* 15 *vertices having girth* 8.

(ii) *[8] If* $\Delta(G) \geq \lceil (t+1)/2 \rceil$ *and* $\delta(G) \geq 2$ *then the girth of* G *is* $g(G) \leq t+2$.

(iii) *[8] For* $t \geq 7$ *and* $n \geq \Big(2(t-2)^{t-2} + t - 5\Big)/(t-3) + 1$, *the girth of* G *is* $g(G) = t+1$.

(iv) *[8] Let* $x = \lceil (t+1)/2 \rceil$. *For* $t \geq 7$ *and* $n \geq \Big(2(x-2)^{t-2} + x - 5\Big)/(x-3) + 1$, *the girth of* G *is* $g(G) \leq t+2$.

We know that, for particular values of girth and order, there do exist some graphs with largest number of edges having minimum degree 1, for example, there exist graphs in $EX(11,4)$ with the degree sequences $\{4^1, 3^9, 1\}$, as well as $\{3^{10}, 2\}$ or $\{4, 3^8, 2^2\}$, on the other hand, in general, it is believed that the degrees are not far from the average degree [87]. This observation relates the problem of constructing EX graphs to the problem of constructing cages. However, as pointed out in many papers, these two classes of graphs are not the same. For example, the $(5;5)$-cage has order $n = 30$, and 75 edges, while $EX(30;4)$ has 76 edges. An example of a degree sequence of $EX(30;4)$ is $\{6^6, 5^{20}, 4^4\}$ [87].

Another extremal problem in graph theory regarding the girth is the construction of *Cage*. A $(d;g)$-*graph* is a d-regular graph of girth g. A $(d;g)$-*cage* is a $(d;g)$-*graph* with the smallest possible number of vertices. The term *cage* is used to mean a $(d;g)$-cage for any values of d and g.

Although some cages were discussed by Tutte [85] already in 1947, these graphs have been intensively studied only after Erdös and Sachs [32] showed the existence of cages for all d and g, and Hoffman and Singleton [48] showed the nonexistence of certain Moore graphs.

3.4 Connectivity

This thesis deals with strong connectivity. From now on, by connectivity we shall mean strong connectivity. Connectivity is one of the basic concepts of graph theory. It is divided into two areas, namely 'vertex-connectivity' κ and 'edge-connectivity' (respectively, 'arc-connectivity' in case of directed graphs) λ. *Vertex-connectivity*, or simply, *connectivity*, is defined as the minimum number of vertices that must be removed in order to disconnect a graph or a digraph. Analogously, the edge-connectivity (respectively, arc-connectivity) equals the minimum number of edges (respectively, arcs), whose removal disconnects the graph (respectively, digraph). Naturally, it is desirable to have large connectivity for graphs and digraphs. In order to construct reliable and fault-tolerant networks, we are interested in finding sufficient conditions for graphs and digraphs to possess large connectivity. In this chapter, we summarize the known sufficient conditions on some parameters for a graph or a digraph to have large connectivity. The parameters that we consider are order n, girth g, minimum degree δ, maximum degree Δ, and diameter D.

3.4.1 Undirected graphs

Recall that a graph G is *connected* if there is a path between any two vertices of G. A *vertex-cut* (respectively, *edge-cut*) of a connected graph G is a set of vertices (respectively, edges), whose removal disconnects the graph. Every graph that is not complete has a vertex-cut. The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not a complete graph, and we define $\kappa(G) = r - 1$ if $G = K_r$. Similarly, the *edge-connectivity*, denoted by $\lambda = \lambda(G)$, is the minimum number of edges whose deletion disconnects the graph. A well-known result [42] relating connectivity to the minimum degree $\delta(G)$ states

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Thus, a graph with minimum degree $\delta(G)$ is *maximally connected* (respectively, *maximally edge-connected*) if $\kappa(G) = \delta(G)$ (respectively, $\lambda(G) = \delta(G)$).

One area of major interest for researchers in extremal graph theory has been providing sufficient conditions to guarantee lower bounds of κ (respectively, lower bounds of λ). Several results in this area gave sufficient conditions on the degree of pairs of vertices, on the minimum degree in terms of the order, and on the diameter. The following theorem summarizes some of the most relevant results, which appeared in various papers, as indicated.

Theorem 3.4.1 *Let G be a connected graph of order n with minimum degree $\delta \geq 3$ and maximum degree Δ, diameter D, and connectivity parameters λ and κ. The following statements hold.*

(i) [25] If $\delta \geq \lfloor \frac{n}{2} \rfloor$ then $\lambda = \delta$.

(ii) [59] If $d(u) + d(v) \geq n - 1$, for every nonadjacent vertices u, v, then $\lambda = \delta$.

(iii) [71] If $D \leq 2$ then $\lambda = \delta$.

In 1985 and 1987, Soneoka et al. [80, 81] provided sufficient conditions on the diameter of a graph in terms of its girth. Notice that an improved version of Theorem 3.4.2 (iii) will be given in Chapter 7.

Theorem 3.4.2 *Let G be a connected graph of order n with minimum degree $\delta \geq 3$ and maximum degree Δ, diameter D, girth g, and connectivity parameters λ and κ. The following statements hold.*

(i) [81] If $\begin{cases} D \leq g - 1, & g \text{ odd} \\ D \leq g - 2 & g \text{ even} \end{cases}$ *then $\lambda = \delta$.*

(ii) *[80] If* $\begin{cases} D \le g-2, & g \text{ odd} \\ D \le g-3 & g \text{ even} \end{cases}$ *then* $\kappa = \delta$.

The notion of superconnectedness of undirected graphs was proposed independently in [13, 16, 17], and this notion aims at advancing the analysis of connectivity properties of graphs beyond the original notion of connectivity. A graph is *superconnected*, for short, *super-*κ, if every minimal vertex-cut is the neighbourhood of one vertex, see Boesch [16], and Tindell [17] and Fiol, Fàbrega and Escudero [37]. Observe that a superconnected graph is necessarily maximally connected, $\kappa = \delta$, but the converse is not true. For example, a cycle $\mathcal{C}_g$ of length g, with $g \ge 6$, is a maximally connected graph that is not superconnected. Analogously, a graph is *edge-superconnected* if the deletion of every minimal edge-cut isolates a vertex of degree δ. Lesniak [59], Soneoka [78] and Fiol [36] bring some results that guarantee a graph to be edge-superconnected in terms of the degree of pairs of vertices, or the order, or the diameter.

Theorem 3.4.3 *Let G be a connected graph of order n, with minimum degree δ, maximum degree Δ, diameter D. Then G is edge-superconnected if any one of the following statements holds:*

(i) *[59]* $d(u) + d(v) \ge n+1$, *for every pair of nonadjacent vertices* u, v.

(ii) *[78]* $n > \delta(\frac{(\Delta-1)^{D-1}-1}{\Delta-2} + 1) + (\Delta-1)^{D-1}$.

(iii) *[36]* $D = 2$ *and G contains no clique* $K_{\delta+1}$.

(iv) *[36]* $\delta \ge \lfloor n/2 \rfloor + 1$.

In this thesis, we are more interested in 'trivial versus non-trivial vertex-cut'. A *trivial vertex-cut* X is defined as a vertex-cut X that contains the whole neighbourhood $N(u)$ of some vertex $u \notin X$. Similarly, an edge-cut S of G is called a *trivial edge-cut* if S contians $E(u)$ of a vertex u, where $u \in V$. Otherwise, the vertex-cut is called a *non-trivial vertex-cut* (respectively, the edge-cut is called a *nontrivial edge-cut*). There exist some superconnected graphs for which every vertex-cut or edge-cut is trivial. For example, in a complete bipartite graph $K_{x,y}$, with $x, y \geq 2$, every vertex-cut is trivial. In addition, some graphs contain both non-trivial and trivial vertex-cuts (respectively, edge-cuts). Provided that some non-trivial vertex-cut exists, the *superconnectivity* of G, denoted by κ_1, respectively, the edge-superconnectivity of G, denoted by λ_1, were defined in [5, 37] as

(a) $\kappa_1 = \kappa_1(G) = min\{|X| : X \subset V$ is a nontrivial vertex-cut$\}$

(b) $\lambda_1 = \kappa_1(G) = min\{|S| : S \subset E$ is a nontrivial edge-cut$\}$

A nontrivial vertex-cut X is called a κ_1-*cut* if $|X| = \kappa_1$ (respectively, a nontrivial edge-cut S is called a λ_1-*cut* if $|S| = \lambda_1$). Notice that if $\kappa_1 \leq \delta$, then $\kappa_1 = \kappa$ and that $\kappa_1 > \delta$ is a sufficient and necessary condition for G to be super-κ. Similarly, if $\lambda_1 \leq \delta$, then $\lambda_1 = \lambda$, and $\lambda_1 > \delta$ is a sufficient and necessary condition for G to be super-λ. In 1989, Fàbrega and Fiol [34] showed that a graph is superconnected if $D \leq 2\lfloor (g-1)/2 \rfloor - 2$ (respectively, edge-superconnected if $D \leq 2\lfloor (g-1)/2 \rfloor - 1$).

Theorem 3.4.4 [34] *Let G be a connected graph of minimum degree $\delta \geq 3$, diameter D, girth g, and the cardinality of the minimum nontrivial vertex-cut κ_1 and the cardinality of the minimum nontrivial edge-cut λ_1. Then*

$$\lambda_1 > \delta \text{ if } \begin{cases} D \leq g-2, & g \text{ odd} \\ D \leq g-3 & g \text{ even} \end{cases} \tag{3.3}$$

$$\kappa_1 > \delta \text{ if } \begin{cases} D \leq g-3, & g \text{ odd} \\ D \leq g-4 & g \text{ even} \end{cases} \tag{3.4}$$

In 1990, Fàbrega and Fiol [37] proved that if $D \leq 2\lfloor (g-1)/2 \rfloor - 2$ then a graph is superconnected and $\kappa_1 \geq 2\delta - 2$ (respectively, if $D \leq 2\lfloor (g-1)/2 \rfloor - 1$ then a graph is edge-superconnected and $\lambda_1 \geq 2\delta - 2$).

Theorem 3.4.5 [37] *Let G be a connected graph with minimum degree $\delta \geq 3$, diameter D, girth g, the cardinality of the minimum nontrivial vertex-cut κ_1 and the* *cardinality of the minimum nontrivial edge-cut* λ_1. *Then*

$$\lambda_1 \geq 2\delta - 2 \text{ if } \begin{cases} D \leq g-2, & g \text{ odd} \\ D \leq g-3 & g \text{ even} \end{cases} \tag{3.5}$$

$$\kappa_1 \geq 2\delta - 2 \text{ if } \begin{cases} D \leq g-3, & g \text{ odd} \\ D \leq g-4 & g \text{ even} \end{cases} \tag{3.6}$$

In 2006, Balbuena et al. [9] showed that if $D \leq g-2$ then a graph is edge-superconnected and λ_1 is equal to $2\delta - 2$.

Theorem 3.4.6 [9] *Let G be a connected graph with minimum degree $\delta \geq 2$, diameter D, girth g, the cardinality of a minimum nontrivial vertex-cut κ_1, and the cardinality of a minimum nontrivial edge-cut λ_1. Then*

$$\lambda_1 = 2\delta - 2 \text{ if } D \leq g-2. \tag{3.7}$$

Recently, Balbuena et al. [7] improved Theorem 3.4.5 as follows.

Theorem 3.4.7 [7] *Let G be a connected graph with minimum degree $\delta \geq 2$, diameter D, girth g, the cardinality of a minimum nontrivial vertex-cut κ_1, and the cardinality of a minimum nontrivial edge-cut λ_1. Then*

$$\kappa_1 \geq 2\delta - 2 \text{ if } D \leq g-3. \tag{3.8}$$

3.4.2 Directed graphs

In this section we deal with digraphs. We allow loops but not parallel arcs. Recall that when there exists a $u \to v$ path for any pair $u, v \in V(G) = V$, a digraph G is said to be *strongly connected*. A subset of vertices F, whose deletion results in a digraph $G - F$ that is not strongly connected, will be referred to as a *vertex-disconnecting set*, or simply a *vertex-cut*. If G is not a complete digraph, the *strong* *vertex-connectivity* of G, $\kappa = \kappa(G)$, is the minimum cardinality of a vertex-cut. Analogously, an *arc-cut* is a subset $S \subset A(G)$ whose deletion from G results in a nonstrongly connected digraph $G - S$, and the *arc-connectivity*, $\lambda = \lambda(G)$, is the minimum cardinality of an arc-cut.

Recall that

$$\kappa(G) \leq \lambda(G) \leq \delta(G). \tag{3.9}$$

For more details, see the results provided by Geller and Harary [42]. When $\kappa = \delta$ (respectively, $\lambda = \delta$), the digraph is said to be *maximally connected* (respectively, *maximally arc-connected*).

Results of digraphs being maximally connected (respectively, maximally arc-connecte d) are often presented as sufficient conditions on the diameter of a digraph in terms of its girth, order, minimum or maximum degree. The following theorem summarizes some of the most relevant results given by Ayoub and Frisch [3], Jolivet [51] and Imase et al. [49].

Theorem 3.4.8 *Let G be a connected digraph of order n, with minimum degree $\delta \geq 2$, maximum degree Δ, diameter D, girth g, and connectivity parameters λ and κ. The following statements hold.*

(i) [3] If $g \geq 3$ and $\delta \geq \lfloor \frac{n+2}{4} \rfloor$ then $\lambda = \delta$.

(ii) [51] If $D \leq 2$ then $\lambda = \delta$.

(iii) [49] If $n > (\delta - 1)\frac{\Delta^{D-1}+\Delta^2-2}{\Delta-1}$ then $\lambda = \delta$.

(iv) [49] If $n > (\delta - 1)\frac{\Delta^{D}+\Delta^2-\Delta-1}{\Delta-1}$ then $\kappa = \delta$.

A vertex-cut $F \subset V$ is called *trivial* if either $N^+(x)$ or $N^-(x)$ is contained in F, for some vertex $x \in V \setminus F$. An arc-cut $S \in A(G)$ is called *trivial* if S is the arc set $A^+(x)$ or $A^-(x)$ of some $x \in V$. A vertex-cut (respectively, arc-cut) that is not trivial will be called *nontrivial*. A maximally connected (respectively, arc-connected) digraph with minimum degree δ is *superconnected*, for short, *super-κ*

(respectively, *arc-superconnected*, for short, *super*-λ) if all its disconnecting sets of minimum cardinality (respectively, arc-disconnecting sets of minimum cardinality), with cardinality equal to δ, are trivial; see Boesch and Tindell [17] and Fábrega and Fiol [34]. The study of arc-superconnected digraphs has a particular significance in the design of reliable networks [16] because attaining arc-superconnectivity implies minimizing the number of minimum arc-disconnecting sets [79]. In 1992, Fiol [36] provided results on the diameter of a digraph and the degree of a pair of vertices, defined as sum of degrees of a pair of vertices that are sufficient for a digraph to be arc-superconnected.

Theorem 3.4.9 *[36] Let G be a connected digraph of order n, with minimum degree δ, maximum degree Δ, and diameter D. Then G is arc-superconnected if either of the following statements holds.*

(i) $d^+(u) + d^-(v) \geq n + 1$, *for every pair of nonadjacent vertices* u, v.

(ii) $D = 2$ *and* G *contains no directed clique* $\overrightarrow{K_{\delta+1}}$.

There exist some superconnected digraphs in which every vertex-cut is trivial, in other words, such a digraph has no nontrivial vertex-cuts. For example, the complete symmetric digraph $\overrightarrow{K_n}$ and the complete symmetric bipartite digraph $\overrightarrow{K_{n,m}}$ are of this type [45]. In a certain sense, such digraphs can be regarded as optimally superconnected, since they cannot possibly be disconnected unless one vertex is isolated. For this reason, we only consider digraphs containing both nontrivial and trivial vertex-cuts (respectively arc-cuts). Thus we consider the following parameters, defined in [37] and also in [5]:

(a) $\kappa_1 = \kappa_1(G) = min\{|F| : F \subset V$ is a nontrivial vertex-cut$\}$,

(b) $\lambda_1 = \lambda_1(G) = min\{|S| : S \subset A$ is a nontrivial arc-cut$\}$.

Notice that if $\kappa_1 \leq \delta$ (respectively, $\lambda_1 \leq \delta$) then $\kappa_1 = \kappa$ (respectively, $\lambda_1 = \lambda$). The condition $\kappa_1 > \delta$ (respectively, $\lambda_1 > \delta$) means that all the minimum vertex-cuts of cardinality equal to δ must be trivial, and the digraph G is superconnected (respectively, arc-superconnected). Hence, κ_1 (respectively, λ_1) can be seen as a measure of the superconnectivity (respectively, arc-superconnectivity) of G.

In order to study the connectivity of digraphs, a new parameter related to the number of shortest paths was introduced in [34] called 'parameter ℓ', which has recently received the more descriptive name of 'semigirth', see The Handbook of Graph Theory [43]:

Definition 3.4.1 Let G be a digraph without loops, of diameter D. Then the *semigirth* $\ell = \ell(G)$, $1 \leq \ell \leq D$, is defined as the greatest integer so that, for any two vertices u, v,

(a) if $d(u, v) < \ell$ then the shortest $u \to v$ path is unique and there is no $x \to y$ path of length $d(u, v) + 1$;

(b) if $d(u, v) = \ell$ then there is only one shortest $u \to v$ path.

More generally, we have

Definition 3.4.2 Let G be a digraph of diameter D, and let $\pi \geq 0$ be an integer. The π-*semigirth* $\ell^\pi = \ell^\pi(G)$, $1 \leq \ell^\pi \leq D$, is defined as the greatest integer so that, for any two vertices u, v,

(a) if $d(u, v) < \ell^\pi$ then the shortest $u \to v$ path is unique and there are at most π paths $u \to v$ of length $d(u, v) + 1$,

(b) if $d(u, v) = \ell^\pi$ then there is only one shortest $u \to v$ path.

Clearly, according to the definition, semigirth and 0-semigirth have the same meaning.

In terms of semigirth ℓ, various sufficient conditions on the diameter D and order n have been studied in order to give a lower bound for the connectivity or superconnectivity parameters. In 1989, Fábrega and Fiol [34] provided some sufficient conditions for maximum connected and maximum arc-connected digraphs.

Theorem 3.4.10 *[34] Let G be a connected digraph, diameter D, semigirth $\ell = \ell(G)$, connectivity κ and arc-connectivity λ. Then*
(i) $\lambda = \delta$ if $D \leq 2\ell$.
(ii) $\kappa = \delta$ if $D \leq 2\ell - 1$.

Theorem 3.4.11 *Let G be a connected digraph of order n and size m, with minimum degree $\delta \geq 3$, diameter D and semigirth $\ell = \ell(G)$, superconnectivity κ_1 and arc-superconnectivity λ_1.*
(i) $\kappa_1 \geq 2\delta - 2$ and G is superconnected if any of the following conditions holds

(a) [37] $D \leq 2\ell - 2$.

(b) [35] G is bipartite, $\ell \geq 2$, and $D \leq 2\ell - 1$.

(ii) $\lambda_1 \geq 2\delta - 2$ and G is arc-superconnected if any of the following conditions holds:

(a) [37] $D \leq 2\ell - 1$.

(b) [35] G is bipartite, $\ell \geq 2$, and $D \leq 2\ell$.

In terms of ℓ^π, various sufficient conditions on the diameter have been studied in order to give a lower bound for connectivity or superconnectivity parameters. Some of them are summarized in the following proposition.

Theorem 3.4.12 *Let G be a connected digraph with minimum degree δ, diameter D and π-semigirth ℓ^π. Then, for any integer π, such that $0 \leq \pi \leq \delta - 2$, the following assertions hold.*

(i) [60] For either $\delta \geq 7$ and $1 \leq \pi \leq \lfloor \delta/2 \rfloor$, or $\delta \leq 6$ and $1 \leq \pi \leq \lfloor (\delta - 1)/2 \rfloor$, we have
$\kappa = \delta$ if $D \leq 2\ell^\pi - 2$ *and $\ell \geq 2$; and*
$\lambda = \delta$ if $D \leq 2\ell^\pi - 1$.

(ii) [69] For $\delta \geq 4$, $\ell \geq 2$ and $\pi = 1$, we have
$\kappa_1 \geq 2\delta - 2$ if $D \leq 2\ell^1 - 3$; *and*
$\lambda_1 \geq 2\delta - 2$ if $D \leq 2\ell^1 - 2$.

(iii) *[60] For* $\delta \geq 5$ *and* $1 \leq \pi \leq \lfloor(\delta-2)/2\rfloor$, *we have*

$\kappa_1 \geq 2(\delta-\pi)$ if $D \leq 2\ell^\pi - 2$ *and* $\ell \geq 2$; *and*

$\lambda_1 \geq 2(\delta-\pi)$ if $D \leq 2\ell^\pi - 1$.

Observe that $\kappa = \delta$ or $\lambda = \delta$ can be also assured through Theorem 3.4.10 (*i*) and (*ii*), whenever (*i*) $\pi = 0$, or (*ii*) $\pi = 1$, provided that the corresponding condition on the diameter holds. Moveover, Theorem 3.4.10 (*iii*) improves (*i*) and (*ii*), since it holds also for $\pi \geq 2$ (if $\delta \geq 5$). Observe also that if $\delta \geq 5$ and $\pi = 1$, then the sufficient conditions on the diameter in (*v*) guaranteeing either $\kappa_1 \geq 2(\delta-\pi)$ or $\lambda_1 \geq 2(\delta-\pi)$ are less restrictive than (*iv*) of Theorem 3.4.10.

Next, we will consider the connectivity of a 'line digraph'. The line digraph ($L(G)$) of a digraph G is defined as follows. In the *line digraph* $L(G)$ of a digraph G, each vertex corresponds to an arc of G. Thus, $V(L(G)) = \{uv : (u,v) \in A(G)\}$. A vertex uv is adjacent to a vertex wz iff $v = w$, that is, when the arc (u,v) is adjacent to the arc (w,z) in G. The *k-iterated line digraph*, $L^k(G)$, is defined recursively by $L^k(G) = L(L^{k-1}(G))$. Let $G = (V,A)$ be a digraph different from a cycle. It is well known that, for any integer $k \geq 1$, the relationship between the diameters is

$$D(L^k(G)) = D(G) + k. \tag{3.10}$$

For more details about line digraphs, see, for instance [38]. From the definition of $L(G)$, it is clear that $\delta(L^k(G)) = \delta(G) = \delta$ for any $k \geq 1$. Also, it can be shown that

$$\kappa(L(G)) = \lambda(G), \tag{3.11}$$

and therefore, by (3.9),

$$\kappa(G) \leq \lambda(G) = \kappa(L(G)) \leq \lambda(L(G)) \leq \delta. \tag{3.12}$$

Hence, line digraph iteration tends to increase both connectivities. Of course, when

$\kappa(G) = \lambda(G) = \delta$, (3.12) gives

$$\kappa(L(G)) = \lambda(L(G)) = \delta \tag{3.13}$$

for any order of the iteration k.

Proposition 3.4.1 [34] *For any digraph G without loops and different from a cycle,*

$$\ell(L^k(G)) = \ell(G) + k. \tag{3.14}$$

The interest in considering k-iterated line digraphs stems from the fact that, if k is large enough, the conditions on the diameter in (i) of Theorem 3.4.10 are satisfied. More precisely, from (3.10) and (3.14),

$$\begin{aligned} D(L^k(G)) &\leq 2\ell(L^k(G)) \text{ iff } k \geq D(G) - 2\ell(G); \\ D(L^k(G)) &\leq 2\ell(L^k(G)) - 1 \text{ iff } k \geq D(G) - 2\ell(G) + 1. \end{aligned} \tag{3.15}$$

From (3.15) and Theorem 3.4.10, it follows

Corollary 3.4.1 [34] *The connectivities of the iterated line digraphs $L^k(G)$ satisfy*

$$\begin{aligned} \lambda(L^k(G)) &= \delta \text{ if } k \geq D - 2\ell; \\ \kappa(L^k(G)) &= \delta \text{ if } k \geq D - 2\ell + 1. \end{aligned} \tag{3.16}$$

From (3.10), (3.14) and Theorem 3.4.11, it follows

Corollary 3.4.2 [37] *Let G be a connected digraph with no loops, minimum degree $\delta \geq 3$, $\ell(G) = \ell$, and diameter D. Let $\kappa_1(L^k(G))$ be the minimum cardinality of a nontrivial vertex-cut of $L^k(G)$ and $\lambda_1(L^k(G))$ be the minimum of a nontrivial arc-cut* *of $L^k(G)$. Then*

$$\lambda_1(L^k(G)) \geq 2\delta - 2 \text{ if } k \geq D - 2\ell + 1; \tag{3.17}$$
$$\kappa_1(L^k(G)) \geq 2\delta - 2 \text{ if } k \geq D - 2\ell + 2.$$

Proposition 3.4.2 [34] *For any digraph G different from a cycle,*

$$\ell^\pi(L^k(G)) = \ell^\pi(G) + k. \tag{3.18}$$

As before, if k is large enough, the conditions on the diameter of (i) of Theorem 3.4.10 are satisfied. More precisely, from (3.10) and (3.18), we get

$$D(L^k(G)) \leq 2\ell^\pi(L^k(G)) \text{ if } k \geq D(G) - 2\ell^\pi(G); \tag{3.19}$$
$$D(L^k(G)) \leq 2\ell^\pi(L^k(G)) - 1 \text{ if } k \geq D(G) - 2\ell^\pi(G) + 1.$$

Last but not least, we provide a link in connectivity between directed graphs and undirected graphs. Let us deal with a (simple) undirected graph G by considering its *associated symmetric digraph* G^*, that is, the digraph obtained from G by replacing each edge $(x, y) \in E(G)$ by the two arcs (x, y) and (y, x) forming a '*digon*'. The basic relationship is that $\kappa(G^*) = \kappa(G)$ and $\lambda(G^*) = \lambda(G)$, since a minimum arc-disconnecting set cannot contain digons. Observe that if G is a graph with girth $g(G)$, then the semigirth of the associated symmetric digraph is

$$\ell(G^*) = \lfloor (g(G) - 1)/2 \rfloor \tag{3.20}$$

while the girth $g(G^*) = 2$.

In particular, according to (3.20), for any digraph, Theorem 3.4.10 (i) and (ii) can be rewritten as Theorem 3.4.2 (i) and (ii), for any graph.

Chapter 4

Construction of Graphs and Digraphs Close to Moore

4.1 Introduction

As mentioned in Chapter 3, Moore graphs exist only for diameter $D = 1$ and maximum degree $\Delta \geq 1$, or $D = 2$ and $\Delta = 2, 3, 7$ and possibly 57. Moore digraphs exist only for $d = 1$ or $D = 1$. Consequently, we are interested in studying the existence of large graphs and digraphs which are in some way 'close' to Moore graphs and digraphs. It is possible to relax in turn one or two of the three parameters, namely, the order n, the maximum degree Δ (respectively, maximum out-degree d for digraphs) and the diameter D, in order to get close to Moore graphs and digraphs. For constructions of some graphs and digraphs of orders which are close to the Moore bounds, see Sections 4.2 and 4.5. Relaxing the diameter, Knor [55] proved that there exist regular *radially Moore digraphs*, which have the maximum possible order n, regular degree d, radius s, and diameter D that does not exceed $s + 1$. In this chapter, we will discuss relaxation of maximum degree in graphs of order equal to the Moore bound (respectively, maximum out-degree in digraphs of order equal to the directed Moore bound), in order to construct 'nearly Moore graphs' (respectively, 'nearly Moore diagraphs'), which have $n = M_{\Delta,D}$ (respectively, $M_{\delta,D}$) and diameter D.

4.2 Graphs of order close to the Moore bound

Since Damerell [28] showed that no Moore graphs exist for $\Delta \geq 3$ and $D \geq 3$, the study of the existence of large graphs of given diameter and maximum degree focuses on graphs whose order is 'close' to the Moore bound, that is, graphs of order $M_{\Delta,D} - s$. The parameter s is called the *defect*. The most usual understanding of 'small defect' is that $s \leq \Delta$. For convenience, by a (Δ, D)-graph we will understand any graph of maximum degree at most Δ and of diameter D; if such a graph has the number of vertices equal to $M_{\Delta,D} - s$ then it will be referred to as a *(Δ, D)-graph of defect s*.

Erdős, Fajtlowitcz and Hoffman [31] proved that there are no graphs of degree Δ, diameter 2 and defect 1, that is, of order one less than the Moore bound, except the cycle C_4. Bannai and Ito [11] generalised this to all diameters. For $\Delta = 2$, (Δ, D)-graphs are the cycles C_{2D}, and for all $\Delta \geq 3$, there are no (Δ, D)-graphs of defect 1. It follows that, for $\Delta \geq 3$, we have $n_{\Delta,D} \leq M_{\Delta,D} - 2$. At present, there are only five known graphs of order $n = M_{\Delta,D} - 2$, maximum degree Δ. Two $(3, 2)$-graphs of order 8 (see Figure 4.1 and 4.2), one $(4, 2)$-graph of order 15 (see Figure 4.4) [66], one $(3, 3)$-graph of order 20 (see Figure 4.3) [66], and one $(5, 2)$-graph of order 24 (see Figure 4.5) [66].

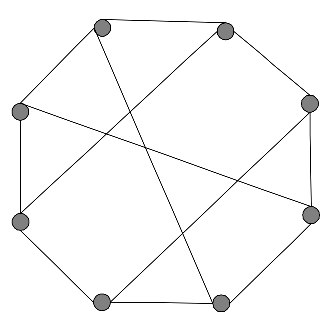


Figure 4.1: $(3, 2)$-graph of defect 2.

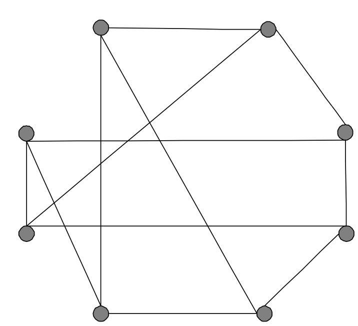


Figure 4.2: $(3, 2)$-graph of defect 2.

The study of the existence of graphs with defects larger than two is still going on and a few results are known. For example, Jorgensen [52] proved that a graph with maximum degree 3 and diameter $D \geq 4$ cannot have defect 2, which shows that

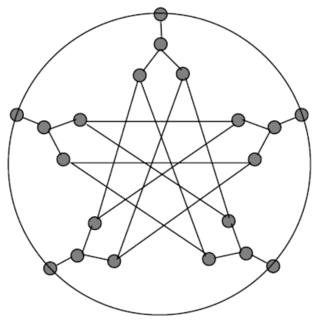


Figure 4.3: $(3, 3)$-graph of defect 2.

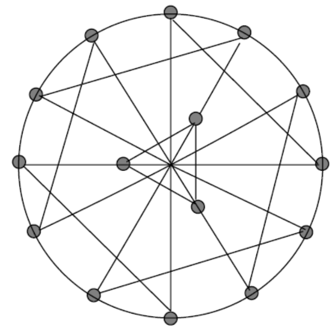


Figure 4.4: $(4, 2)$-graph of defect 2.

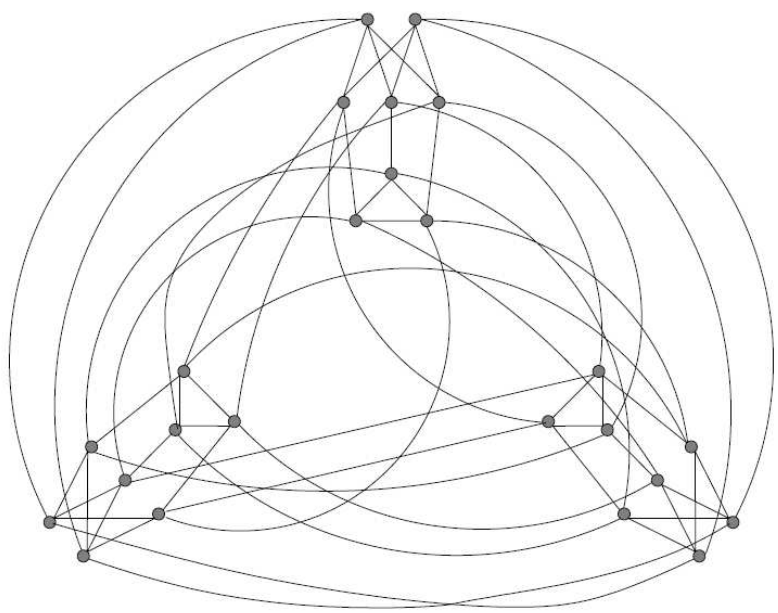


Figure 4.5: $(5, 2)$-graph of defect 2.

$n_{3,D} \leq M_{3,D} - 3$ if $D \geq 4$. Molodtsov [67] showed that the upper bound of the order on a graph with maximum degree 6 and diameter $D = 2$, $n_{2,6}$, is equal to $M_{6,2} - 5$. Conde and Gimbert [27] proved that when $7 \leq \Delta \leq 50$, there does not exist any graph of order $M_{\Delta,2} - 2$. In addition, Buset [23] proved that a graph of maximum degree 3 and diameter $D = 4$ cannot have defect 7, which gives $n_{3,4} \leq M_{3,4} - 8$. Since graphs with $n = M_{3,4} - 8$, degree 3 and diameter 4 exist, we have $n_{3,4} = M_{3,4} - 8$.

We summarise our current knowledge about the upper bound of $n_{\Delta,D}$ on the order of graphs of degree Δ and diameter D in Table 4.1.

Diameter D	***Maximum Degree*** Δ	$n_{\Delta,D}$
1	≥ 1	$M_{\Delta,1}$
2	$2, 3, 7$	$M_{\Delta,2}$
	$4, 5$	$M_{\Delta,2} - 2$
	6	27
	57	≤ 3250
	$7 \leq \Delta \leq 50$	$\leq M_{\Delta,2} - 3$
	other $\Delta \geq 2$	$\leq M_{\Delta,2} - 2$
3	2	7
	3	20
	≥ 4	$\leq M_{\Delta,3} - 2$
4	2	9
	3	46
	≥ 4	$\leq M_{\Delta,4} - 2$
≥ 5	2	$2D + 1$
	3	$\leq M_{3,D} - 3$
	≥ 4	$\leq M_{\Delta,D} - 2$

Table 4.1: Current best upper bounds for the order of graphs of maximum degree Δ and diameter D.

4.3 Constructions of large graphs

Another way to study graphs close to the Moore bound is to construct large graphs in order to improve the current lower bound of $n_{\Delta,D}$, the maximum possible order of graphs of given Δ and D.

Table 4.2 gives a summary of the order of the current largest known graphs for degree $\Delta \leq 16$ and diameter $D \leq 10$. This table can be found on the website 'http://www-mat.upc.es/grup_de_grafs/grafs/taula_delta_d.html', which is kept and updated regularly by Francesc Comellas.

4.4 Nearly Moore graphs

The problem of relaxing the maximum degree to get close to Moore graphs is still open. Here we would like to look for graphs which are 'close' to Moore graphs, by relaxing the maximum degree Δ and having some particular degree sequences. This type of degree sequence is defined as: $\mathcal{D} = ((\Delta+x)^{t_x}, \ldots, (\Delta+2)^{t_2}, (\Delta+1)^{t_1}, \Delta^{t_0})$. Let define a graph called $(n, \mathcal{D}, D)$-graph.

Definition 4.4.1 A graph G is called a $(n, \mathcal{D}, D)$-graph if G has order n, diameter D and degree sequence $\mathcal{D}$.

Next, we introduce some further notation used throughout the rest of this section. In a $(n, \mathcal{D}, D)$-graph G, G has β vertices of degree larger than Δ, $\beta = t_x + \ldots + t_2 + t_1$, and the degree excess is $\alpha = t_1 + 2t_2 + \ldots + xt_x$. We are interested not only in α, but also in the distribution of these excess degrees in the graph.

For $(n, \mathcal{D}, D)$-graphs, we are interested in the following problems.

(a) What is the minimum value α of a $(n, \mathcal{D}, D)$-graph with $n = M_{\Delta,D}$ vertices and given diameter D?

(b) What is the minimum value β of a $(n, \mathcal{D}, D)$-graph with $n = M_{\Delta,D}$ vertices and given diameter D?

D / Δ	2	3	4	5	6	7	8	9	10
3	10	20	38	70	132	196	336	600	1250
4	15	41	96	364	740	1 320	3 243	7 575	17 703
5	24	72	210	624	2 772	5 516	17 030	53 352	164 720
6	32	110	390	1 404	7 917	19 282	75 157	295 025	1 212 117
7	50	168	672	2 756	11 988	52 768	233 700	1 124 990	5 311 572
8	57	253	1100	5 060	39 672	130 017	714 010	4 039 704	17 823 532
9	74	585	1 550	8 200	75 893	270 192	1 485 498	10 423 212	31 466 244
10	91	650	2 223	13 140	134 690	561 957	4 019 736	17 304 400	104 058 822
11	98	715	3 200	18 700	156 864	971 028	5 941 864	62 932 488	250 108 668
12	133	786	4 680	29 470	359 772	1 900 464	10 423 212	104 058 822	600 105 100
13	162	851	6 560	39 576	531 440	2 901 404	17 823 532	180 002 472	1 050 104 118
14	183	916	8 200	56 790	816 294	6 200 460	41 894 424	450 103 771	2 050 103 984
15	186	1 215	11 712	74 298	1 417 248	8 079 298	90 001 236	900 207 542	4 149 702 144
16	198	1 600	14 640	132 496	1 771 560	14 882 658	104 518 518	1 400 103 920	7 394 669 856

Table 4.2: Order of the current largest known graphs of maximum degree Δ and given diameter D.

(c) With respect to β, one extreme case is when $\beta = 1$, that is, when the graph has one vertex of maximum degree $\Delta + \alpha$, and the rest of the vertices all have degree Δ. The corresponding degree sequence can be described as

$$\mathcal{D}_1 = (\Delta + \alpha, \Delta^{M_{\Delta,D}-1}).$$

(d) With respect to $\Delta + x$, one extreme case is when the graph has β vertices of degree $\Delta + 1$, and the rest of the vertices all have degree Δ. The corresponding degree sequence is

$$\mathcal{D}_2 = ((\Delta + 1)^{\beta}, \Delta^{M_{\Delta,D}-\beta}).$$

In particular, we propose two open problems of obtaining $(n, \mathcal{D}, D)$-graphs with either one above special degree sequence $\mathcal{D}_1$ or another above extreme special degree sequences $\mathcal{D}_2$. A graph $G \in (n, \mathcal{D}, D)$-graph is called *nearly Moore graphs* if G has the degree sequence $\mathcal{D}_1$ and α is smallest possible (respectively, the degree sequence $\mathcal{D}_2$ and β is smallest possible).

Problem 4.4.1 *What is the minimum value of α, if G is a $(n, \mathcal{D}, D)$-graph with $M_{\Delta,D}$ vertices, given diameter D, and degree sequence $\mathcal{D}_1 = (\Delta + \alpha, \Delta^{M_{\Delta,D}-1})$?*

Problem 4.4.2 *What is the minimum number of vertices, denoted by β, if G is a $(n, \mathcal{D}, D)$-graph with $M_{\Delta,D}$ vertices, given diameter D, and degree sequence $\mathcal{D}_2 = ((\Delta + 1)^{\beta}, \Delta^{M_{\Delta,D}-\beta})$?*

Using our algorithm HSAGA (for details, see Appendix A), we obtained a graph G, which is close to a nearly Moore graph, of order 17, two vertices of degree 10, and the rest of the vertices of degree 4, and diameter 2, denoted by $G(17, (10^2, 4^{15}), 2)$ (see Figure 4.6).

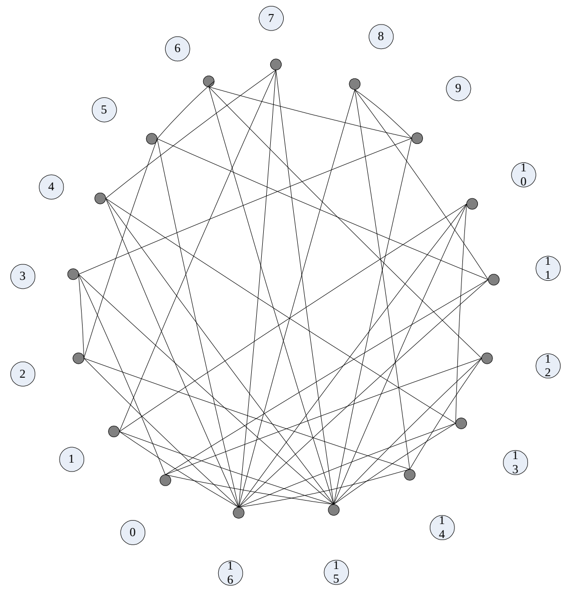


Figure 4.6: $G \in G(17, (10^2, 4^{15}), 2)$

Due to Bannai and Ito [11], we know that there are no graphs of degree $\Delta \geq 2$, diameter $D \geq 2$ and order one less than the Moore bounds, apart from the cycle C_{2D}. Hence we know that a graph with order 9, maximum degree 3 and diameter 2 does not exist. Similarly, a graph with order 16, maximum degree 4, diameter 2 does not exist. If we allow to have one vertex of larger degree than the others, we are able to obtain an optimal graph G of order 9, one vertex of degree 4, and the rest of the vertices of degree 3, and diameter 2, denoted by $G \in G(9, (4, 3^8), 2)$ (see Figure 4.7). Furthermore, we obtain a graph of order 16, one vertex of degree 10, and the rest of the vertices of degree 4, and diameter 2, denoted by $G \in G(16, (10, 4^{15}), 2)$ (see Figure 4.8). Restricting the value of the maximum degree to $\Delta+1$, we have obtained a graph of order 16 that has 6 vertices of degree 5, and the rest of the vertices of degree 4, with diameter still equal to 2, the graph is denoted by $G \in G(16, (5^6, 4^{10}), 2)$ (see Figure 4.9).

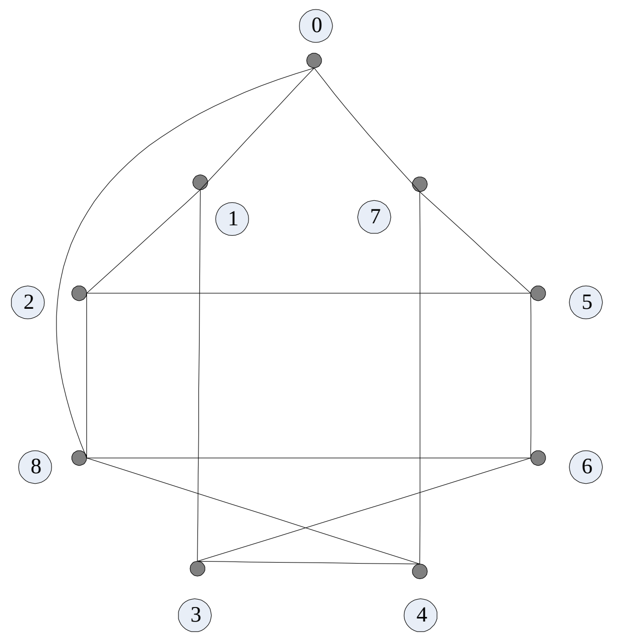


Figure 4.7: $G \in G(9, (4, 3^8), 2)$

4.5 Digraphs of order close to the Moore bound

Currently, the best general upper bound for the maximum order of a digraph of maximum out-degree d and diameter D is $\overrightarrow{M_{d,D}} - 1$. A digraph G is called *almost Moore digraph* for a pair $d \geq 2$ and $D \geq 2$ if G has maximum out-degree d, diameter at most D, and order $\overrightarrow{M_{d,D}} - 1$.

There are some results on the existence or nonexistence of almost Moore digraphs for small values of d or D. When $D = 2$, almost Moore digraphs exist for all $d \geq 1$. When $D \geq 3$, Miller and Fris [64] proved that there are no almost Moore digraphs of maximum out-degree 2, for any $D \geq 3$. Furthermore, Baskoro et al. [12] showed that there are no almost Moore digraphs of maximum out-degree 3 and any diameter greater than or equal to 3. The question of whether or not equality can hold in $n_{d,D} \leq \overrightarrow{M_{d,D}} - 1$, for $d \geq 4$ and $D \geq 4$, is still widely open.

The study of the existence of large digraphs continues by considering the existence of digraphs of order two less than the Moore bound. We call such digraphs *digraphs*

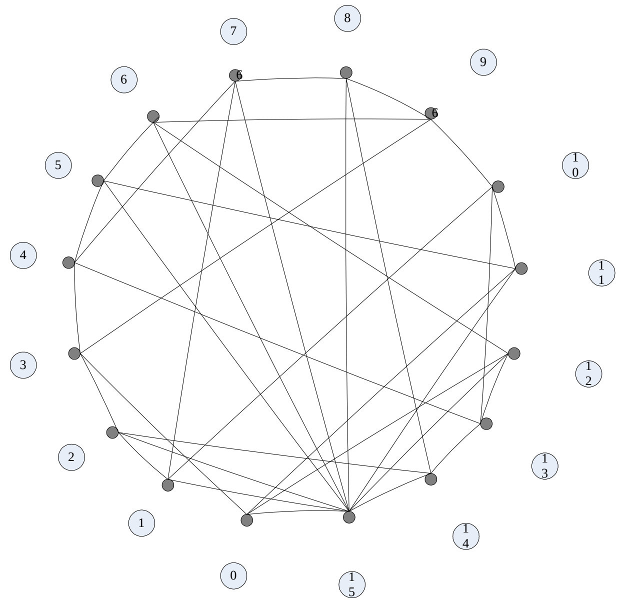


Figure 4.8: $G \in G(16,(10,4^{15}),2)$

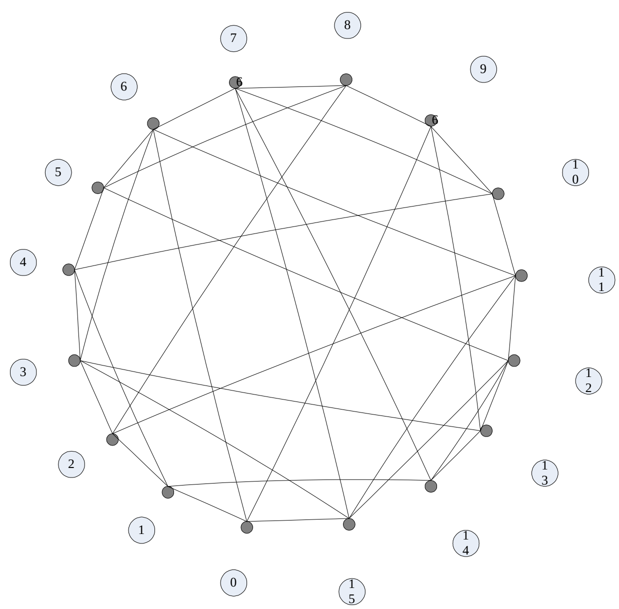


Figure 4.9: $G \in G(16,(5^{6},4^{10}),2)$

of defect two. Early research focused on the study of digraphs of defect two with out-degree $d = 2$. Miller and Širáň [65] proved that digraphs of defect two do not exist for out-degree $d = 2$ and all $D \geq 3$. For the remaining values of $D \geq 3$ and $d \geq 3$, the question of whether digraphs of defect two exist or not remains widely open.

Our current knowledge of the upper bound of $n_{d,D}$, the maximum order of digraphs of out-degree d and diameter at most D, is summarized in Table 4.3.

Diameter D	***Maximum Out-degree*** d	$n_{d,D}$
1	≥ 1	$\overrightarrow{M_{d,1}}$
2	1	3
	≥ 2	$\leq \overrightarrow{M_{d,2}} - 1$
≥ 3	1	$D + 1$
	2	$\overrightarrow{M_{2,D}} - 3$
	3	$\overrightarrow{M_{3,D}} - 2$
	≥ 4	$\leq \overrightarrow{M_{d,D}} - 1$

Table 4.3: Current best upper bounds on the order of digraphs of out-degree d and diameter D.

4.6 Constructions of large digraphs

The current best lower bound of $n_{d,D}$, the order of maximum digraphs of out-degree d and diameter D, is as follows. For out-degree $d \geq 2$ and diameter $D \geq 4$,

$$n_{d,D} \geq 25 \times 2^{D-4}. \tag{4.1}$$

This lower bound is obtained from the *Alegre digraph* of out-degree 2, diameter 4 and order 25 (see Figure 4.10 [66]), and from its iterated line digraphs. For the remaining values of out-degree and diameter, a general lower bound is

$$n_{d,D} \geq d^D + d^{D-1}. \qquad (4.2)$$

This bound is obtained from *Kautz digraphs*, denoted by K_d^D, which is a directed graph of degree d and diameter D, with $d^D + d^{D-1}$ vertices labeled by all possible strings $s_0, \ldots, s_{D-1}$ of length D, which are composed of characters s_i chosen from an alphabet A containing $d+1$ distinct symbols, subject to the condition that adjacent characters in the string cannot be equal ($s_i \neq s_{i+1}$). Notice that a Kautz digraph K_d^D has $d^{D+1} + d^D$ arcs [53]. For example, Figure 4.11 [66] shows the Kautz digraph of out-degree 2, diameter 4 and order 24 [77].

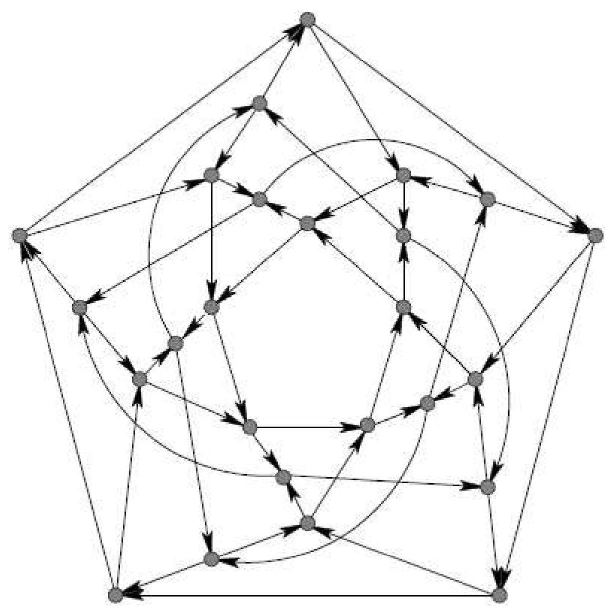


Figure 4.10: the Alegre digraph.

Table 4.4 from [65] gives a summary of the current largest known digraphs for maximum out-degree $d \leq 13$ and diameter at most $D \leq 11$.

4.7 Nearly Moore digraphs

The approach of relaxing the out-degree to get close to the non-existent Moore digraphs is new. Here we would like to find digraphs which are 'close', by relaxing the maximum out-degree d. This type of out-degree sequence is defined as: $\mathcal{D}^+ =$

d \ D	2	3	4	5	6	7	8	9	10	11
2	**6**	**12**	**25**	**50**	**100**	**200**	**400**	**800**	**1 600**	**3 200**
3	**12**	**36**	**108**	**324**	**972**	**2 916**	**8 748**	**26 244**	**78 732**	**236 196**
4	**20**	**80**	**320**	**1 280**	**5 120**	**20 480**	**81 920**	**327 680**	**1 310 720**	**5 242 880**
5	**30**	**150**	**750**	**3 750**	**18 750**	**93 750**	**468 750**	**2 343 750**	**11 718 750**	**58 593 750**
6	**42**	**252**	**1 512**	**9 072**	**54 432**	**326 592**	**1 959 552**	**11 757 312**	**70 543 872**	**423 263 232**
7	**56**	**392**	**2 744**	**19 208**	**134 456**	**941 192**	**6 588 344**	**46 118 408**	**322 828 856**	**2 259 801 992**
8	**72**	**576**	**4 608**	**36 864**	**294 912**	**2 359 296**	**18 874 368**	**150 994 944**	**1 207 959 552**	**9 663 676 416**
9	**90**	**810**	**7 290**	**65 610**	**590 490**	**5 314 410**	**47 829 690**	**430 467 210**	**3 874 204 890**	**34 867 844 010**
10	**110**	**1 100**	**11 000**	**110 000**	**1 100 000**	**11 000 000**	**110 000 000**	**1 100 000 000**	**11 000 000 000**	**110 000 000 000**
11	**132**	**1 452**	**15 972**	**175 692**	**1 932 612**	**21 258 732**	**233 846 052**	**2 572 306 572**	**28 295 372 292**	**311 249 095 212**
12	**156**	**1 872**	**22 464**	**269 568**	**3 234 816**	**38 817 792**	**465 813 504**	**5 589 762 048**	**67 077 144 576**	**804 925 734 912**
13	**182**	**2 366**	**30 758**	**399 854**	**5 198 102**	**67 575 326**	**878 479 238**	**11 420 230 094**	**148 462 991 222**	**1 930 018 885 886**

Table 4.4: Order of the largest known digraphs of maximum out-degree d and diameter D.

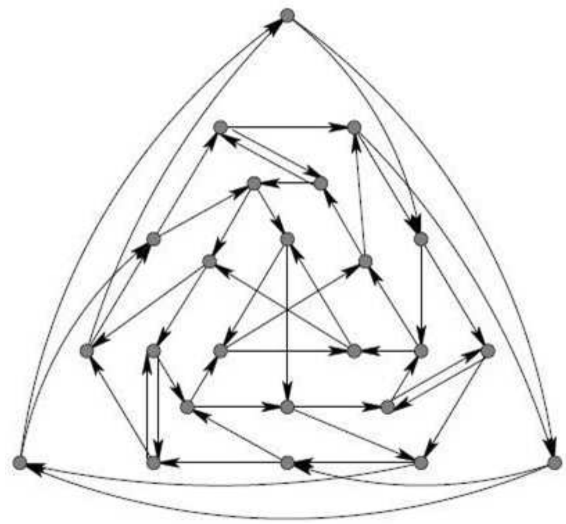


Figure 4.11: a Kautz digraph.

$((d+x)^{t_x}, \ldots, (d+2)^{t_2}, (d+1)^{t_1}, d^{t_0})$. Similarly to the undirected case, we have the following definition of a $(n, \mathcal{D}^+, D)$-digraph.

Definition 4.7.1 A digraph G is called a $(n, \mathcal{D}^+, D) - digraph$, if G has order n and diameter D, and out-degree sequence $\mathcal{D}^+$.

Next, we introduce some further notation used throughout the rest of this section. In a $(n, \mathcal{D}^+, D)$-digraph G, G has β vertices of out-degree larger than d, $\beta = t_x + \ldots + t_2 + t_1$, and the total number of out-degree excess is $\alpha = t_1 + 2t_2 + \ldots + xt_x$. We are interested not only in the number of out-degree excess α, but also in the distribution of the out-degree excess.

For $(n, \mathcal{D}^+, D)$-digraphs, we are particularly interested in the following problems.

(a) What is the minimum value α of a $(n, \mathcal{D}^+, D)$-digraph with $n = \overrightarrow{M_{d,D}}$ vertices and given diameter D?

(b) What is the minimum value β of a $(n, \mathcal{D}^+, D)$-digraph with $n = \overrightarrow{M_{d,D}}$ vertices and given diameter D?

(c) With respect to β, one extreme case is when $\beta = 1$, that is, when the digraph has one vertex of maximum out-degree $d + \alpha$, and the rest of the vertices all have out-degree d. The corresponding out-degree sequence can be described as

$$\mathcal{D}_1^+ = (d+\alpha, d^{\overrightarrow{M_{d,D}}-1}).$$

(d) With respect to $d+x$, one extreme case is when the digraph has β vertices of out-degree $d+1$, and the rest of the vertices of out-degree d. The corresponding out-degree sequence is

$$\mathcal{D}_2^+ = ((d+1)^{\beta}, d^{\overrightarrow{M_{d,D}}-\beta}).$$

In particular, we are interested in the problem of obtaining $(n, \mathcal{D}^+, D)$-digraphs with either above extreme special sequences $\mathcal{D}_1^+$ or $\mathcal{D}_2^+$. A digraph $G \in (n, \mathcal{D}^+, D)$-digraph is called *nearly Moore digraph* if G has degree sequence $\mathcal{D}_1^+$ and α is smallest possible (respectively, degree sequence $\mathcal{D}_2^+$ and β is smallest possible).

Problem 4.7.1 *What is the minimum value α, if G is a $(n, \mathcal{D}^+, D)$-digraph with $\overrightarrow{M_{d,D}}$ vertices and given diameter D, and out-degree sequence $\mathcal{D}_1^+ = (d+\alpha, d^{\overrightarrow{M_{d,D}}-1})$?*

Problem 4.7.2 *What is the minimum number of vertices, denoted by β, if G is a $(n, \mathcal{D}^+, D)$-digraph with $\overrightarrow{M_{d,D}}$ vertices, given diameter D, and out-degree sequence $\mathcal{D}_2^+ = ((d+1)^{\beta}, d^{\overrightarrow{M_{d,D}}-\beta})$?*

To construct nearly Moore digraphs, we have developed an optimization algorithm method [83]: the *Hybrid Simulated Annealing and Genetic Algorithm* (HSAGA) (see Appendix *A* for a detailed description). Here we give only an overall of HSAGA and its performance. The general idea of HSAGA is that a random graph is created at the beginning, and used as the initial graph input into *Simulated Annealing*, denoted by SA. If the iteration reaches the maximum generation of attempted moves at the last step of reaching the maximum frozen then SA will terminate and the current population will be transferred to the *Genetic Algorithm*, called GA. Otherwise, a candidate solution will be obtained and be saved into the population. Furthermore, the set of elite individuals of the population is chosen by a selection procedure of GA according to their evaluation fitness values, following genetic operations consisting of crossover and mutation.

We use HSAGA to construct nearly Moore digraphs. In order to make comparisons with other important optimization algorithms, including Simulating Annealing (SA) and Genetic Algorithm (GA), which could be used to attack the degree/diameter problems, we have performed various experiments. The best experimental results that we have obtained are given in Table 4.5. A central vertex, denoted by *ce*, is a vertex of eccentricity equal to the radius of the digraph. It is easy to observe that neither SA nor GA alone has been effective for the degree/diameter problem. In other words, SA and GA used separately have not given us the minimum diameter and maximum number of central vertices for orders greater than 10. However, combining simulated annealing and genetic algorithm has produced more encouraging results.

Using HSAGA, we have obtained nearly Moore digraphs of diameter $2 \leq D \leq 6$ and most, but not all, vertices with out-degree $d = 2$ and with order equal to the Moore bound $M_{2,D}$. For example, see the nearly Moore digraphs in Figures 4.12 - 4.14. Notice that the nearly Moore digraph in Figure 4.12 is optimal when order $n = 7$ and diameter $D = 2$. Interestingly, we found three non-isomorphic nearly Moore digraphs with the same out-degree sequences, diameter $D = 3$ and order 15, but with different in-degree sequences (see Figure 4.14).

In order to summarise the results concerning our nearly Moore digraphs and the current results on the order-relaxed Moore digraphs, we present Table 4.6. In this table, given out-degree $d = 2$ and diameter D, with $2 \leq D \leq 6$, we list current regular large digraphs of order n, together with the gap, denoted by γ, between this order and $\overrightarrow{M_{d,D}}$. We also show nearly Moore digraphs of order $\overrightarrow{M_{d,D}}$, and either the current minimum number of extra arcs α or the current minimum number β of vertices of degree $d+1$, plus their corresponding out-degree sequences $\mathcal{D}^+$.

	SA		**GA**		**HSAGA**	
n	ce	D	ce	D	ce	D
1	0	0	0	0	0	0
2	2	1	2	1	2	1
3	3	1	3	1	3	1
4	4	2	4	2	4	2
5	4	3	5	2	5	2
6	6	2	6	2	6	2
7	7	3	7	3	7	3
8	8	3	8	3	8	3
9	9	3	9	3	9	3
10	9	4	9	4	**10**	**3**
11	7	4	6	4	**11**	**3**
12	5	4	5	4	**12**	**3**
13	12	5	13	4	**13**	**4**
14	14	4	14	4	**14**	**4**
15	13	5	14	5	**15**	**4**
16	11	5	14	5	**16**	**4**
17	10	5	9	5	**17**	**4**
18	8	5	10	5	**18**	**4**

Table 4.5: The number of central vertices and the minimum value of diameter D obtained from our tests when $d = 2$ and $n \leq 18$.

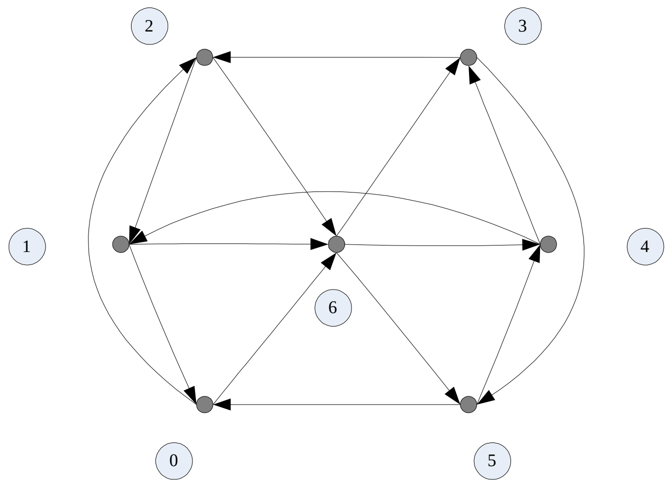


Figure 4.12: $G \in G(7, (3, 2^6), 2)$.

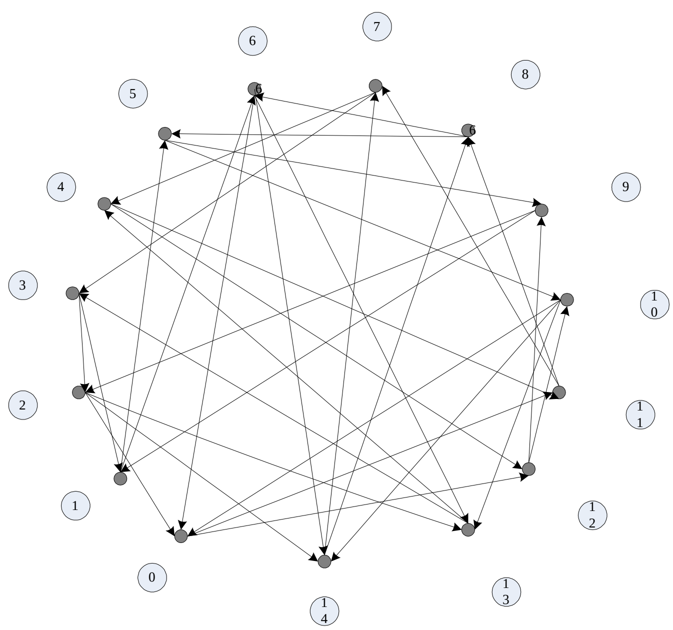
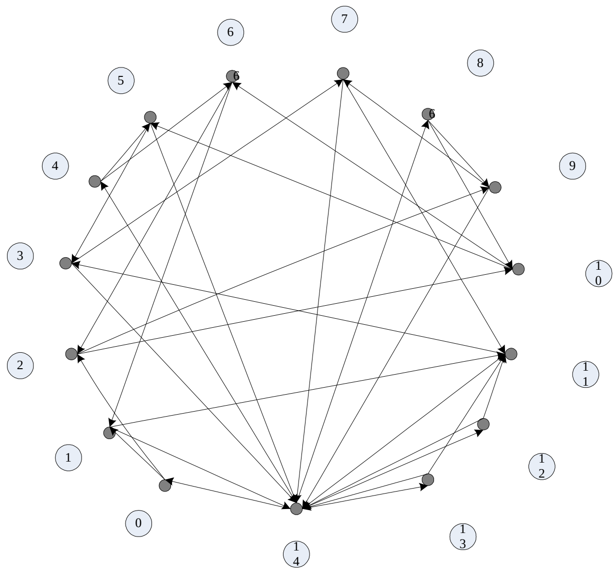


Figure 4.13: $G \in G(15, (3^3, 2^{12}), 3)$.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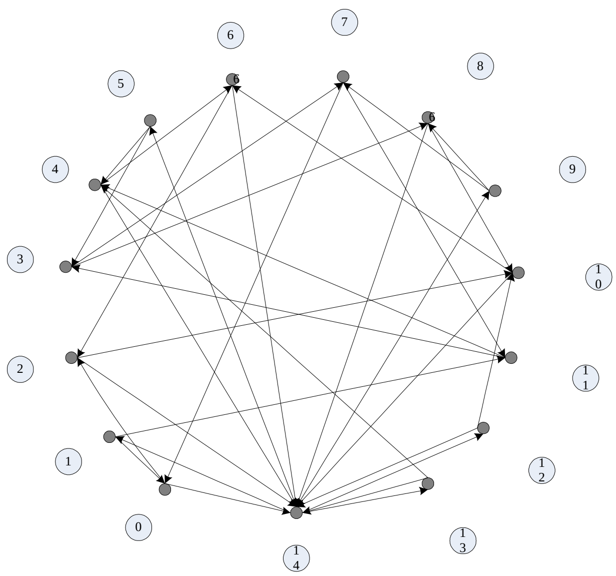
(a) G_1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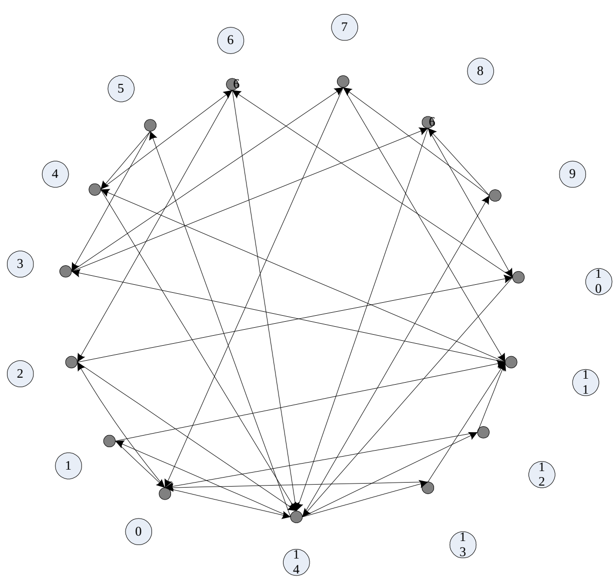
(b) G_2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14

(c) G_3

	Out-Degree-Relaxed			Order-Relaxed	
$\overrightarrow{M_{2,D}}$	$\mathcal{D}^+$	α	β	D	γ
7	$(3, 2^6)$	1	1	2	1
15	$(3^3, 2^{12})$ $(5, 2^{14})$	3 3	3 1	3	3
31	$(3^4, 2^{27})$ $(10, 2^{14})$	4 8	4 1	4	6
63	$(3^6, 2^{57})$	6	6	5	13
127	$(3^9, 2^{118})$	9	9	6	27

Table 4.6: Current results on nearly Moore digraphs and order-relaxed Moore digraphs of out-degree 2 an diameter $D, 2 \leq D \leq 6$.

4.8 Relaxing the out-degree of digraphs of order n, when $l_{d,D} < n \leq u_{d,D}$

There are usually large gaps between the best current lower and upper bounds on the order $n_{d,D}$. Let $l_{d,D}$ represent the current best lower bound for $n_{d,D}$, and let $u_{d,D}$ represent the current best upper bound for $n_{d,D}$. We are interested in improving the lower bounds to decrease these gaps by obtaining digraphs of order equal to values between $l_{d,D}$ and $u_{d,D}$, given diameter and maximum out-degree. However, we have not been able to obtain such digraphs and so we begin by obtaining large digraphs, of orders equal to values between $l_{d,D}$ and $u_{d,D}$, having a given diameter D. Obtaining these large digraphs is expected to produce structures which may be useful in the construction of new digraphs in order to improve the lower bounds for the maximum order of digraphs.

We present in Tables 4.7, 4.8 and 4.9, the outstanding potential values of orders larger than those obtained so far, for diameter D up to 10, and for maximum out-degree $d = 2, 3$ and 4. The 'Largest Known Order' column gives the order of the current largest known digraph of the given maximum out-degree d and diameter D.

The possible larger orders, between the current known lower bounds and current best upper bounds, are listed under the heading 'Possible Larger Values of Order'.

D	Largest Known Order	Feasible Larger Values of Order
2	6	–
3	12	–
4	25	26 - 28
5	50	51 - 60
6	100	101 - 124
7	200	201 - 252
8	400	401 - 508
9	800	801 - 1,020
10	1,600	1,601 - 2,044

Table 4.7: Feasible values of $n_{2,D}$ for $2 \leq D \leq 10$.

Definition 4.8.1 A digraph G is an *out-degree-relaxed digraph*, denoted by $(n, \mathcal{D}^+, D)$-digraph, if G has n vertices, $l_{d,D} < n \leq u_{d,D}$, given diameter D, and $\mathcal{D}^+ = ((d+x)^{t_x}, \ldots, (d+2)^{t_2}, (d+1)^{t_1}, d^{t_0})$.

Currently, we are dealing with special out-degree sequences $\mathcal{D}^+$ in the $(n, \mathcal{D}^+, D)$-digraph. Let G have β vertices of out-degree larger than d, $\beta = (t_1 + \ldots + (t_x)$, minimum out-degree d and maximum out-degree $d+x$, $0 < x \leq n-d-1$. The out-degree sequence $\mathcal{D}^+$ can then be described as $(d+x)^{t_x}, \ldots, (d+2)^{t_2}, (d+1)^{t_1}, d^{n-\beta}$. The number of extra arcs in a $(n, \mathcal{D}^+, D)$-digraph is $\alpha = t_1 + 2t_2 + \ldots + xt_x$.

Problem 4.8.1 *What is the minimum value of α if G is a $(n, \mathcal{D}^+, D)$-digraph, with $l_{d,D} < n \leq u_{d,D}$, and the out-degree sequence $\mathcal{D}^+ = (d+\alpha, d^{n-1})$?*

Problem 4.8.2 *What is the minimum number of vertices, denoted by β, if G is a $(n, \mathcal{D}^+, D)$-digraph, with $l_{d,D} < n \leq u_{d,D}$, and the out-degree sequence $\mathcal{D}^+ = ((d+1)^{\beta}, d^{n-\beta})$?*

D	Largest Known Order	Feasible Larger Values of Order
2	12	–
3	36	37 - 38
4	108	109 - 119
5	324	325 - 362
6	972	973 - 1,091
7	2,961	2,962 - 3,278
8	8,748	8,749 - 9,839
9	26,244	26,245 - 29,522
10	78,732	78,733 - 88,571

Table 4.8: Feasible values of $n_{3,D}$ for $2 \leq D \leq 10$.

D	Largest Known Order	Feasible Larger Values of Order
2	20	–
3	80	81 - 84
4	320	321 - 340
5	1,280	1,281 - 1,364
6	5,120	5,121 - 5,460
7	20,480	20,481 - 21,844
8	81,920	81,921 - 87,380
9	327,680	327,681 - 349,524
10	1,310,720	1,310,721 - 1,398,100

Table 4.9: Feasible values of $n_{4,D}$ for $2 \leq D \leq 10$.

Using HSAGA, we have obtained some interesting out-degree-relaxed digraphs, given diameter $D = 4$. For example, we obtained an out-degree-relaxed $(26, (5, 2^{25}), 4)$-digraph. Another outcome found by HSAGA is out-degree-relaxed $(26, (3^2, 2^{24}), 4)$-digraphs (see Figures 4.15 - 4.18).

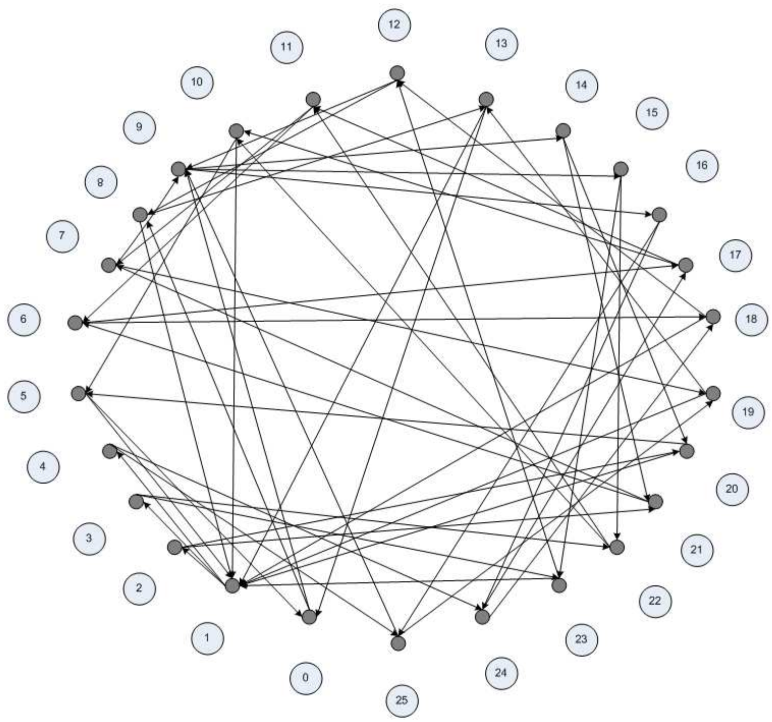

Figure 4.15: $G \in G(26, (3^2, 2^{24}), 4)$.

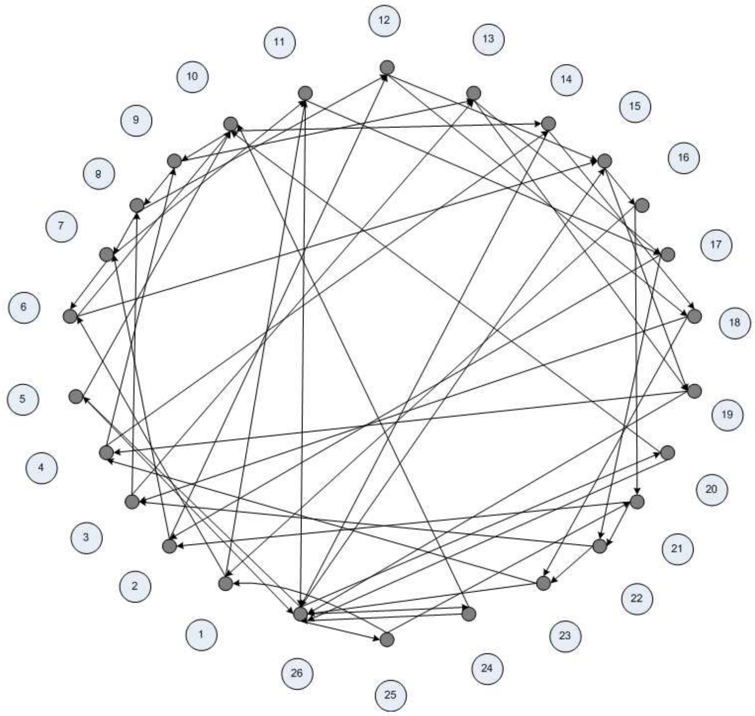

Figure 4.16: $G \in G(26, (5, 2^{25}), 4)$.

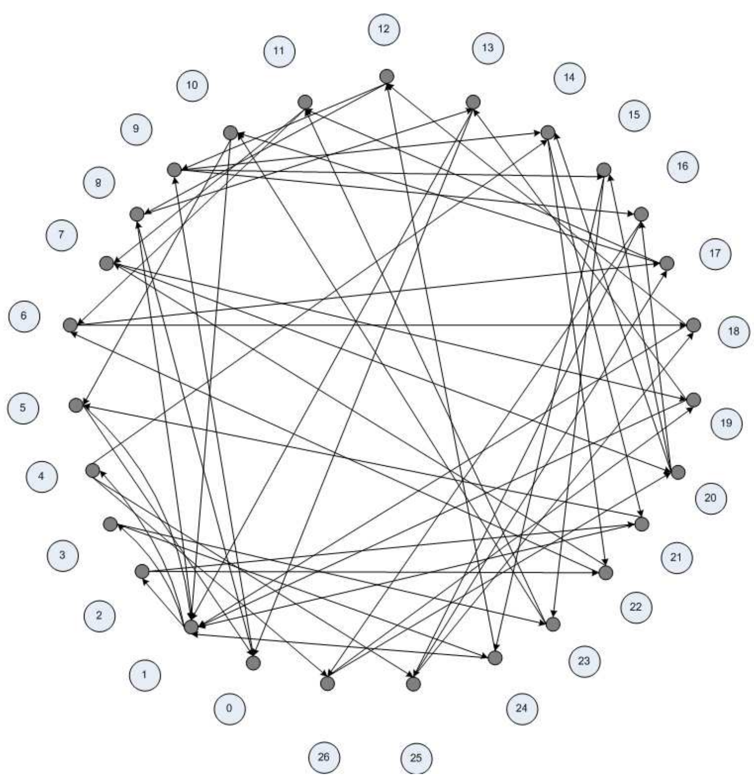

Figure 4.17: $G \in G(27, (3^3, 2^{24}), 4)$.

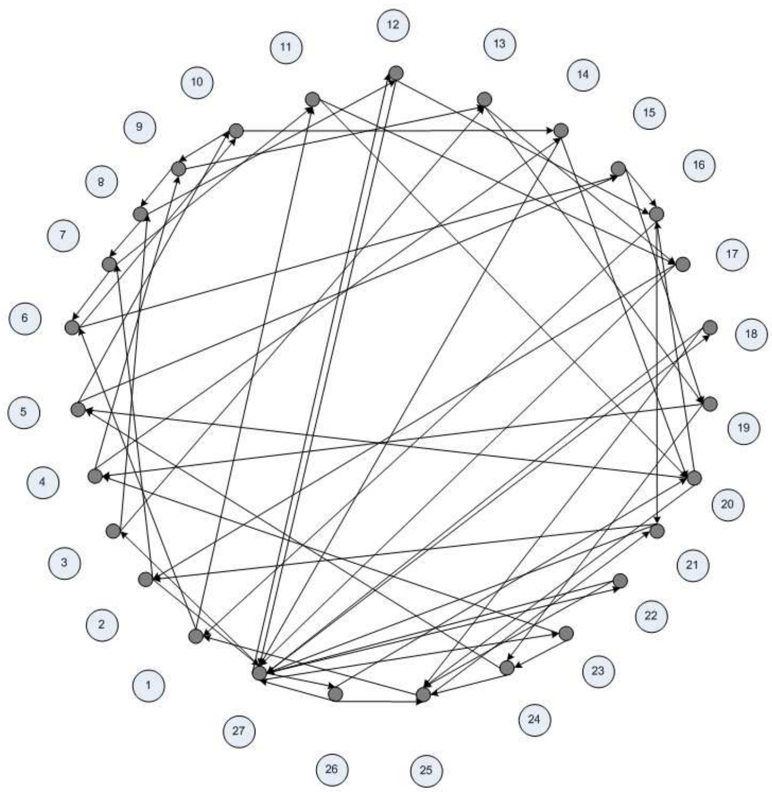

Figure 4.18: $G \in G(27, (6, 2^{26}), 4)$.

To summarize, we list the current minimum number of extra arcs α, current minimum β, and the corresponding out-degree sequence $\mathcal{D}^+$, in Table 4.10.

Out-degree-Relaxed			
n	$\mathcal{D}^+$	α	β
26	$(3^2, 2^{24})$	2	2
	$(5, 2^{25})$	3	1
27	$(3^3, 2^{24})$	3	3
	$(6, 2^{26})$	4	1
28	$(3^3, 2^{25})$	3	3
	$(8, 2^{27})$	6	1

Table 4.10: The current minimum number of extra arcs α, current minimum β, and the out-degree sequence $\mathcal{D}^+$, where $d = 2$, $D = 4$ and n, such that $l_{2,4} < n \leq u_{2,4}$.

Chapter 5

Construction of Cages

5.1 Introduction

Since we know little about the structure or even the number of vertices in a cage, it is very interesting to study the existence of graphs of given girth and regular degree, of order as close as possible to the unknown order of cages. Although so far research in directed cages is not very popular compared to the interest in undirected cages, we still believe it is worth mentioning the current knowledge of directed cages, including two conjectures in the last section of this chapter.

5.2 Lower and upper bounds on the order of cages

Let $n(d;g)$ be the number of vertices in a cage. In general, computing the value of $n(d;g)$ is very difficult. A lower bound of $n(d;g)$ when $d \geq 3$, denoted by $n_l(d;g)$, was given by Tutte [86] and Bollobás [19] as follows.

$$n_l(d;g) = \begin{cases} (d(d-1)^{(g-1)/2} - 2)/(d-2) & \text{for } g \text{ odd} \\ (2(d-1)^{g/2} - 2)/(d-2) & \text{for } g \text{ even} \end{cases} \tag{5.1}$$

Note that in the context of cages, some researchers call the lower bounds of cages: Moore bounds for both odd girth g and even girth g, although Moore bounds correspond only to odd girths.

Any cage that actually meets these lower bounds is very special - it is a Moore graph if g is odd and a 'generalized polygon' if g is even.

The definition of a *generalized polygon or (x-gon)*:
Let P (the set of vertices) and B (the set of edges) be disjoint non-empty sets, and let I (the vertex-edge incidence relation) be a subset of $P \times B$. Let $\mathcal{I} = (P, B, I)$, and let $G(\mathcal{I})$ be the associated bipartite graph on $P \cup B$ with edges joining the vertices from P to their incident edges in B ($p \in P$ is adjacent to $b \in B$ whenever $(p, b) \in I$). The ordered triple (P, B, I) is said to be a generalized polygon subject to the following four regularity conditions:

- There exist $s \geq 1$ and $h \geq 1$ such that every edge is incident to exactly $s + 1$ vertices and every vertex is incident to exactly $h + 1$ edges.
- Any two distinct edges intersect in at most one vertex and there is at most one edge through any two distinct vertices.
- The diameter of the incidence graph $G(\mathcal{I})$ is x.
- The girth of $G(\mathcal{I})$ is $2x$.

For example, a 2-gon is a complete bipartite graph $K_{s+1,h+1}$. Figure 5.1 shows a complete bipartite graph $G \in K_{2,3}$, the diameter of G is 2 and the girth of G is 4.

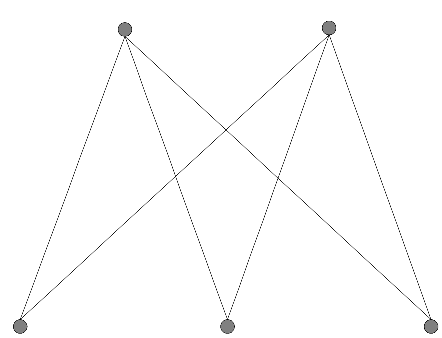

Figure 5.1: 2-gon when $s = 1$ and $h = 2$.

The existence of cages was proved by Erdős and Sachs [32] and they obtained the first upper bound for $n(d; g)$.

So far, the best current upper bound, denoted by $n_u(d;g)$, was given by Sauer [74].

$$n_u(3;g) = \begin{cases} 4/3 + (29(2^{(g-2)})/12) & \text{for } g \text{ odd} \\ 2/3 + (29(2^{(g-2)})/12) & \text{for } g \text{ even} \end{cases} \tag{5.2}$$

$$n_u(d;g) = \begin{cases} 2(d-1)^{(g-2)} & \text{for } g \text{ odd and } d \geq 4 \\ 4(d-1)^{(g-3)} & \text{for } g \text{ even and } d \geq 4 \end{cases} \tag{5.3}$$

5.3 Graphs of order close to cages

We list some family of known cages with small values of d and g.

- For $d = 2$, the $(2;g)$-cage is the g-cycle.
- For $g = 2$, the $(d;2)$-cage has just two vertices and they are joined by exactly d edges.
- For $g = 3$, the $(d;3)$-cage is the complete graph K_{d+1}.
- For $g = 4$, the $(d;4)$-cage is the complete bipartite graph $K_{d,d}$.

From now on, we focus on $(d;g)$-cages with $d \geq 3$ and $g \geq 5$. In this chapter, we present two tables of the current best upper bounds on $n(d;g)$.
The following two tables, which are kept by Royle [73], list the current knowledge of cages or smallest known graphs close to either the Moore graphs whose order are the lower bounds on the order of cages with odd girth, or generalized polygons whose order are the lower bounds on the order of cages with even girth.

Table 5.1, lists the current best values for the order of graphs close to cubic cages. For certain small values of g, the cages themselves are all known, and are written with bold font. For larger values of g, a range is given: the lower value, denoted by "Current lower bound", is either the Moore bound $n_l(3;g)$ or a bound which has been laboriously increased by extensive computation. The higher value, denoted by 'Current upper bound', is simply the order of the smallest known cubic graph of that girth, which is given explicitly. Under the column labeled ‘Number’, we list the number of known non-isomorphic graphs known that meet the upper bound.

In some cases, there exists unique cages for given degree d and girth g. For example, the Number of $(3;5)$-cages is 1, and the unique $(3;5)$-cage is the Petersen graph with 10 vertices. In other cases, there exist multiple cages for a given combination of d and g. For instance, there are three non-isomorphic $(3;10)$-cages, each with 70 vertices. Numbers that are not known to be exact are followed by the + symbol (so 1+ means that one example is known, but there may be more). In this table, the known cubic cages start from $g = 4$ and go up to $g = 12$. For instance, for the $(3;11)$-cage, the lower bound was 94, but this lower bound has been increased by McKay et al. [62] who, using standard backtracking technique, demonstrated that a $(3;11)$-cage must have exactly 112 vertices. They showed in 2003 [73] that the $(3;11)$-cage is unique. The fact that an 11-cage has order 112 was known some time earlier by Balaban [4]. For $g > 14$ of cubic cages, the smallest known graphs of regular degree and fixed girth, whose order is as close as currently possible to the corresponding cubic cages, are shown in Table 5.1. For example, for $(3;13)$-cages, computations by McKay et al. [62] have lifted the lower bound of the order from 190 to 202. The best current upper bound on the order of $(3;13)$-cages is 272, found by Hoa, and described by Biggs [15].

The Table 5.2 summarizes the best known upper bounds for degree $d \leq 14$ and girth g up to 12, and some of them are the smallest values of $n(d;g)$. This table can be found on the website “http://people.csse.uwa.edu.au/gordon /cages/allcages.html”, which is kept by Royle [73]. Most of the values in this table are exact, either meeting the Moore bound or calculated by extensive computations. These exact values are represented by bold font, for example, the unique $(4;5)$-cage, with 19 vertices, called the *Robertson graph*. The values displayed in italic font represent the order of the current smallest graph, which has regular degree and fixed girth and order close to the corresponding Moore bound. For instance, the lower bound on the order of a $(4;7)$-cage is 53, the first $(4;7)$-graph on 70 vertices is due to McKay and Yuanshen [73]. In 2007, Exoo et al. improved on this by producing a $(4;7)$-graph of order

(*d;g*)	**Current upper bound**	**Current lower bound**	**Number**
(3;3)	**4**	4	1
(3;4)	**6**	6	1
(3;5)	**10**	10	1
(3;6)	**14**	14	1
(3;7)	**24**	24	1
(3;8)	**30**	30	1
(3;9)	**58**	58	18
(3;10)	**70**	70	3
(3;11)	**112**	112	1
(3;12)	**126**	126	1
(3;13)	272	202	1+
(3;14)	384	258	1+
(3;15)	620	382	1+
(3;16)	960	510	1+
(3;17)	2 176	766	1+
(3;18)	2 640	1 022	1+
(3;19)	4 324	1 534	1+
(3;20)	6 048	2 046	1+
(3;21)	16 028	3 070	1+
(3;22)	16 206	4 094	1+
(3;23)	49 482	6 142	1+
(3;24)	49 608	8 190	1+
(3;25)	109 010	12 286	1+
(3;26)	109 200	12 286	1+
(3;27)	285 852	24 574	1+
(3;28)	415 104	32 766	1+
(3;29)	1 143 408	49 150	1+
(3;30)	1 227 666	65 534	1+
(3;31)	3 649 794	98 302	1+
(3;32)	3 650 304	131 070	1+

Table 5.1: Cubic cages of small girth.

$d \backslash g$	5	6	7	8	9	10	11	12
3	**10**	**14**	**24**	**30**	**58**	**70**	**112**	**126**
4	**19**	**26**	*67*	**80**	*275*	*384*		**728**
5	**30**	**42**	*152*	**170**				**2 730**
6	**40**	**62**	*294*	**312**				**7 812**
7	**50**	**90**						
8	*80*	**114**		**800**				**39 216**
9	*98*	**146**		**1 170**				**74 898**
10	*126*	**182**		**1 640**				**132 860**
11	*160*	*240*						
12	*203*	**266**		**2928**				**354 312**
13	*240*							
14	*312*	**366**		**4760**				**804 468**

Table 5.2: Current best upper bounds on the order of cages.

67 [73]. A blank cell in Table 5.2 means that any known $(d; g)$-graphs for given degree d and girth g have order that is far from the lower bounds on the order of a $(d; g)$-cage.

5.4 Directed cages

In Section 5.3, we have discussed the problem of finding undirected cage graphs. The situation with regard to directed cage graphs is quite different. For simplicity, the digraphs we consider here contain no circuits of length less than three (directed or otherwise) [20]. We say that a d-regular digraph G has *directed girth* g if each vertex has in-degree and out-degree d and if the shortest directed cycle is of length g. A *directed* $(d;g)$*-cage* is thus a smallest d-regular digraph of girth g.

In the undirected case, there are many results on cage graphs. See Wang's survey [88], and Royle's web site [73]. However, there are not many results on directed cage graphs. An upper bound on the number of vertices of directed cages, denoted by $n_u(d;g)$, has been known from Behzad et al. [14].

Theorem 5.4.1 *[14] For each $d \geq 1$ and $g \geq 2$, the number $n_u(d;g)$ exists, and $n_u(d;g) \leq d(g-1)+1$.*

They also provided a lower bound on the number of vertices of directed cages, called $n_l(d;g)$, for girth $g = 4$.

Theorem 5.4.2 *[14] For $d > 1$, $n_l(d,4) \geq (5d+4)/2$.*

Moreover, they conjectured that $n_u(d;g) = d(g-1)+1$ for directed cages. An equivalent formulation of their conjecture is as follows:

Conjecture 5.4.1 *[14] Let G be a d-regular digraph on n vertices. Then the directed girth of G is at most $\lceil n/d \rceil$.*

Caccetta and Häggkvist [24] proposed a generalization of the above conjecture.

Conjecture 5.4.2 *[24] Let G be a digraph on n vertices, each vertex with out-degree at least d. Then the directed girth g of G is at most $\lceil n/d \rceil$.*

This last conjecture has been verified in several particular cases, namely, for

- $d = 1$ (trivial);
- $d = 2$, by Caccetta and Häggkvist [24];
- $d = 3$, by Hamidoune [44];
- $d = 4, 5$, by Hoàng and Reed [47];

- $d \leq \sqrt{n/2}$, by Shen [75].

In the next chapter, we consider graphs that are related to cages, called EX graphs.

Chapter 6

Construction of EX Graphs

6.1 Introduction

Let us recall the definitions given in Section 3.3.1. In particular, let $ex(n;t) = ex(n;\{C_3, C_4, \ldots, C_t\})$ denote the maximum number of edges in a graph of order n and girth $g \geq t+1$. By $EX(n;t) = EX(n;\{C_3, C_4, C_5, \ldots, C_t\})$ we denote extremal graphs of order n, girth at least $t+1$, having the number of edges equal to $ex(n;t)$. The notation $ex_l(n;t)$ means the best current lower bound on the size of extremal graphs, and $ex_u(n;t)$ represents the best current upper bound on the size of extremal graphs.

In this chapter, we construct some large graphs of given order n and girth at least $t+1$, and with size larger than the previous best lower bounds of $ex(n;t)$. Additionally, we prove that the extremal number $ex(29;6) = 45$, and we improve upper bounds of $ex(n;6)$ and $ex(n;7)$, when $n \leq 40$.

6.2 Constructing graphs close to EX graphs

In general, constructing EX graphs has been known to be hard and has turned out to be useful in different problems in extremal graphs theory, for instance, in studies of graphs with a high degree of symmetry [22]; and in the design of communication networks [26].

The exact value of $ex(n;t)$ is known only for small values of t or n, for example, $ex(n;3) = \lfloor n^2/4 \rfloor$, and the corresponding graphs are $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Even for $t = 4$, the conjecture by Erdős [30] stating that $ex(n;4) = (1/2 + o(1))^{3/2} n^{3/2}$ is still open. Recently, researchers have obtained some constructive lower bounds of $ex(n;4)$; these lower bounds resulted in improvements to the previous lower bounds for $ex(n;4)$. In [56] Garnic et al. developed algorithms which combined hill-climbing and backtracking techniques to generate graphs with order up to 201 and $t = 4$. The size of these graphs gives support to Erdős' conjecture. Wang et al. [**?**] used simulated annealing to generate graphs for several values of n and t. For $t \geq 5$, most values of $ex(n;t)$ are unknown, except in 2007, Abajo and Diánez [1] proved a few exact values of $ex(n;t)$ when $t \geq 5$.

Theorem 6.2.1 *[1] Given integers n and k such that $n \geq 4$ and $k \geq 0$, we define $n_k(t) = min\{n; ex(n;t) - n = k\}$. Then the following equalities hold.*

(i) $n_0(t) = t + 1.$

(ii) $n_1(t) = \lfloor 3t/2 \rfloor + 1.$

(iii) $n_2(t) = 2t.$

(iv) $n_3(t) = \begin{cases} \lceil 9t/4 \rceil & \textit{if } t \textit{ is even.} \\ \lfloor 9t/4 \rfloor & \textit{if } t \textit{ is odd.} \end{cases}$

(v) $n_4(t) = \begin{cases} \lceil (8t-2)/3 \rceil & \textit{if } t \textit{ is even when } t \neq 4. \\ \lfloor (8t-2)/3 \rfloor & \textit{if } t \textit{ is odd.} \end{cases}$

(vi) $n_6(4) = 12, n_6(5) = 14, n_6(6) = 19, n_6(7) = 21.$

The software HSAGA described in Appendix A.1 was first developed for constructing nearly Moore digraphs. However, with some modifications, we are able to make use of HSAGA in the construction of EX graphs. The basic processes of HSAGA and the details of each process are described in Appendix A.1. Each result provided by HSAGA consists of three parts, namely, the maximum number of edges, the adjacency list and the degree sequence; see the output in Table 6.1.

Adjacency List			
$\{0,1\}$	$\{0,10\}$	$\{0,23\}$	
$\{1,0\}$	$\{1,2\}$	$\{1,29\}$	
$\{2,1\}$	$\{2,3\}$	$\{2,8\}$	$\{2,16\}$
$\{3,2\}$	$\{3,11\}$	$\{3,13\}$	
$\{4,12\}$	$\{4,20\}$	$\{4,21\}$	
$\{5,9\}$	$\{5,24\}$	$\{5,28\}$	
$\{6,9\}$	$\{6,10\}$	$\{6,18\}$	$\{6,27\}$
$\{7,15\}$	$\{7,22\}$	$\{7,23\}$	
$\{8,2\}$	$\{8,9\}$	$\{8,14\}$	
$\{9,5\}$	$\{9,6\}$	$\{9,8\}$	
$\{10,0\}$	$\{10,6\}$	$\{10,19\}$	
$\{11,3\}$	$\{11,24\}$	$\{11,26\}$	
$\{12,4\}$	$\{12,14\}$	$\{12,23\}$	
$\{13,3\}$	$\{13,15\}$	$\{13,18\}$	
$\{14,8\}$	$\{14,12\}$	$\{14,17\}$	
$\{15,7\}$	$\{15,13\}$	$\{15,25\}$	
$\{16,2\}$	$\{16,21\}$	$\{16,22\}$	
$\{17,4\}$	$\{17,25\}$	$\{17,26\}$	
$\{18,6\}$	$\{18,13\}$	$\{18,20\}$	
$\{19,10\}$	$\{19,21\}$	$\{19,25\}$	
$\{20,4\}$	$\{20,18\}$	$\{20,29\}$	
$\{21,4\}$	$\{21,16\}$	$\{21,19\}$	
$\{22,7\}$	$\{22,16\}$	$\{22,27\}$	
$\{23,0\}$	$\{23,7\}$	$\{23,12\}$	$\{23,24\}$
$\{24,5\}$	$\{24,11\}$	$\{24,23\}$	
$\{25,15\}$	$\{25,17\}$	$\{25,19\}$	$\{25,28\}$
$\{26,11\}$	$\{26,17\}$	$\{26,27\}$	
$\{27,6\}$	$\{27,22\}$	$\{27,26\}$	
$\{28,5\}$	$\{28,25\}$	$\{28,29\}$	
$\{29,1\}$	$\{29,20\}$	$\{29,28\}$	

Table 6.1: A graph on 30 vertices with girth 7 and 47 edges. This graph has the degree sequence $\mathcal{D} = \{4^4, 3^{26}\}$.

Using HSAGA, we produced a graph on 30 vertices, degree sequence $\mathcal{D} = \{4^4, 3^{26}\}$, size $e = 47$, having girth 7 (see Figure 6.1).

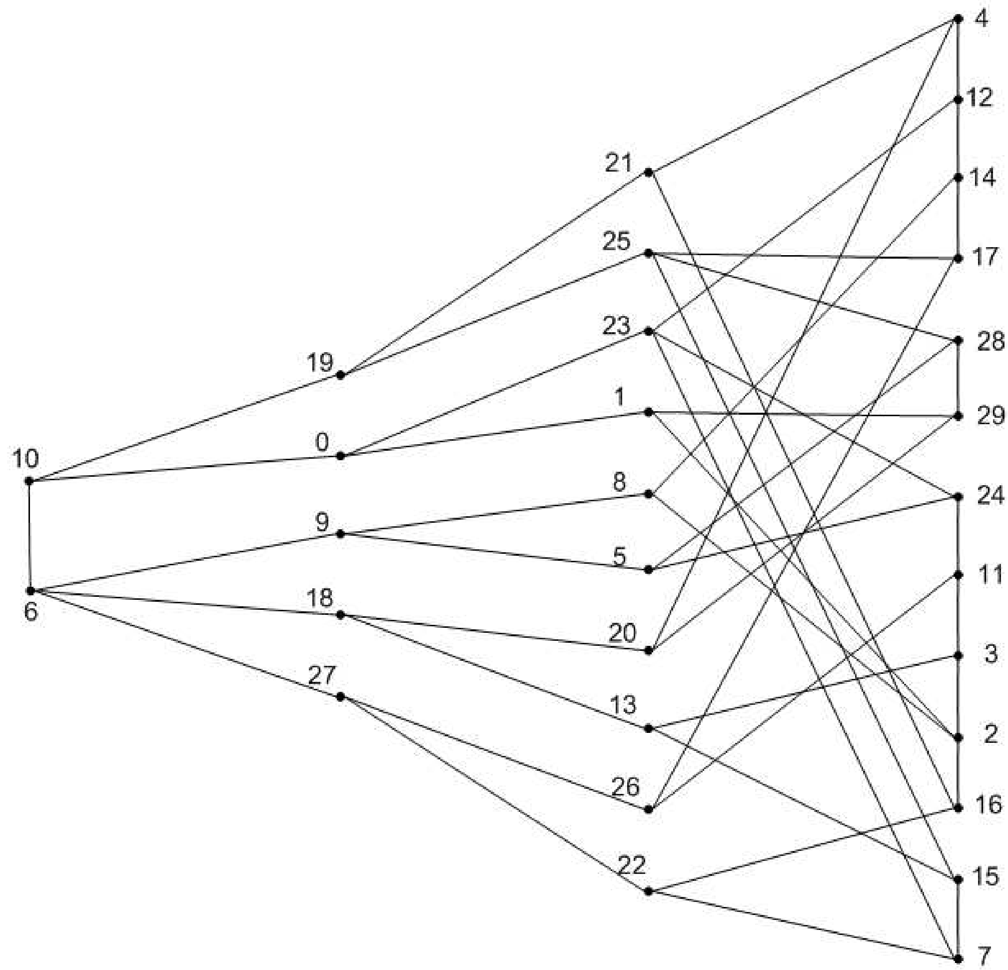

Figure 6.1: Graph with $n = 30$, $g = 7$ and $e = 47$.

Then a question asked in [8] as to whether the $(3;8)$-cage does or does not belong to EX$(30;6)$ is answered as stated in the next observation.

* **Observation 6.2.1** *$ex(30;6) \geq 47$ and so the $(3;8)$-cage does not belong to $EX(30;6)$.*

From Theorem 3.3.4 (i) and Observation 6.2.1 the following corollary is immediate.

* **Observation 6.2.2** *For $n \geq 12$, $n \notin \{15, 80, 170\}$, the girth of $G \in EX(n;6)$ is 7, and there exists a graph in $EX(15;6)$ with 18 edges and girth 8.*

Some of the obtained extremal graphs are cages. For instance, in Table 4.5, if $n = 14$, the corresponding graph is the $(3; 6)$-cage, and if $n = 26$, the corresponding graph is the $(4; 6)$-cage. Furthermore, in Table 6.3, if $n = 24$, the corresponding graph is the $(3; 7)$-cage, and in Table 6.4, if $n = 30$, the corresponding graph is the $(3; 8)$-cage. This shows that the computed lower bounds by using HSAGA are reasonable.

After we produced new results for extremal graphs with small girth, we also ran the program for large girths. For example, it is known that for $m \geq 12$, $ex(t+2; t)$=$2t+4$, and there exists $G \in EX(n; t)$ with $g(G) = t + 2$ (see Theorem 3.3.3(iii)). Wang et al. [87] used pure simulated annealling to generate an optimal solution in the above case for $n \geq 15$ and $g(G) \geq t + 1$. By using HSAGA, we also generated optimal solutions for $t \geq 12$, $n = t + 2$ and $g(G) = t + 2$.

6.3 Upper bounds on the size of EX graphs

Since the maximum degree is at least as big as the average degree, we obtain

$$\Delta(G) \geq \lceil \bar{d} \rceil = \lceil 2e(G)/n(G) \rceil. \tag{6.1}$$

We have constructed a graph of order $n = 29$, girth $g = 7$, and size 45, giving $ex_l(29; 6) = 45$ (see Table 6.3). Next we will prove that $ex(29; 6) = 45$.

* **Theorem 6.3.1** *Let $G \in EX(29; 6)$. Then $e(G) = ex(29; 6) = 45$.*

Proof First we show that $ex_u(29; 6) \leq 46$. Assume that a graph $G \in EX(29; 6)$ has 47 edges. Notice that $\delta(G) \geq 4$, otherwise, if there is a vertex x of degree 3, then $G - \{x\}$ would have 44 edges, contradicting $ex(28; 6) = 43$ [82]. Then the number of edges in G would be at least $(n\delta)/2 = 58$, which is a contradiction to our assumption.

Next we show that $ex(29; 6) \neq 46$.

Assume that a graph $G \in EX(29; 6)$ has 46 edges. In this case, $\delta = \delta(G) \geq 3$, otherwise, if there were a vertex x of degree 2, then $G - x$ would have 44 edges, contradicting $ex(28;6) = 43$ [82]. If x has degree 1, then $G - x$ has 45 edges, again a contradiction to $ex(28;6) = 43$ [82]. Also $\delta \leq 3$, otherwise, if $\delta \geq 4$ then G would have at least $(n\delta)/2 = 58$ edges. Therefore, $\delta = 3$.

Furthermore, $\Delta = \Delta(G) \geq 4$ by (6.1). If $\Delta \geq 5$, let x be a vertex with $d(x) \geq 5$. Since there is no cycle of length less than 7 in G, within at most 3 steps, x must reach at least $n(G) \geq 1 + |N(x)| + |N_2(x)| + |N_3(x)| \geq 1 + 5 + 10 + 20 = 36$ distinct vertices, a contradiction with $n = 29$. Hence, $\Delta = 4$.

Therefore, $\delta = 3$, $\Delta = 4$, and the graph G contains 24 vertices of degree 3 and 5 vertices of degree 4. These vertices of degree 4 are called *special vertices*.

Now we provide a claim which will be useful in finishing the proof of the theorem.

* **Claim** 1 *If a graph* $G \in EX(29;6)$ *and* $e(G) = 46$*, then the eccentricity of any special vertex must be* 3.

The justification of Claim 1 is that if a spanning tree of graph G starts from any special vertex, say y, then $n(G) = 1 + |N(y)| + |N_2(y)| + |N_3(y)| = 29$, that is, all the vertices of G are within distance 3 from each special vertex.

In particular

* **Claim** 2 *If a graph* $G \in EX(29;6)$ *and* $e(G) = 46$*, then every vertex of degree* 3 *is within* 3 *steps from any special vertex.*

In the following figures, a vertex of degree 3 is represented by a triangle $\triangle$, a vertex of degree 4 is denoted by a cycle $\bigcirc$, and a vertex of degree either 3 or 4 is shown as a square box.

All the five special vertices have to appear in a spanning tree T because of Claim 1.

Consider a vertex x of degree 3. Let $N(x) = \{x_1, x_2, x_3\}$. All the five special vertices cannot appear at distance 3 from x. Assume otherwise (see Figure 6.2). Consider the tree of depth 3 from vertex x_3. In view of Claim 1, we can place at most 4 special vertices at distance at most 3 from x_3 without creating a cycle of length less than 7 (see Figure 6.3).

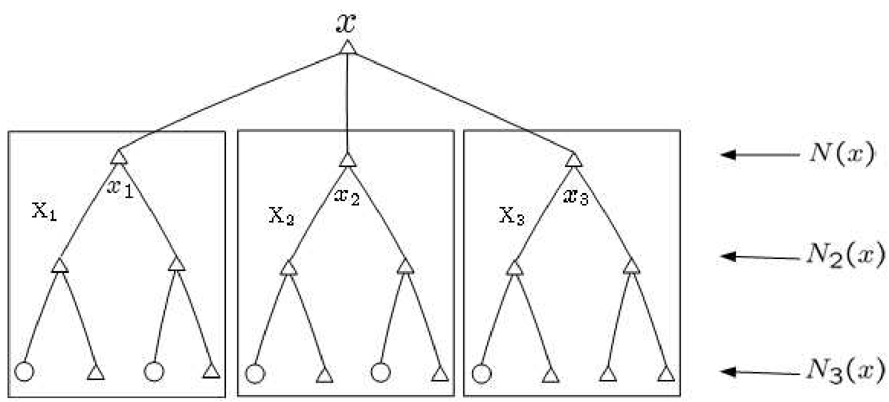

Figure 6.2: Tree T with all special vertices at $N_3(x)$.

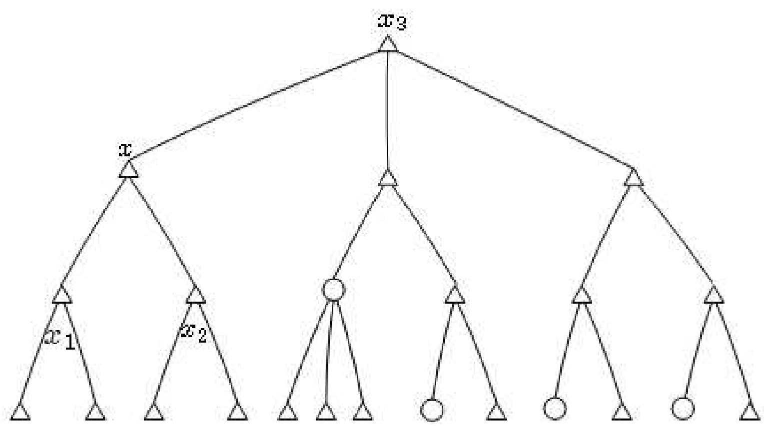

Figure 6.3: Tree started from x_3 with at most 4 special vertices.

Therefore, for each vertex x of degree 3, there must be a special vertex in $N(x)$ or $N_2(x)$. Since it is not possible for all 24 vertices of degree 3 to be adjacent to a special vertex, to prove the theorem it suffices to show that it is not possible to have a vertex of degree 3 at distance 2 from a special vertex.

Let x be a vertex of degree 3 which is not adjacent to a special vertex in the spanning tree T. Assume that there is a special vertex in $N_2(x)$ (see Figure 6.4). It is easy to see that each subtree X_i in $T - x$(with root vertex x_i) of T contains at most two special vertices because of Claim 1. Hence, there exists one subtree of T containing only one special vertex because the tree T has three subtrees and there are only five special vertices in total. Consider one special vertex in $N_2(x)$, denoted by p, belonging to the subtree X_3, which only has one special vertex p. In

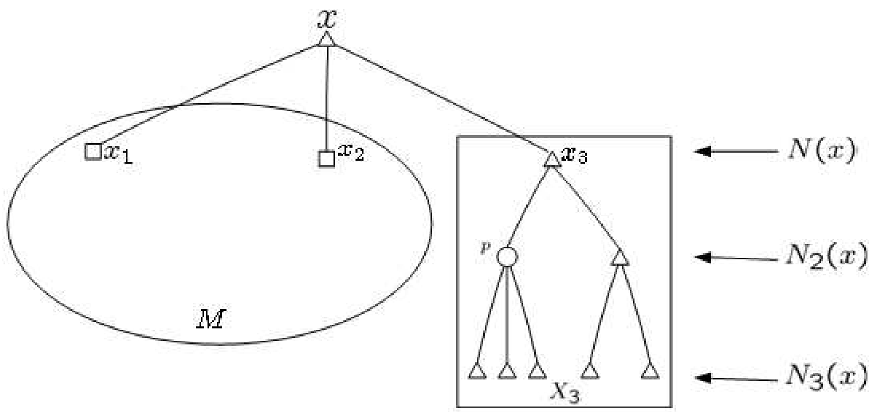

Figure 6.4: Tree T when there exists a special vertex p in $N_2(x)$.

the tree starting from vertex x_3, there exist two special vertices in the vertex set $M = (N(x) \cup N_2(x)) \cap (X_1 \cup X_2)$, by Claim 1 and Claim 2. This implies two cases should be considered.

Case 1. There is one special vertex in $N(x) \cap X_1$, and another special vertex is located at $N_2(x) \cap X_2$ (see Figure 6.5). This implies the subtree X_2 must have three special vertices, which is a contradiction, by Claim 1.

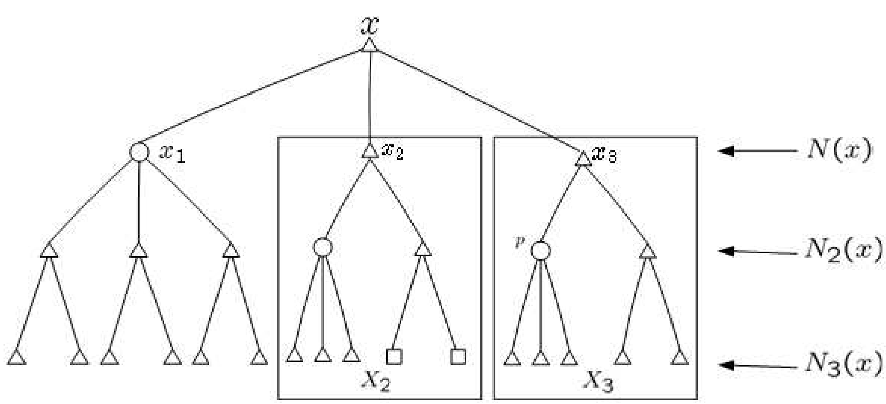

Figure 6.5: Case 1

Case 2. There is one special vertex in $N_2(x) \cap X_1$, denoted by m, and another special vertex in $N_2(x) \cap X_2$, represented by o. Consider a vertex z such that the degree of z is 3 and $z \in N(x_1) \setminus \{m, x\}$ (see Figure 6.6). Consider the tree starting from the

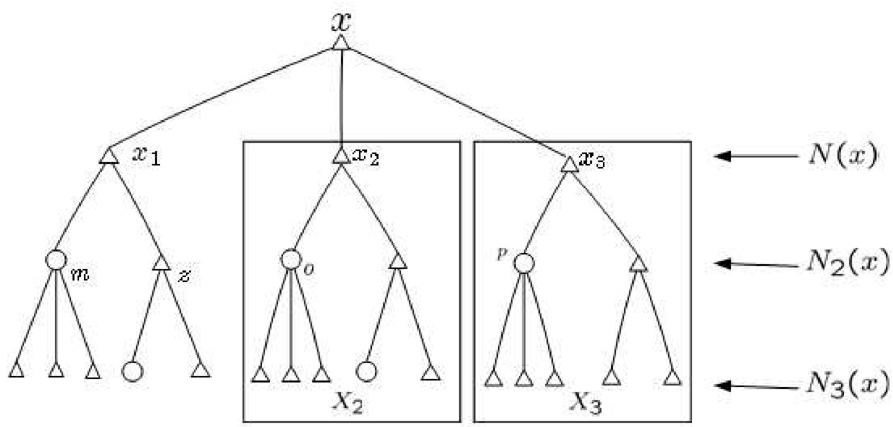

Figure 6.6: Case 2

vertex z (see Figure 6.7). The middle subtree, denoted by Z_2, must contain three special vertices, which is impossible by Claim 1.

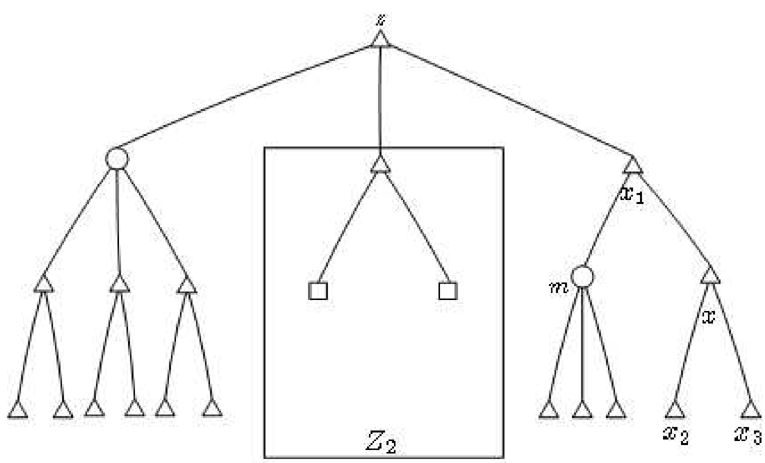

Figure 6.7: A tree starting from vertex z in Case 1b.

Therefore, $ex(29; 6) = 45$. ■

Next we improve upper bounds of $ex_u(n; t)$, for some particular values of n and t.

* **Theorem 6.3.2** *The following statements hold.*

(i) $ex_u(30; 6) = 48$*;*

(ii) $ex_u(31; 6) = 51$*;*

(iii) $ex_u(37;7) = 58$;

(iv) $ex_u(38;7) = 60$.

Proof (i) Suppose there exists a graph $G \in EX(30;6)$ with 49 edges. In this case, $\delta \geq 3$, $\Delta \leq 4$, and the number of vertices of degree 3 is 22. Further, G cannot contain a pair of adjacent vertices x and y each having degree 3, since $G - \{x, y\}$ would have 44 edges, contradicting $ex(28;6) = 43$. Hence, the set of three edges incident at each of the 22 vertices of degree 3 of G are pairwise disjoint, implying that $e(G) \geq 66$, which is a contradiction.

(ii) Let us assume that a graph $G \in EX(31;6)$ has 52 edges. Since $ex_u(30;6) = 48$, then $\delta(G) \geq 4$. Therefore, the number of edges in G is at least 62, which is a contradiction with our assumption.

(iii) Let us assume that a graph $G \in EX(37;7)$ has 59 edges. Since $ex(36;7) = 55$, then $\delta(G) \geq 4$. Therefore, the number of edges in G is at least 74, which is a contradiction with our assumption.

(iv) Assume that a graph $G \in EX(38;7)$ has 61 edges. In this case, $\delta = 3$, $\Delta = 4$. Further, G cannot contain a pair of adjacent vertices x and y each having degree 3, since $G - \{x, y\}$ would have 56 edges, contradicting $ex(36;7) = 55$. Let us draw a spanning tree of the graph G starting with an edge xy such that $d(x) = 4$ and $d(y) = 3$. We obtain that $n(G) \geq 2 + |N_1(x)| + |N_2(x)| + |N_3(x)| + |N_1(y)| + |N_2(y)| + |N_3(y)| \geq 49$, a contradiction. ■

6.4 Summary tables of EX graphs

To construct extremal graphs of given order n and no cycles of length $\leq t$, we used the *hybrid simulated annealing and genetic algorithm* (HSAGA), originally developed to create graphs of given order, diameter and maximum degree or maximum out-degree. For details of HSAGA, see Appendix A.1. We have modified the HSAGA algorithm, and then used it with different values of the parameters *cooling rate* and *crossover rate*. We present the new results for $t = 6$ we have obtained in Tables 6.2, 6.3 and 6.4.

Tables 6.2, 6.3 and 6.4 give the exact value of extremal numbers $ex(n;5)$, $ex(n;6)$ and $ex(n;7)$, for $n < 40$. In these tables, n represents the order of graphs; $ex(n;t)$ written with bold font denotes the value of the extremal number of EX graphs of given n and t. We use italics and bold font for the value of $ex(29;6)$ to denote that it is new. 'Sample $\mathcal{D}$' is a degree sequence of some corresponding EX graphs. For instance, in Table 6.2, $ex(26;5) = 52$, and its degree sequence is 4^{26}, in fact, this graph is a $(4;6)$-cage.

n	$ex(n;5)$	Sample $\mathcal{D}$	n	$ex(n;5)$	Sample $\mathcal{D}$
6	**6**	$\{2^6\}$	**24**	**45**	$\{4^{18}, 3^6\}$
7	**7**	$\{3, 2^5, 1\}$	**26**	**52**	$\{4^{26}\}$
8	**9**	$\{3^2, 2^6\}$	**25**	**48**	$\{4^{21}, 3^4\}$
9	**10**	$\{3^2, 2^7\}$	**27**	**53**	$\{5, 4^{24}, 3, 2\}$
10	**12**	$\{3^4, 2^6\}$	**28**	**56**	$\{4^{28}\}$
11	**14**	$\{3^6, 2^5\}$	**29**	**58**	$\{5^2, 4^{26}, 2\}$
12	**16**	$\{3^8, 2^4\}$	**30**	**61**	$\{5^4, 4^{24}, 3^2\}$
13	**18**	$\{3^{10}, 2^3\}$	**31**	**64**	$\{5^6, 4^{23}, 3^2\}$
14	**21**	$\{3^{14}\}$	**32**	**67**	$\{5^6, 4^{26}\}$
15	**22**	$\{3^{14}, 2\}$	**33**	**70**	$\{5^{10}, 4^{21}, 3^2\}$
16	**24**	$\{3^{16}\}$	**34**	**74**	$\{4^{22}, 3^{12}\}$
17	**26**	$\{4^3, 3^{12}, 2^2\}$	**35**	**77**	$\{5^{15}, 4^{19}, 3\}$
18	**29**	$\{4^4, 3^{14}\}$	**36**	**81**	$\{5^{18}, 4^{18}\}$
19	**31**	$\{4^6, 3^{12}, 2\}$	**37**	**84**	$\{5^{15}, 4^{21}, 3\}$
20	**34**	$\{4^8, 3^{12}\}$	**38**	**88**	$\{5^{24}, 4^{14}\}$
21	**36**	$\{4^{10}, 3^{10}, 2\}$	**39**	**92**	$\{5^{28}, 4^{11}\}$
22	**39**	$\{4^{12}, 3^{10}\}$	**40**	**96**	$\{5^{32}, 4^8\}$
23	**42**	$\{4^{15}, 3^8\}$			

Table 6.2: Optimal values of $ex(n;5)$ for $5 \leq n \leq 40$.

n	$ex(n;6)$	Sample $\mathcal{D}$	n	$ex(n;6)$	Sample $\mathcal{D}$
7	**7**	$\{2^7\}$	**19**	**25**	$\{3^{12}, 2^7\}$
8	**8**	$\{2^8\}$	**20**	**27**	$\{3^{14}, 2^6\}$
9	**9**	$\{3, 2^7, 1\}$	**21**	**29**	$\{3^{16}, 2^5\}$
10	**11**	$\{3^2, 2^8\}$	**22**	**31**	$\{3^{18}, 2^4\}$
11	**12**	$\{3^3, 2^7, 1\}$	**23**	**33**	$\{3^{20}, 2^3\}$
12	**14**	$\{3^4, 2^8\}$	**24**	**36**	$\{3^{24}\}$
13	**15**	$\{3^5, 2^7, 1\}$	**25**	**37**	$\{3^{24}, 2\}$
14	**17**	$\{3^6, 2^8\}$	**26**	**39**	$\{4^2, 3^{22}, 2^2\}$
15	**18**	$\{4^1, 3^4, 2^{10}\}$	**27**	**41**	$\{5, 4^{24}, 3, 2\}$
16	**20**	$\{3^8, 2^8\}$	**28**	**43**	$\{5^2, 4^{23}, 3^2, 2\}$
17	**22**	$\{3^{10}, 2^7\}$	**29**	***45***	$\{4^6, 3^{20}, 2^3\}$
18	**23**	$\{4, 3^8, 2^9\}$			

Table 6.3: Values of $ex(n;6)$, for $7 \leq n \leq 29$.

Tables 6.5 and 6.6 show the current lower bounds and upper bounds of $ex(n;5)$, $ex(n;6)$ and $ex(n;7)$, for $n \leq 40$. In these tables, $ex_l(n;t)$ means the lower bounds on the size of EX graphs, and $ex_u(n;t)$ represents the upper bounds on the size of EX graphs. Furthermore, any new values of lower bounds and upper bounds are written in italic font. For instance, in Table 6.5, we generated a graph of order $n = 31$ with girth $g = 7$ and size 49, which is the new lower bound of $ex(31;6)$; the degree sequence of this graph is $\mathcal{D} = \{4^5, 3^{26}\}$, and the new upper bound of $ex(31;6)$ is 51.

n	$ex(n;7)$	Sample $\mathcal{D}$	n	$ex(n;7)$	Sample $\mathcal{D}$	n	$ex(n;7)$	Sample $\mathcal{D}$
8	**8**	$\{2^8\}$	**18**	**22**	$\{3^8, 2^{10}\}$	**28**	**40**	$\{3^{24}, 2^4\}$
9	**9**	$\{2^9\}$	**19**	**24**	$\{3^{10}, 2^9\}$	**29**	**42**	$\{3^{26}, 2^3\}$
10	**10**	$\{3^2, 2^6, 1^2\}$	**20**	**25**	$\{3^{10}, 2^{10}\}$	**30**	**45**	$\{3^{30}\}$
11	**12**	$\{3^2, 2^9\}$	**21**	**27**	$\{3^{12}, 2^9\}$	**31**	**46**	$\{3^{30}, 2\}$
12	**13**	$\{3^2, 2^{10}\}$	**22**	**29**	$\{3^{14}, 2^8\}$	**32**	**47**	$\{3^{30}, 2^2\}$
13	**14**	$\{3^3, 2^9, 1\}$	**23**	**30**	$\{3^{15}, 2^7, 1\}$	**33**	**49**	$\{4^2, 3^{28}, 2^3\}$
14	**16**	$\{3^4, 2^{10}\}$	**24**	**32**	$\{3^{16}, 2^8\}$	**34**	**51**	$\{4^3, 3^{28}, 2^3\}$
15	**18**	$\{3^6, 2^9\}$	**25**	**34**	$\{3^{18}, 2^7\}$	**35**	**53**	$\{4^5, 3^{26}, 2^4\}$
16	**19**	$\{3^7, 2^8, 1\}$	**26**	**36**	$\{3^{20}, 2^6\}$	**36**	**55**	$\{5, 4^5, 3^{25}, 2^5\}$
17	**20**	$\{3^7, 2^9, 1\}$	**27**	**38**	$\{3^{22}, 2^5\}$			

Table 6.4: Current values $ex(n;7)$ with $8 \leq n \leq 36$.

n	$ex_l(n;6)$	$ex_u(n;6)$	Sample $\mathcal{D}$
30	*47*	*48*	$\{4^4, 3^{26}\}$
31	*49*	*51*	$\{4^5, 3^{26}\}$
32	*51*	54	$\{4^7, 3^{24}, 2\}$
33	*53*	56	$\{4^9, 3^{22}, 2^2\}$
34	55	58	$\{4^8, 3^{26}\}$
35	*58*	61	$\{4^{11}, 3^{24}\}$
36	59	63	$\{4^{11}, 3^{24}, 2\}$
37	*61*	65	$\{4^{24}, 3^{12}, 2\}$
38	*63*	68	$\{4^{14}, 3^{22}, 2^2\}$
39	*65*	70	$\{4^{15}, 3^{22}, 2^2\}$
40	67	73	$\{4^{14}, 3^{26}\}$

Table 6.5: Values of $ex_l(n;6)$ and $ex_u(n;6)$, for $30 \leq n \leq 40$.

n	$ex_l(n;7)$	$ex_u(n;7)$	Sample $\mathcal{D}$
37	56	58	$\{4^4, 3^{30}, 2^3\}$
38	58	*60*	$\{4^5, 3^{30}, 2^3\}$
39	60	63	$\{4^7, 3^{28}, 2^4\}$
40	62	65	$\{4^4, 3^{36}\}$

Table 6.6: Current values $ex_l(n;7)$ and $ex_u(n;7)$ for $37 \leq n \leq 40$.

Chapter 7

Connectivity of Graphs

7.1 Introduction

In Chapter 3, we have given a brief summary of known results regarding the connectivity of graphs. In this chapter, we will focus on the connectivity of two particular types of graphs: EX graphs and regular graphs. More specifically, we will first prove that every EX graph is at least δ-edge connected and then we give more new results about the connectivity of EX graphs. In the second part of this chapter, we will show a new result about supperconnectivity of regular graphs of odd girth with respect to small diameter.

7.2 Connectivity of EX graphs

In general, it is difficult to find the exact values of the size of EX graphs. We believe that structural properties of EX graphs, such as connectivity, will help us to construct EX graphs.

For a graph G of minimum degree δ to be maximally connected or maximally edge-connected, sufficient conditions have been given in terms of its diameter and its girth. Recall that the following result is contained in [34, 49].

$$\kappa = \delta \quad \text{if} \quad D \leq 2\lfloor (g-1)/2 \rfloor - 1. \tag{7.1}$$

Edge-connectivity is a coarse measure of the connectedness of a graph. A newer, more refined notion, the *restricted edge connectivity*, was proposed by Esfahanian and Hakimi [33] who denoted it by $\lambda'(G)$. For a connected graph G, the restricted edge connectivity is defined as the minimum cardinality of a set W of edges such that $G-W$ is not connected and W does not contain the set of incident edges of one vertex of the graph. Then obviously $G-W$ does not contain any isolated vertices. The restricted edge connectivity has been studied in terms of *super edge connectivity*. This is a stronger measure of connectivity than the standard edge connectivity, and was proposed by Boesch [16] and Boesch and Tindell [17]. A graph is *super edge-connected*, or *super-*λ, if every minimum edge cut consists of a set of edges incident with one vertex. See [16, 17] for more details. Clearly, $\lambda'(G) > \delta(G)$ is a sufficient and necessary condition for G to be super edge connected. The *edge-degree* of edge $uv \in E(G)$ is defined as $\{d(uv) = d(u) + d(v) - 2\}$. It was shown [33] that $\lambda'(G)$ exists and $\lambda'(G) \leq \xi(G)$, if G is not a star and its order is at least 4, where $\xi = \xi(G)$ denotes the minimum edge-degree of G. The following result was proved in [9].

$$\lambda' = \xi \quad \text{if} \quad D \leq g - 2. \tag{7.2}$$

Regarding the diameter of $EX(n;t)$, a useful theorem is given below.

Theorem 7.2.1 *[6] Let $G \in EX(n;t)$, $t \geq 3$ and $n \geq t+1$. Then the diameter of an extremal graph G is $D(G) \leq t-1$.*

Applying these results to extremal graphs, we obtain the following result.

* **Corollary 7.2.1** *Every graph $G \in EX(n;t)$ has $\lambda' = \xi$.*

Proof By Theorem 7.2.1, the diameter is $D \leq t-1 \leq g-2$, because $g \geq t+1$. Therefore, from (7.2) it follows that $\lambda' = \xi$. ■

Using Theorem 7.2.1, we next prove the following theorem.

* **Theorem 7.2.2** *Let $G \in EX(n;t)$. Then G is at least δ-vertex connected when t is even.*

Proof Let F be a smallest vertex cut, and let $G - F$ contain two components G_1 and G_2. Let us assume $|F| < \delta$. It is easy to see that $|V(G_1)| > 1$ and $|V(G_2)| > 1$.

Considering component G_1, since $G_1 \neq \emptyset$, there exists a vertex u_1 which is at distance 1 to F. We know that $|F| < \delta \leq |N(u_1)|$. This implies that there exists at least one vertex $u_2 \in N(u_1)$ which is at distance two to F. Otherwise, all neighbors of u_1 would have distance 1 to F, and since $|F| < \delta$, either two neighbors of u_1 will have distance one to the same vertex $f \in F$ or two neighbors of u_1 will be adjacent, thus form a cycle of either size three or four in the graph, contradicting the assumption.

Next we assume $t \geq 6$ and consider the neighbors of u_2. With the same reasoning, there exists a vertex u_3 which is at distance three to F. We continue the process until we find a vertex which is farthest from F. If t is even, we shall find a vertex $u_{t/2}$ which is at distance $t/2$ to F. If t is odd, then we shall find a vertex $u_{(t-1)/2}$ which is at distance $(t-1)/2$ to F. In other words, we will find a vertex $u_{\lfloor t/2 \rfloor}$ which has distance $\lfloor t/2 \rfloor$ to F. Similarly, in G_2, we find a vertex $v_{\lfloor t/2 \rfloor}$ which is at distance $\lfloor t/2 \rfloor$ to F. Thus we find in G two vertices which are $t/2 + t/2 = t$ apart for t even, in contradiction to Theorem 7.2.1.
Note that this approach does not work for t odd, since the vertices we find in G_1 and G_2 might be at distance $t - 1$ apart when t is odd. ■

Next, we will show that a graph $G \in EX(n;t)$ is at least δ-vertex connected when t is odd and and the minimum degree δ is small ($\delta < 5$).

To begin with, we will establish some useful lemmas that will be used in the proofs of Theorems 7.2, 7.2 and 7.2.

Given a subset of vertices F of a graph $G = (V, E)$, let $u \in V \setminus F$ such that $d(u, F) = d$. For every integer $s \geq d$, $F_s(u) = \{f \in F : d(u, f) \leq s\}$.

* **Lemma 7.2.1** *Let $G \in EX(n;t)$, t odd and $\delta \geq 2$. Let $F \subset V$ be a cut set with $|F| < \delta$. Let $G - F$ contain two components C and C', such that each C and C' contains some vertex u with $d(u, F) = (t-1)/2$. Then*

(i) there exist at least two vertices $u_1, u_2 \in N(u)$ such that $d(u_1, F) = d(u_2, F) = (t-1)/2$, and $F_s(u_1) \cap F_s(u_2) \neq \emptyset$ when $s = (t-1)/2$;

(ii) $F_s(u) \cap F_s(u_1) = \emptyset$ if $s \leq (t-1)/2$ and $u_1 \in N(u)$.

Proof (i) Since $|F| < \delta$ and $d(u) \geq \delta$, it is possible to have two paths from u to a common vertex f in F, namely, $u, u_1 \to f$ of length $(t+1)/2$ and $u, u_2 \to f$ of length $(t+1)/2$ when $u_1, u_2 \in N(u)$. Otherwise, these two paths must form a cycle whose length is at most t. Therefore, $d(u_1, F) = (t-1)/2$ and $d(u_2, F) = (t-1)/2$.

(ii) Assume $F_s(u) \cap F_s(u_1) \neq \emptyset$, since $F < \delta$ and $d(u) \geq \delta$, there are two paths, namely, a shortest path $u \to f$ of length $(t-1)/2$ and $uu_1, u_1 \to f$ of length $(t+1)/2$, which together form a cycle of length t. This is a contradiction. ■

Now we are ready to prove the following theorems.

* **Theorem 7.2.3** *Let $G \in EX(n; t)$. Then G is 2-vertex connected when $\delta \geq 2$ and t is odd.*

Proof Let us consider G being 1-connected when $\delta \geq 2$. Assume G contains a cut vertex f. From the proof of Theorem 7.2, it is clear that when t is odd, we find a vertex u which is at distance $(t-1)/2$ to f. Since $\delta \geq 2$ and girth g is at least $t+1$, there are two different paths from vertex u to f, namely, the shortest path $u \to f$ of length $(t-1)/2$, and the path $u, u_1 \to f$ of length at least $(t+3)/2$ when $u_1 \in N(u)$. So, the path $u_1 \to f$ has length $(t+1)/2$ which contradicts the assumption that u is a farthest vertex from f because of $D \leq t-1$. ■

* **Theorem 7.2.4** *Let $G \in EX(n; t)$. Then G is 3-vertex connected when $\delta = 3$ and t is odd.*

Proof We consider G being 2-connected, when $\delta \geq 3$. Assume G contains a cut set F, and $F = \{f_1, f_2\}$. From Theorem 7.2, it is clear that when t is odd, we have a vertex u which is at distance $(t-1)/2$ to F. By the same reasoning, we can find a vertex v which is at distance $(t-1)/2$ to F in another component of $G - F$. Assume $d(u, f_1) = d(v, f_1) = (t-1)/2$(as $D \leq t-1$). By Lemma 7.2.1(i), there exist two vertices, say, u_1 and u_2, such that $u_1, u_2 \in N(u)$, $d(u_1, F) = (t-1)/2$ and $d(u_2, F) = (t-1)/2$. Now we need to discuss two cases based on the distance between the neighborhoods of u and v.

Case 1(see Figure 7.1). Assume $f_2 \in F_{(t-1)/2}(u_2)$, then $d(v, f_2) \geq (t+1)/2$. Therefore, $d(u_2, v) \geq d(u_2, f_2) + d(v, f_2) = (t-1)/2 + (t+1)/2 = t$. This is a contradiction since $D \leq t-1$.

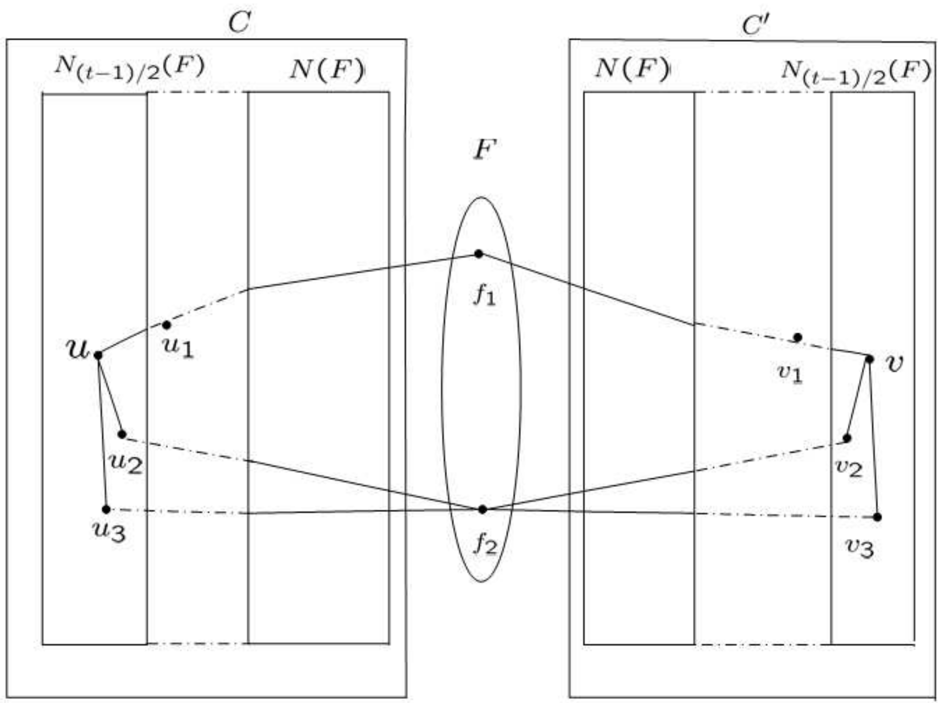

Figure 7.1: Case 1 when $\delta = 3$.

Case 2(see Figure 7.2). By Lemma 7.2.1(i), there exists a vertex, say v_2, such that $v_2 \in N(v)$ and $d(v_2, F) = (t-1)/2$. Suppose $f_2 \in F_{(t-1)/2}(v)$. Then $d(v_2, f_2) \geq (t+1)/2$. Therefore, $d(u_2, v_2) \geq d(u_2, f_2) + d(f_2, v_2) = t$. This is a contradiction since $D \leq t-1$. ■

* **Theorem 7.2.5** *Let $G \in EX(n;t)$. Then G is 4-vertex connected when $\delta = 4$ and t is odd.*

Proof Let G be 3-connected when $\delta \geq 4$. Assume G contains a cut set F, and $|F| = 3$. By Lemma 7.2.1(i), there exists a vertex, say v_3, such that $v_3 \in N(v)$ and

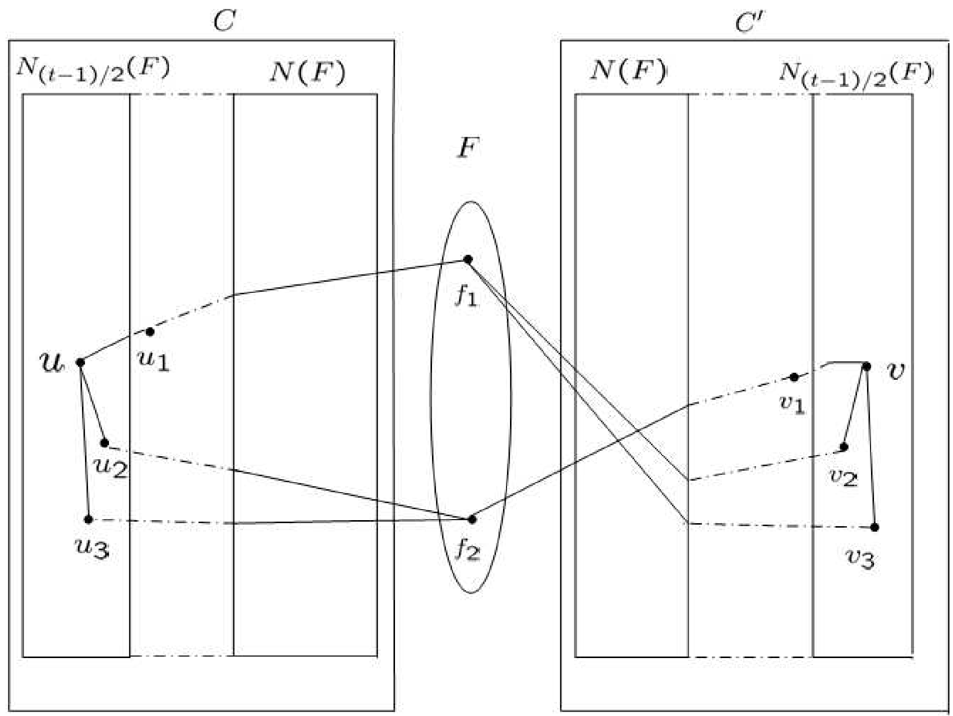

Figure 7.2: Case 2 when $\delta = 3$.

$d(v_3, F) = (t-1)/2$. Since $\delta = 4$ and there are at least two vertices from $N(u)$ of distance $(t-1)/2$ to F, by Lemma 7.2.1, we need only consider three different cases of how many neighbours of u and v are at distance $(n-3)/2$ to F.

Case 1. Assume that there is only one shortest path from u to F, and that there exists only one shortest path from v to F. Let $f_1 \in F$ such that $d(u, f_1) = (t-1)/2$. In addition, $d(v, f_1) = (t-1)/2$, by our assumption. Otherwise, $d(u, v) \geq t$, which is a contradiction (see Figure 7.3). Therefore, $d(f_1, v_3) \geq (t+1)/2$, by Lemma 7.2.1(ii). Hence $d(u, v_3) \geq d(u, f_1) + d(f_1, v_3) = t$, which is impossible since $D \leq t-1$.

Case 2. Assume that there is only one shortest path from u to F, and that there exist two vertices from $N(v)$ of distance $(t-3)/2$. Since $D \leq t-1$, there exists a vertex $f_1 \in F_{(n-1)/2}(u)$ and $F_{(n-1)/2}(v)$. By our Lemma 7.2.1(ii), $d(f_1, v_3) \geq (t+1)/2$. Since we assume $d(u, F) = d(u, f_1)$, it follows that $d(u, v_3) \geq d(u, f_1) + d(f_1, v_3) = t$, which is a contradiction because of $D \leq t-1$ (see Figure 7.4).

Case 3. Assume that there are two shortest paths from u to F, and there exist two vertices from $N(v)$ at distance $(t-3)/2$ to F. Let us look at two sub-cases according

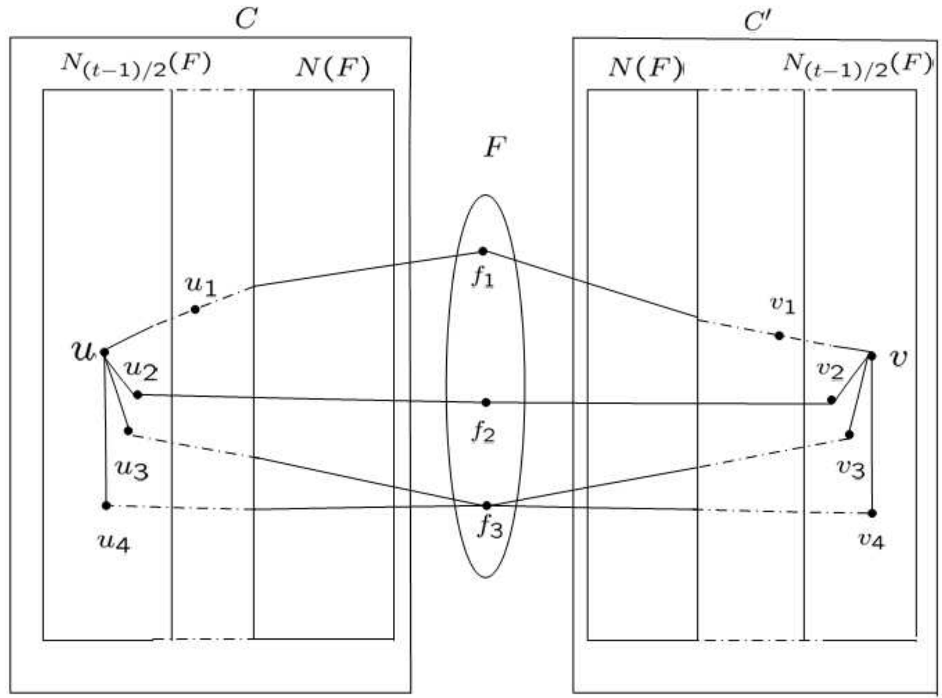

Figure 7.3: Case 1 when $\delta = 4$.

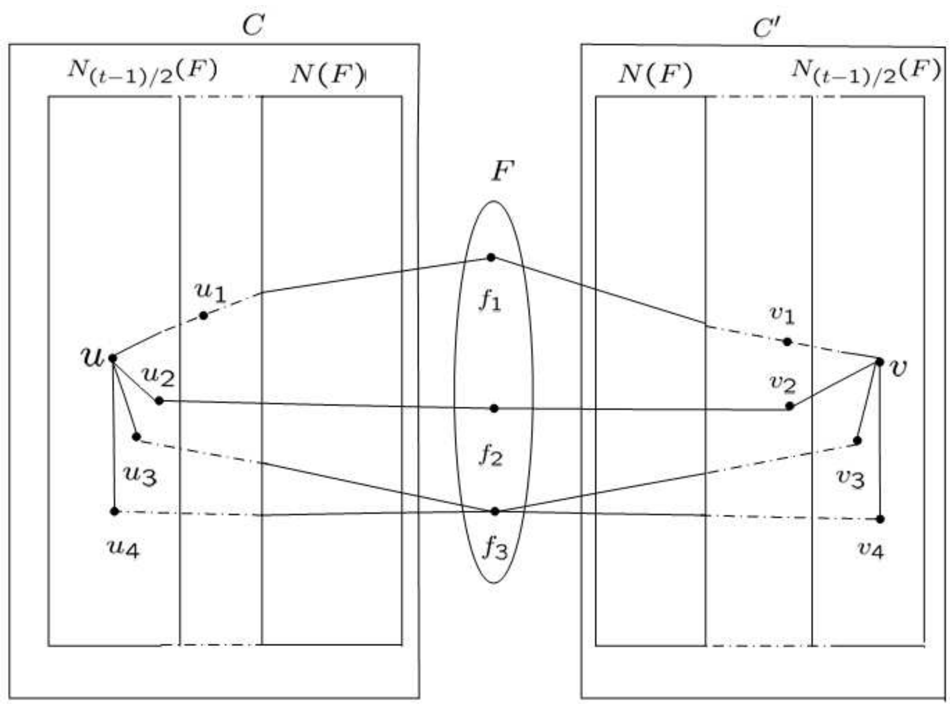

Figure 7.4: Case 2 when $\delta = 4$.

to whether or not $F_{(t-1)/2(u)} = F_{(t-1)/2(v)}$.

(a) Assume $F_{(t-1)/2(u)} = F_{(t-1)/2(v)}$. Then there exists a vertex, say f_1, such that $f_1 \in F_{(n-1)/2}(u)$ and $F_{(n-1)/2}(v)$. By Lemma 7.2.1(ii), $d(f_1, v_3 \geq (t+1)/2$. Then $d(u, v_3 \geq d(u, f_1) + d(f_1, v_3) = t$, which is impossible (see Figure 7.5).

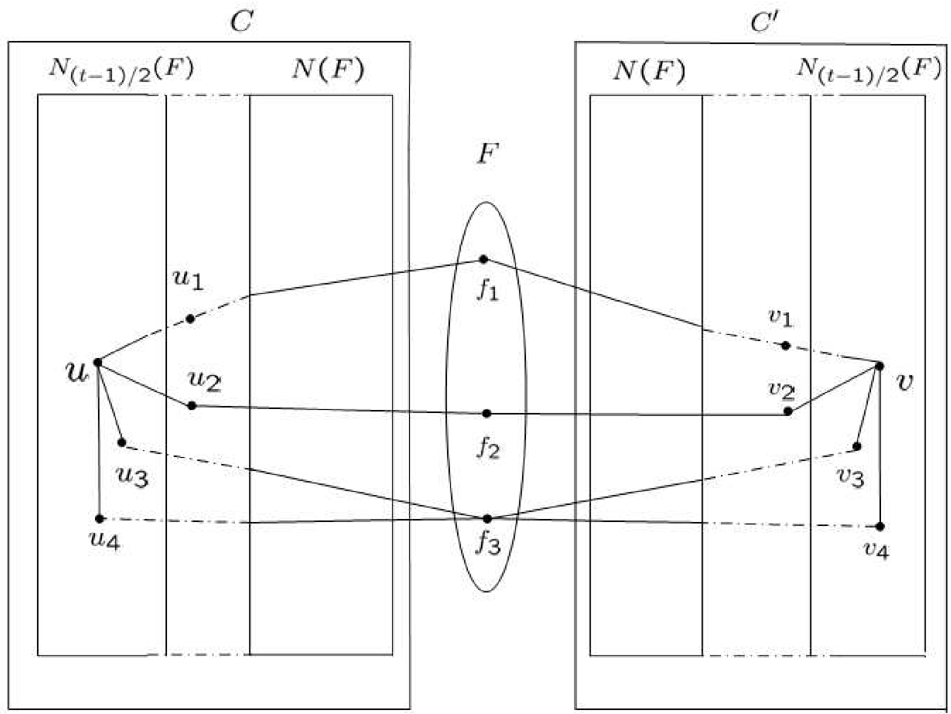

Figure 7.5: Case $3a$ when $\delta = 4$.

(b) Assume $F_{(t-1)/2(u)} \neq F_{(t-1)/2(v)}$ (see Figure 7.6). By Lemma 7.2.1(i), there exists a vertex, say u_3, such that $u_3 \in N(u)$ and $d(u_3, F) = (t-1)/2$. In this case, $d(u_3, v_3) \geq t$, which is a contradiction since $D \leq t - 1$. ■

7.3 Superconnectivity of regular graphs with odd girth

Recall that a graph is super-κ if its diameter is at most $g - 3$, when g is odd (respectively, if its diameter is at most $g - 4$, when g is even) [34]. In this section

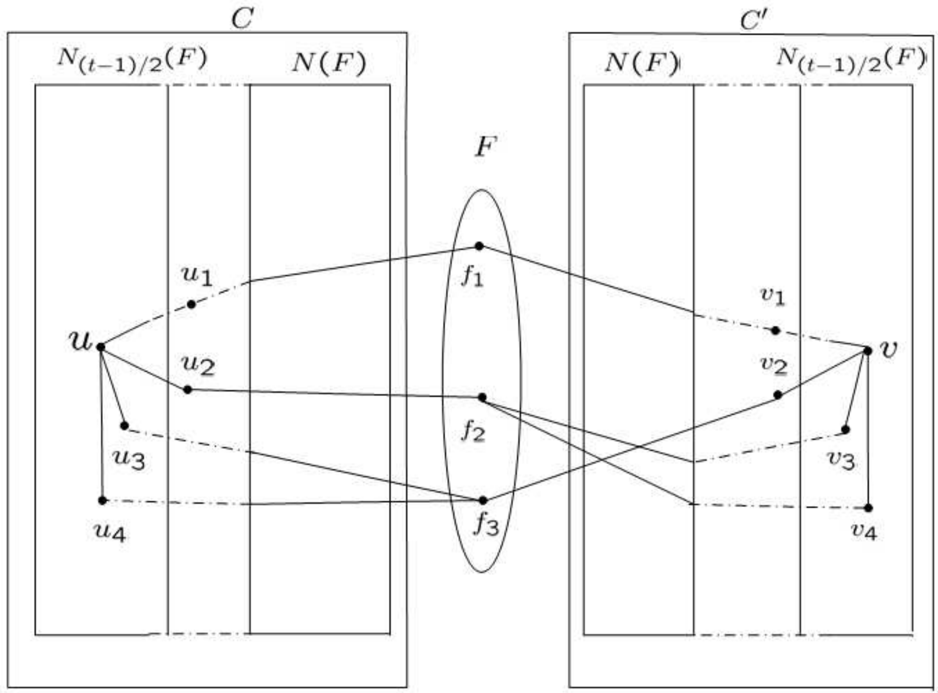

Figure 7.6: Case $3b$ when $\delta = 4$.

we improve this result by proving that a r-regular graph G with $r \geq 3$ and diameter at most $g-2$ is super-κ when g is odd.

To prove that a r-regular graph G with $r \geq 3$ and diameter at most $g-2$ is super-κ when g is odd, we require the following known result.

Proposition 7.3.1 *[7] Let $G = (V, E)$ be a connected graph with girth g and minimum degree $\delta \geq 2$. Let $X \subset V$ be a κ_1-cut with cardinality $|X| < \xi(G)$. Then for each connected component C of $G - X$ there exists some vertex $u_0 \in V(C)$ such that $d(u_0, X) \geq \lceil (g-3)/2 \rceil$ and $|N_{\lceil (g-3)/2 \rceil}(u_0) \cap X| \leq 1$.*

We use Proposition 7.3.1 to prove some structural properties of a component C when g is odd and $\max\{d(u, X) : u \in V(C)\} = (g-3)/2$.

* **Lemma 7.3.1** *Let G be a κ_1-connected graph with odd girth and minimum degree $\delta \geq 3$. Let X be a κ_1-cut with $|X| = \delta$ and assume that there exists a connected component C of $G - X$ such that* $\max\{d(u, X) : u \in V(C)\} = (g-3)/2$. *Then the following assertions hold:*

(i) *If* $u \in V(C)$ *is such that* $d(u, X) = (g-3)/2$ *and* $|N_{(g-3)/2}(u) \cap X| = 1$ *then* $d(u) = \delta$ *and* u *has* $\delta - 1$ *neighbors* z *such that* $d(z, X) = (g-3)/2$ *and* $|N_{(g-3)/2}(z) \cap X| = 1$. *Moreover,* $|N_{(g-1)/2}(u) \cap X| = \delta - 1$ *and* X *is a set of independent vertices.*

(ii) *There exists a* $(\delta-1)$*-regular subgraph* Γ *such that, for every vertex* $w \in V(\Gamma)$, $d_G(w) = \delta$ *and* $d(w, X) = (g-3)/2$.

(iii) *If* $g = 5$ *then* $|N(X) \cap V(C)| \geq \delta(\delta - 1)$.

(iv) *If* $g \geq 7$ *then* $|N(X) \cap V(C)| \geq (\delta - 1)^2 + 2$.

Proof Set $\mu = (g-3)/2$. Notice that $g \geq 5$ since $\mu \geq 1$.

(i) Given a vertex u in C such that $|N_\mu(u) \cap X| = 1$, let x_1 be a vertex in X such that $d(u, X) = d(u, x_1) = \mu$ and let $z_1 \in N(u)$ be such that $d(z_1, x_1) = \mu - 1$. Every vertex in $N(u) \setminus \{z_1\}$ is located in $N_\mu(X) \cap V(C)$, since otherwise, there are at least two vertices, say z_j and z_k, such that $d(z_j, X) = d(z_k, X) = \mu - 1$ and there exist two paths of length μ from u to x_1, namely $u, z_j, \ldots, x_1$ and $u, z_k, \ldots, x_1$, which form a cycle of length at most $2\mu = 2(g-3)/2 < g$. Therefore, there are $|N(u) \setminus \{z_1\}| = d(u) - 1$ vertices $z \in N(u) \cap N_\mu(X)$. Moreover, the sets $N_\mu(z_i) \cap X$, where $z_i \in N(u) \setminus \{z_1\}$ and $i = 2, \ldots, d(u)$, are pairwise disjoint (see Figure 7.7), because otherwise there exist at least two vertices, say z_j and z_k in $N(u) \setminus \{z_1\}$ and a vertex $x_k \in X$ such that the $z_j - x_k$ path and the $z_k - x_k$ path both have length μ. Thus a cycle of length at most $2 + 2\mu = 2 + 2(g-3)/2 < g$ is created through the vertices z_j, u, z_k and x_k. Hence, by the Pigeonhole Principle

$$\begin{aligned} |X| = \delta \;&\geq\; |N_\mu(u) \cap X| + \textstyle\sum_{i=2}^{d(u)} |N_\mu(z_i) \cap X| \\ &\geq\; 1 + (d(u) - 1) \\ &=\; d(u) \geq \delta. \end{aligned} \tag{7.3}$$

Hence, the inequalities are forced to be equalities, that is,

$$d(u) = \delta, \text{and } \; |\mathrm{N}_\mu(\mathrm{z_i}) \cap \mathrm{X}| = 1,$$

for every vertex $z_i \in N(u) - z_1$, $i = 2, \ldots, \delta$, and

$$X = (N_\mu(u) \cap X) \cup (\cup_{i=2}^{d(u)} (N_\mu(z_i) \cap X))$$

which means that X is a set of independent vertices. Therefore, we obtain that

$$|N_{\mu+1}(u) \cap X| = | \cup_{i=2}^{d(u)} (N_\mu(z_i) \cap X)| = \sum_{i=2}^{d(u)} |N_\mu(z_i) \cap X| = \delta - 1,$$

which concludes the proof of (i).

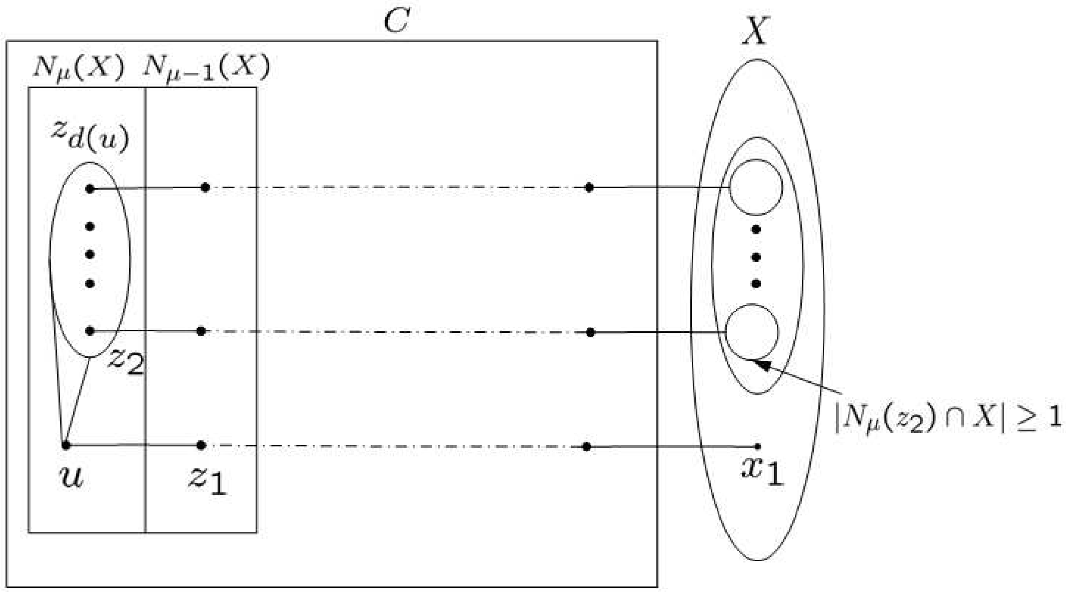

Figure 7.7: Pairwise disjoint sets $N_\mu(z_i) \cap X$.

(ii) From Proposition 7.3.1, it follows that there exists a vertex

$$u_0 \in N_\mu(X) \cap V(C)$$

such that $|N_\mu(u_0) \cap X| = 1$. By (i), the degree of u_0 is $d(u_0) = \delta$ and there are $\delta - 1$ vertices $z_i \in N(u_0) \cap N_\mu(X)$ such that $|N_\mu(z_i) \cap X| = 1$, for $i = 2, \ldots, \delta$. Applying the same reasoning used for proving (i) to the vertices z_i, we obtain $d(z_i) = \delta$, $i = 2, \ldots, \delta$ and each z_i has $\delta - 1$ neighbors $w \in N_\mu(X) \cap V(C)$, such that $|N_\mu(w) \cap X| = 1$.

Iterating this reasoning for each of the neighbors of z_i, we obtain a $(\delta-1)$-regular subgraph Γ in $G[N_\mu(X) \cap V(C)]$, such that every $w \in V(\Gamma)$ has $d_G(w) = \delta$.

(iii) and *(iv)* By *(ii)*, we know that there exists a $(\delta-1)$-regular subgraph Γ in $G[N_\mu(X) \cap V(C)]$, such that every $w \in V(\Gamma)$ has $d_G(w) = \delta$, and by *(i)*,

$$|N_\mu(w) \cap X| = 1,$$

for every $w \in V(\Gamma)$. Let $u \in V(\Gamma)$ and let $T = (\{u\} \cup N(u) \cup N_2(u)) \cap V(\Gamma)$. Then $|N_{\mu-1}(T) \cap N(X) \cap V(C)| \geq |T|$ because otherwise forbidden cycles through u and two different vertices of $N_2(u) \cap V(\Gamma)$ of length at most $2(\mu-1)+4 = g-1$ would be created. Therefore, since $g \geq 5$, we have

$$\begin{aligned} |N_{\mu-1}(T) \cap N(X) \cap V(C)| &\geq |T| \\ &= 1 + (\delta-1) + (\delta-1)(\delta-2) \\ &= 1 + (\delta-1)^2. \end{aligned} \tag{7.4}$$

Since $u \in V(\Gamma)$ then $d_G(u) = \delta$ which implies that there exists a unique vertex $z_1 \in N(u) \cap N_{\mu-1}(X)$. Let $X = \{x_1, x_2, \ldots, x_\delta\}$ denote the elements of the non trivial cut set and $N(u) \cap T = \{z_2, \ldots, z_\delta\}$ the neighbors of u included in T. Without loss of generality, let $N_\mu(u) \cap X = \{x_1\}$ and $N_\mu(z_i) \cap X = \{x_i\}$, for $i = 2, \ldots, \delta$. Since $N_\mu(X) = N(X)$ for $g = 5$, we need to consider the two cases $g = 5$ and $g \geq 7$ separately.

Case $g = 5$. Then $\mu = 1$ and ux_1, $z_i x_i$, $i = 2, \ldots, \delta$ are edges of G. Define the sets $X_i = X \setminus \{x_1, x_i\}$ and $Z_i = N(z_i) \setminus \{u, x_i\}$. Clearly $|X_i| = |Z_i| = \delta - 2$. Since $|N(w) \cap X| = 1$, for every $w \in Z_i$, then there exists a perfect matching between each of the sets Z_i and X_i, for all $i = 2, 3, \ldots, \delta$. Let $w_k^i \in Z_i$ denote the $\delta - 2$ elements of Z_i, such that $w_k^i x_k$, $x_k \in X_i$ are the edges of the matching between Z_i and X_i. Since $d_G(w_k^i) = \delta$ and $\{x_k, z_i\} \subset N(w_k^i)$, then w_k^i must have $\delta - 2$ neighbors more in $N(X)$. Furthermore, w_k^i has at most one neighbor w_h^j in Z_j for each $j \neq i$, because if $w_k^i w_h^j$ and $w_k^i w_t^j$ were two edges of G, the forbidden cycle including vertices $w_k^i, w_h^j, z_j, w_t^j, w_k^i$ of length four would be created. Moreover, if w_k^i

has a neighbor in Z_k, then there exists an edge $w_k^i w_h^k$ which forms a cycle including vertices $w_k^i, x_k, z_k, w_h^k, w_k^i$ of length four, therefore, $N(w_k^i) \cap Z_k = \emptyset$ (see Figure 7.8).

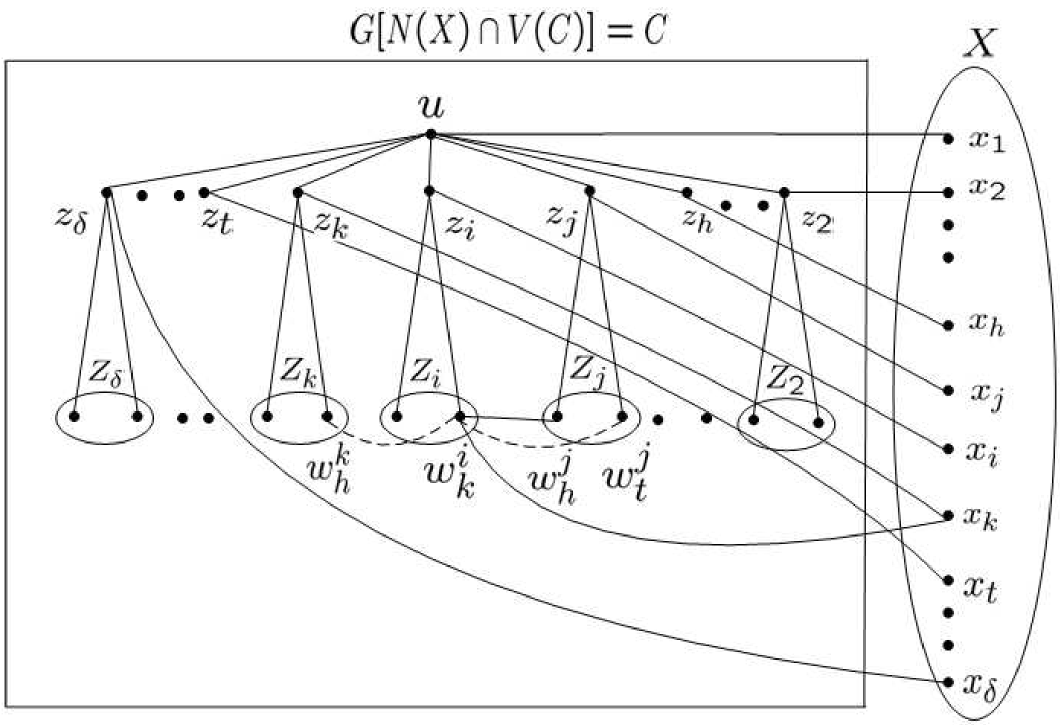

Figure 7.8: Illustration of the proof that $|N(X) \cap V(C)| \geq \delta(\delta - 1)$ for $g = 5$.

Consequently, $|N(w_k^i) \cap (\cup_{j=2}^{\delta} Z_j - \{Z_i, Z_k\})| \leq \delta - 3$, which implies that each $w_k^i \in Z_i$ has at least one new neighbor in $N(X) - T$. (As an illustration, see the graph depicted in Figure 7.10). Therefore,

$|N(X) \cap V(C)| \geq |T| + |Z_i| \geq 1 + (\delta - 1)^2 + (\delta - 2) = \delta(\delta - 1)$, and thus (iii) follows.

Case $g \geq 7$. In this case the subgraph of Γ induced by T is a tree and, by (7.4), we have $|N(X) \cap V(C)| \geq 1 + (\delta - 1)^2$. We reason by contradiction, assuming $|N(X) \cap V(C)| = (\delta - 1)^2 + 1$. Again by (7.4), we know that

$$|N_{\mu-1}(T) \cap N(X) \cap V(C)| = |T| = 1 + (\delta - 1)^2,$$

which implies $|N_{\mu-1}(u) \cap N(X) \cap V(C)| = 1$, $|N_{\mu-1}(z_i) \cap N(X) \cap V(C)| = 1$ and

$|N_\mu(z_i) \cap N(X) \cap V(C)| = \delta - 1$ for $i = 2, \ldots, \delta$. Denote

$$\{z_1''\} = N_{\mu-1}(u) \cap N(X) \cap V(C) = N(x_1) \cap V(C).$$

Since $g \geq 7$, there exists $w \in N_2(z_i) \cap V(\Gamma)$, for some $i \in \{2, \ldots, \delta\}$, such that $w \notin T$ and $z_1'' \notin N_{\mu-1}(w) \cup N_\mu(w)$, because otherwise a forbidden cycle through u, w, z_1'' of length at most $2\mu + 2$ would be created. Therefore, $(N_\mu(w) \cup N_{\mu+1}(w)) \cap X \subseteq X \setminus \{x_1\}$. Applying Lemma 7.3.1 (i) we get $N_{\mu+1}(w) \cap X = \{x_2, \ldots, x_\delta\}$. Hence there exists $x_j \in \{x_2, \ldots, x_\delta\}$, $j \neq i$, such that $x_j \in N_\mu(w) \cap N_{\mu+1}(w)$, creating a cycle through x_j and w of length $2\mu + 1$ which is a contradiction. Therefore $|N(X) \cap V(C)| \geq 2 + (\delta - 1)^2$, as required. ■

As a consequence of Lemma 7.3.1 we obtain Theorem 7.3.1 which is an improvement of Theorem 3.4.2 (ii) for regular graphs of odd girth.

* **Theorem 7.3.1** *Let G be a r-regular graph with $r \geq 3$ and odd girth g. If the diameter $D \leq g - 2$ then G is super-κ when $g \geq 5$, and a complete graph otherwise.*

Proof

For $g = 3$, the diameter is $D \leq g - 2 = 1$ and G is a complete graph. For $g \geq 5$ assume that G is not super-κ. Then $\kappa_1 = \kappa \leq r$ because G is an r-regular graph. By Theorem 3.4.1 (vi), $\kappa_1 = \kappa = r$. Let X be a κ_1-cut with $|X| = \kappa_1 = r$.

Let C and C' denote two components of $G - X$. Let

$$\mu(C) = max\{d(u, X) \, : \, u \in V(C)\}$$

and $\mu(C') = \max\{d(u', X) \, : \, u' \in V(C')\}$, as shown in Figure 7.9. From Proposition 7.3.1 it follows that $\mu(C) \geq (g - 3)/2$ and $\mu(C') \geq (g - 3)/2$. If $\mu(C), \mu(C') \geq (g - 1)/2$ then, given $u \in V(C)$, $u' \in V(C')$, the diameter $D \geq d(u, u') \geq d(u, X) + d(u', X) \geq 2(g - 1)/2 = g - 1$, contradicting our hypothesis that $D \leq g - 2$. Therefore, there exists at most one component, say C', such that $\mu(C') = (g - 1)/2$, and any other component $C \neq C'$ must have $\mu(C) = (g - 3)/2$.

By Lemma 7.3.1, $|N(X) \cap V(C)| \geq r^2 - r$ when $g = 5$, and $|N(X) \cap V(C)| \geq (r-1)^2 + 2$ when $g \geq 7$. Since G is an r-regular graph $|N(X) \cap V(C')| \leq |N(X)| - |N(X) \cap V(C)| \leq r^2 - (r^2 - r) = r$ when $g = 5$, and $|N(X) \cap V(C')| \leq r^2 - ((r-1)^2 + 2) \leq 2r - 3$ when $g \geq 7$.

Let $F' = [X, V(C')]$ denote the set of edges having one vertex in X and the other vertex in $V(C')$. Then F' is an edge-cut of cardinality $|F'| \leq 2r - 3$. Assume that F' is trivial. Then F' contains the edges incident with some vertex $y \in X$ or $y' \in V(C')$ (see Figure 7.9). If $y' \in V(C')$ then X contains the neighborhood of y', which contradicts our hypothesis that X is a non trivial vertex-cut. If $y \in X$ then $X \setminus \{y\}$ is a vertex-cut with cardinality $|X| - 1$, that is, X is non-minimal, again contradicting our hypothesis. Therefore, F' is a nontrivial edge-cut and $\lambda_1 \leq |F'| < 2r - 2 = \xi$. However, we know from Theorem 3.4.6 that $\lambda_1 = \xi$ for $D \leq g - 2$. As a consequence, if $D \leq g - 2$ then $|X| = \kappa_1 > r$ and G is super-κ. ■

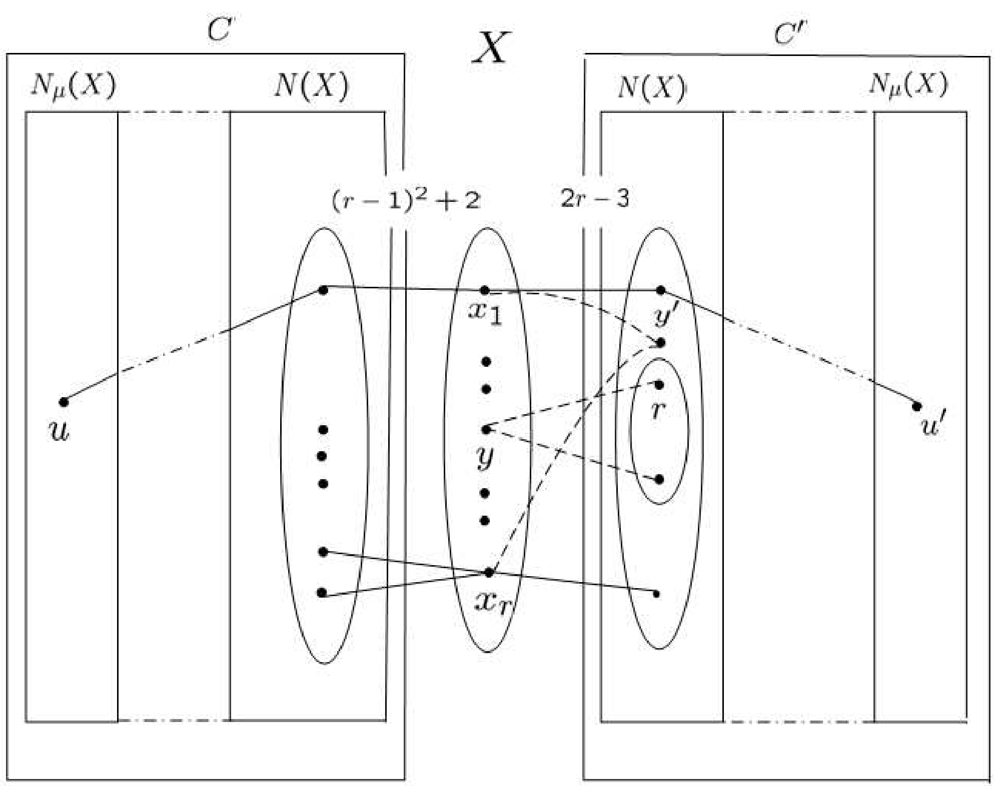

Figure 7.9: Illustration of the proof in Theorem 7.3.1 if $g \geq 7$.

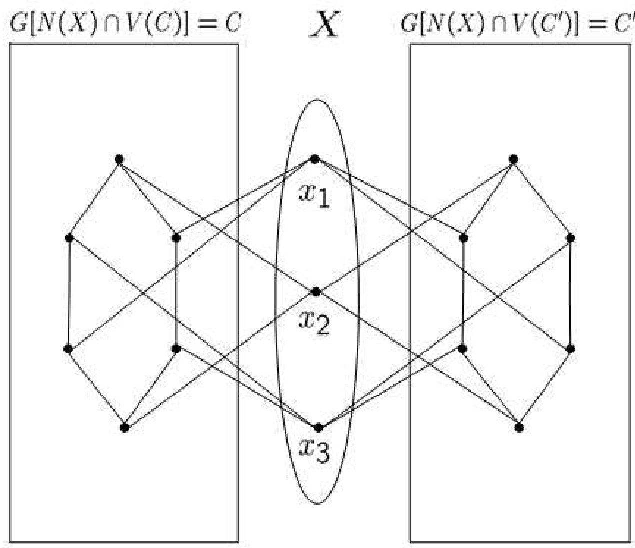

Figure 7.10: Graph with $g = 5$ and $\kappa_1 = \delta = 3$.

The graph depicted in Figure 7.10 shows a non δ-regular graph, with $g = 5$ and $D = 3$, which is non super-κ. Consequently, the hypothesis of regularity is essential to establish Theorem 7.3.1.

Chapter 8

Conclusions and Open Problems

In this thesis, we focused on several extremal graph problems in terms of several parameters, namely, connectivity, order, degree, diameter and girth.

In particular, we considered out-degree-relaxed Moore digraphs, that is, digraphs which are 'close' to Moore digraphs, by relaxing the maximum out-degree d, for given $\overrightarrow{M_{d,D}}$ vertices and diameter D. We constructed some digraphs by using HSAGA algorithms, in order to improve the current upper bounds of out-degree for out-degree-relaxed Moore digraphs. Next, we constructed various large graphs with given order n and girth at least $t+1$, and sizes that increase the current lower bounds of extremal numbers, $ex(n;t)$. Notice that some of our improved lower bounds of extremal numbers are optimal. Then we improved upper bounds of extremal numbers for some particular values of n and t, and we proved that an extremal graph G has optimal restricted edge-connectivity. Moreover, we proved that an extremal graph G is at least δ-connected when t is even, and G is at least δ-connected when $1 \leq \delta \leq 4$ and t odd. Last but not least, we proved that every r-regular graph G with $r \geq 3$ is superconnected when the girth g is odd the diameter D is at most $g-2$.

Not only have we presented research directions and results concerning extremal graph theory, but also we give several open problems for the direction of further research in this area. We start with some open problems in the constructions of nearly Moore graphs and digraphs.

In Chapter 4, we proposed several open problems:

Problem 8.1 *What is the minimum value of α, if G is a $(n, \mathcal{D}, D)$-graph with $M_{\Delta,D}$ vertices, given diameter D, and degree sequence $\mathcal{D}_1 = (\Delta + \alpha, \Delta^{M_{\Delta,D}-1})$?*

Problem 8.2 *What is the minimum number of vertices, denoted by β, if G is a $(n, \mathcal{D}, D)$-graph with $M_{\Delta,D}$ vertices, given diameter D, and degree sequence $\mathcal{D}_2 = ((\Delta + 1)^{\beta}, \Delta^{M_{\Delta,D}-\beta})$?*

The directed version of the above problems is listed below.

Problem 8.3 *What is the minimum value α, if G is a $(n, \mathcal{D}^+, D)$-digraph with $\overrightarrow{M_{d,D}}$ vertices and given diameter D, and out-degree sequence $\mathcal{D}_1^+ = (d + \alpha, d^{\overrightarrow{M_{d,D}}-1})$?*

Problem 8.4 *What is the minimum number of vertices, denoted by β, if G is a $(n, \mathcal{D}^+, D)$-digraph with $\overrightarrow{M_{d,D}}$ vertices, given diameter D, and out-degree sequence $\mathcal{D}_2^+ = ((d + 1)^{\beta}, d^{\overrightarrow{M_{d,D}}-\beta})$?*

In Table 4.6 from Chapter 4, we showed the current minimum number of extra out-degree a, and the current minimum number of vertices β with out-degree $d + 1$, for given $2 \leq D \leq 6$ and $d = 2$. In the future, we want to prove that these current minimum values are optimal.

Next, we notice that in Table 6.5 and 6.6 from Chapter 6, there exist a few small gaps between current lower bounds of extremal number, denoted by $ex_l(n; t)$, and current upper bounds of extremal number, called $ex_u(n; t)$. For example, $ex_l(30; 6) \geq 47$ and $ex_u(30; 6) \leq 48$, and so on. In the future, we plan to work on these small gaps, in order to decrease the size of these gaps. Furthermore, we would like to prove exact values of extremal number, for some particular values of n and t.

Regarding the connectivity of extremal graphs, in Chapter 7, we have proved that every $G \in EX(n; t)$ is maximally connected for all even t, and 4-connected for $\delta \leq 4$ and t odd. Now we propose the following conjecture.

Conjecture 8.1 *Let $G \in EX(n;t)$. Then G is δ-connected when $\delta \geq 5$ and t is odd.*

We mentioned before that, in some cases, $(k;g)$-cages are extremal graphs. In 1997, Fu *et al.* [39] conjectured that every $(k;g)$-cage is k-connected. Daven and Rodgers [29] and Jiang and Mubayi [50] have proved that every $(k;g)$-cage with $k \geq 3$ is 3-connected. Xu, Wang and Wang [89], showed that all $(4;g)$-cages are 4-connected. Recently, Marcote *et al.* showed that $(k;g)$-cages with $g \geq 10$ are 4-connected [61]. We hope that solving our conjecture will shed some light on the conjecture that every $(k;g)$-cage is k-connected.

Appendix A

Algorithm HSAGA

A.1 Introduction

In Chapter 2 we presented known results on graphs and digraphs, and we listed our open problems regarding graphs and digraphs. Currently, our study is focusing on directed graphs. In this chapter, we will describe construction techniques, namely, *line digraphs*, *generalised Kautz digraphs* and *vertex deletion scheme*, which will be used in this study to obtain new large digraphs from 'base digraphs'. A given digraph which is used to create a new digraph is called a *base digraph*. Our methodology also includes several optimization algorithms, such as *simulated annealing*, *genetic algorithms*, as well as our new algorithm method called *hybrid simulated annealing and genetic algorithms*. Using these algorithms in computer search, we are able to obtain large digraphs, which are in some way 'close' to Moore digraphs and some digraphs of number of vertices in the gaps between current known lower bounds and current best upper bounds of digraphs.

A.2 Optimization algorithms

A.2.1 Simulated annealing

The algorithm is based on that of Metropolis *et al.* [63], which was originally proposed as a means of finding the equilibrium configuration of a collection of atoms at a given temperature. The connection between this algorithm and mathematical minimization was first noted by Pincus [70]. However, it was Kirkpatrick *et al.* [54] who proposed that simulated annealing could form the basis of an optimization technique for combinatorial (and other) problems.

Simulated Annealing (SA) is a means of finding good solutions to combinatorial optimization problems. The basic operation in this technique is a *move*. A move is a transition from one element of the solution space to another element. In this paper, a move means inserting an arc between two randomly generated nonadjacent vertices x and y and remove one of arcs from x, in terms of their *cost*. The *cost* of a vertex x, denoted by $c(x)$, is the number of unique vertices reached by x at most in k steps. Assume m and n are out-neighbors of the vertex x and $c(m)$ is less than $c(n)$. If $c(y)$ is greater than $c(m)$, we must remove the arc between x and m, and insert an arc from x to y. Otherwise, we accept the arc $(x \rightarrow y)$ with probability $e^{-\triangle E/T}$, where T is a global time-varying parameter called the *temperature* and $\triangle E$ is the increase in *cost* (i.e., $c(y) - c(m)$) that would result from this prospective move.

The pseudo code [87] for our implementation is given below. In the inner loop, move is selected at random. A limited number of move are accepted at each *temperature* level. For better results in terms of small diameters, we would use larger numbers of move. In our study, we use $20|V(G)|$ as the maximum number of moves. Furthermore, there is a limit on the number of attempted moves at each temperature. For each accepted move, we want to attempt no more than 60 moves. Once the maximum number of accepted moves or the maximum number of attempted moves has been reached, the temperature is lowered and a new iteration begins.

```
Simulated Annealing(G)
   temp = initial_temp = 1.0
   Cool_rate = 0.95
   Max_moves = 20 * |V|                 // maximum numbers of move
   Max_attempted_moves = 60 * max_moves // maximum numbers of attempted move
   Max_frozen = 100
   frozen = 0
   randomly create a digraph based on given order and out-degree
   While (frozen <= Max_frozen)
       moves = 0             // numbers of move
       attempted_moves = 0   // numbers of attempted move
       While ((moves <= Max_moves) and (attempted_moves <= Max_attempted_moves))
          increase attempted_moves
          randomly select two non-adjacent vertices
          If the random vertices are accepted
             do move() and increase moves
             // new digraph's diameter is equal to required diameter k
             If (k(G new) == k required)
               return the improved solution, and end Simulated Anneal(G)
             End if
          End if
       End while
       temp = temp * Cool_rate
       If(attempted_moves > Max_attempted_moves)
          increase frozen
       End if
   End while
End Simulated Annealing(G)
```

A.2.2 Genetic algorithm

Genetic Algorithm (GA) is a search technique for global optimization in a complex search space. As the name suggests, GA employs the concepts of natural selection and genetics [68].

In GA, search space is composed of all the possible solutions to the problem. A solution in the search space is represented by a sequence of $0's$ and $1's$. This solution is referred to as the *chromosome* in the search space. Each chromosome has an associated objective function value called *fitness value*. A good chromosome is one that has high/low fitness value depending on the problem (maximization/minimization). A set of chromosomes and the associated fitness values is called the *population*.

There are five basic functions inside of GA. $Fitness$ is used to evaluate the fitness value of each chromosome in the current population. *Selection* is used to choose two parent chromosomes from the current population according to their fitness values. $Crossover$ is used to cross over the two parents to form two new offspring based on *Crossover_rate*, which is the odds of a parent being selected for the crossover operation. Actually, if no crossover is performed, then offsprings are created as the exact copies of parents. Furthermore, the $Mutation$ is used to mutate new offspring at each position in chromosome in terms of *Mutation_rate*, which specifies the odds that a given position in a offspring will be mutated. Finally, $Test$ is used to evaluate whether or not the new offspring satisfies the end condition.

The pseudo code of the general GA proceeds is given below.

```
Genetic Algorithm(G)
  Crossover_rate = 0.95, Mutation_rate = 0.03
  Curr_pupulation_size = 200
  Create an empty new population
  Found_soulation = false
  do Initial_Population() // Random generate a current population with
                             200 chromosomes.
  While (Found_soulation = false)
    do Fitness()
    While (Curr_population_size < 0)
      do Selection()
      do Crossover()
      do Mutation()
      If (Test() = true)
         Return the improved solution
         Found_solution = true // End Genetic Algorithm(G)
      Else (Test() = false)
         do Accepting()   // Place new offsprings in the new population.
         Curr_population_size = Curr_population_size - 2
      End if
    End while
    do Replace()          // Use new generated population to replace the
                             current population for a further run of the GA.
    Curr_population_size = 200
  End while
End Genetic Algorithm(G)
```

A.2.3 Hybrid simulated annealing and genetic algorithms

We introduced an optimization algorithm method: *Hybrid simulated annealing and genetic algorithms* (HSAGA). The general idea of HSAGA is that an initial digraph is created at the beginning, and used as the initial digraph input into SA. SA will terminate if the generated solution is satisfied in terms of given diameter after move, otherwise, the population of candidate solutions will be obtained. Furthermore, the set of elite individuals of the population is chosen by a selection procedure of GA according to their evaluation fitness values, following genetic operations consisting of crossover and mutation. The basic processes of HSAGA are shown in Figure A.1, and the details of each process can be described as below.

(a) Input parameters into our program, such as the out-degree, required minimum diameter, as well as *cooling rate*, which controls the decreasing of temperature, and *population's size*, that is, the numbers of chromosome, and so on.

(b) Create an initial base digraph in terms of given out-degree and diameter by using construction techniques, known as the generalised Kautz digraphs, and line digraph iterations. In addition, every digraph is represented by an adjacency matrix.

(c) If the current digraph is an improved digraph in terms of given diameter, then we terminate our process and output the result.

(d) Otherwise, put the current digraph into the method called SA. During its processing, SA will execute move to optimize the current digraph. We have a valuation function to test whether or not the diameter of the generated digraph matches the desired given diameter. If yes, then the process will stop and go to Step *c*. Otherwise, it will create a chromosome, based on its fitness value, which is represented by the number of reached central vertices by the current generated digraph, then store each chromosome into the population. If we fix the population size as 200, HSAGA will obtain a population of first 200 best chromosomes, based on their fitness values.

(e) Input the current population into GA functions consisting of selection, crossover and mutation, in order to obtain an improved solution, that is, a digraph whose diameter is equal to the given diameter. It is well known that GA never guarantees to generate a best solution, no matter what is the running time. So if GA could not give us an improved digraph at the end of the running time, we will select a current best chromosome, and input it back to SA, until the improved solution is found with respect to the given diameter.

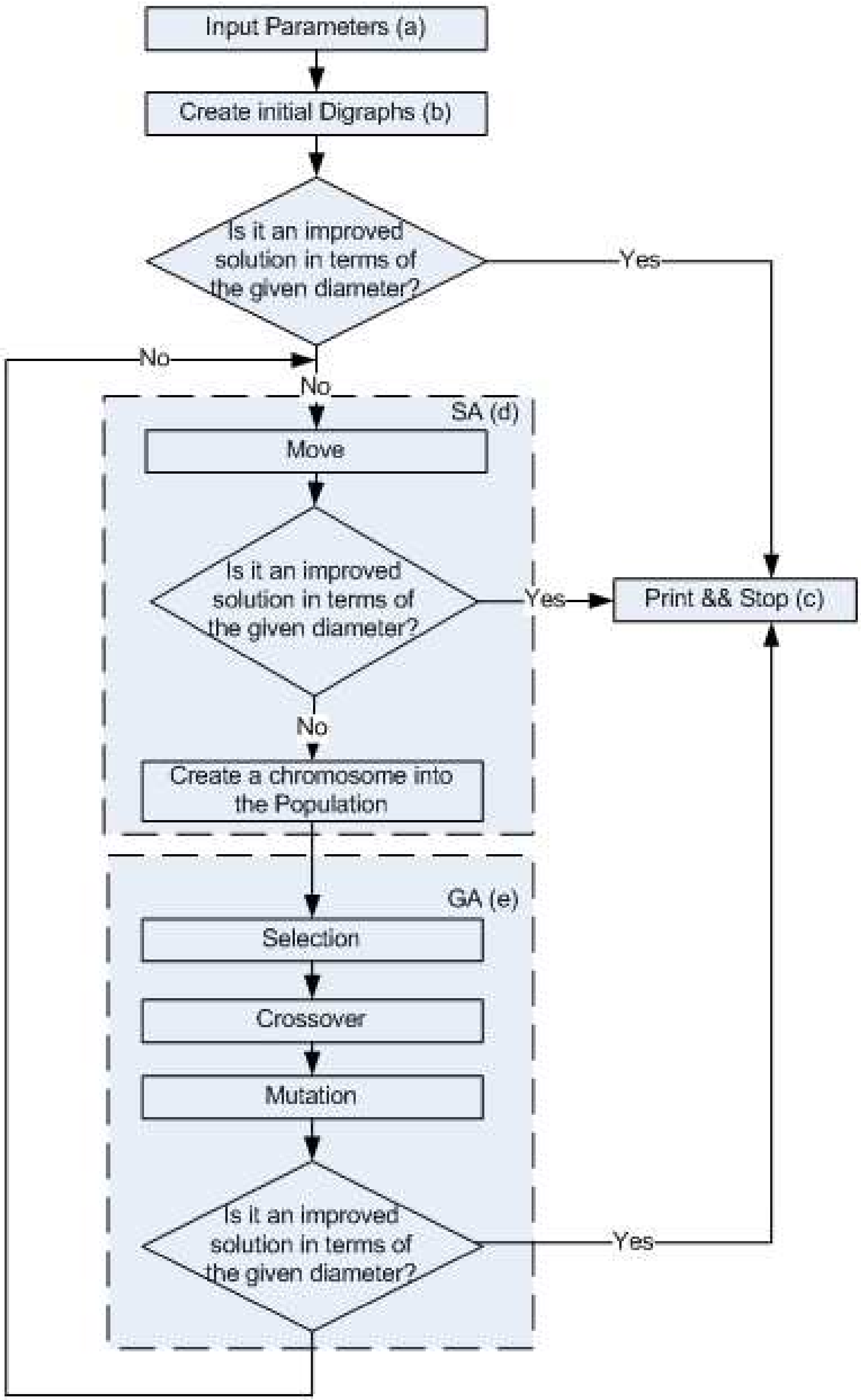

Figure A.1: Basic structure of HSAGA.

Bibliography

[1] E. Abajo and A. Diánez. Size of graphs with high girth. *Electronic Notes in Discrete Mathematics*, 29:179–183, 2007.

[2] N. Alon, S. Hoory, and N. Linial. The moore bound for irregular graphs. *Graphs Combin.*, 18:53–57, 2002.

[3] J.N. Ayoub and I.T. Frisch. On the smallest-branch cuts in directed graphs. *IEEE Trans. Circuit Theory*, CT-17:249–250, 1970.

[4] A.T. Balaban. Trivalent graphs of girth nine and eleven and relationships among the cages. *Rev. Roumaine Math*, 18:1033–1043, 1973.

[5] C. Balbuena and A. Carmona. On the connectivity and superconnectivity of bipartite digraphs and graphs. *Ars. Combin.*, 61:3–21, 2001.

[6] C. Balbuena, M. Cera, A. Diánez, and P. García-Vázquez. On the girth of extremal graphs without shortest cycles. *Discrete Mathematics*, 2007.

[7] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez, and X. Marcote. On the restricted connectivity and superconnectivity in graphs with given girth. *Discrete Mathematics*, 307(6):659–667, 2006.

[8] C. Balbuena and P. García-Vázquez. On the minimum order of extremal graphs to have a prescribed girth. *SIAM J. Discrete Mathematics*, 21(1):251–257, 2007.

[9] C. Balbuena, P. García-Vázquez, and X. Marcote. Suffcient conditions for λ'-optimality in graphs with girth g. *J. Graph Theory*, 52(1):73–86, 2006.

[10] E. Bannai and T. Ito. On finite Moore graphs. *J. Fac. Sci. Tokyo Univ*, 20:191–208, 1973.

[11] E. Bannai and T. Ito. Regular graphs with excess one. *Discrete Mathematics*, 37:147–158, 1981.

[12] E.T. Baskoro, M. Miller, J. Širáň, and M. Sutton. Complete characterisation of almost Moore digraphs of degree three. *J. Graph Theory*, 48:112–126, 2005.

[13] D. Bauer, F. Boesch, C. Suffel, and R. Tindell. *Connectivity extremal problems and the design of reliable probabilistic networks*. The theory and application of graphs, Y. Alavi and G. Chartrand (Editors), Wiley, New York, 1981.

[14] M. Behzad, G. Chartrand, and C.E. Wall. On minimal regular digraphs with given girth. *Fund. Math.*, 69:227–231, 1970.

[15] N.L. Biggs. *Algebraic Graph Theory* (2nd ed.). Cambridge University Press, 1993.

[16] F. Boesch. Synthesis of reliable networks-a survey. *IEEE Trans. Reliability*, 35:227–231, 1986.

[17] F. Boesch and R. Tindell. Circulants and their connectivities. *J. Graph Theory*, 4:487–499, 1984.

[18] B. Bollobas. *Extremal Graph Theory*. Academic Press Inc. London, 1978.

[19] B. Bollobás. *Extremal Graph Theory*. New York: Academic Press (2nd ed.), 1993.

[20] J.A. Bondy. Counting subgraphs: a new approach to the Caccetta-Häggkvist conjecture. *Discrete Mathematics*, 165:71–80, 1997.

[21] W.G. Bridges and S. Toueg. On the impossibility of directed Moore graphs. *J. Combinatorial Theory*, 3:339–341, 1981.

[22] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance Regular Graphs*. Springer, Heidelberg, 1989.

[23] D. Buset. Maxmal cubic graphs with diameter 4. *Discrete Applied Mathematics*, 101:53–61, 2000.

[24] L. Caccetta and R. Hággkvist. On minimal digraphs with given girth. *Congressus Numerantium*, 21:181–187, 1978.

[25] G. Chartrand. A graph-theoretic approach to a communications problem. *SIAM J. Applied Mathematics*, 143:778–781, 1966.

[26] F.R.K. Chung. *Constructing random-like graphs, in: B. Bollobas* (Ed.) *Probabilistic Combinatorics and its Applications*, volume 44. Amer. Math. Soc., 1991.

[27] J. Conde and J. Gimbert. On the existence of graphs of diameter two and defect two. *Discrete Mathematics*, 2008. Article in Press, Corrected Proof.

[28] R.M. Damerell. On moore graphs. *Proc. Cambridge Phil. Soc*, 74:227–236, 1973.

[29] M. Daven and C.A. Rodger. $(k; g)$-cages are 3-connected. *Discrete Mathematics*, 199:207–215, 1999.

[30] P. Erdős. Some recent progress on extremal problems in graph theory. *Congr. Numer.*, 14:3–14, 1975.

[31] P. Erdős, S. Fajtlowicz, and A.J. Hoffman. Maximum degree in graphs of diameter 2. *Networks*, 10:87–90, 1980.

[32] P. Erdős and H. Sachs. Reguläre graphen gegebener taillenweite mitminimaler knotenzahl. *Wiss. Z. Uni. Halle* (Math.Nat.), 12:251–257, 1963.

[33] A.H. Esfahanian and S.L. Hakimi. On computing a conditional edge-connectivity of a graph. *Inf. Process. Lett.*, 27:195–199, 1988.

[34] J. Fábrega and M.A. Fiol. Maximally connected digraphs. *J. Graph Theory*, 13:657–668, 1989.

[35] J. Fábrega and M.A. Fiol. Bipartite graphs and digraphs with maximum connectivity. *Discrete Applied Mathematics*, 69:269–278, 1996.

[36] M.A. Fiol. On super-edge-connected digraphs and bipartite digraphs. *J. Graph Theory*, 16:545–555, 1992.

[37] M.A. Fiol, J. Fábrega, and M. Escudero. Short paths and connectivity in graphs and digraphs. *Ars Combin.*, 29B:17–31, 1990.

[38] M.A. Fiol, J.L.A. Yebra, and I. Alegre. Line digraph iterations and the (d, k) digraph problem. *IEEE Trans. Computers*, C-33:400–403, 1984.

[39] H.L. Fu, K.C. Huang, and C.A. Rodge. Connectivity of cages. *J. Graph Theory*, 24:187–181, 1997.

[40] D. K. Garnick, Y. H. H. Kwong, and F. Lazebnik. Extremal graphs without three-cycles or four-cycles. *J. Graph Theory*, 17:633C645, 1993.

[41] D.K. Garnick and N.A. Nieuwejaar. Non-isomorphic extremal graphs with three-cylces or four-cycles. *J. Combin. Math. Combin. Comput*, 12:33–56, 1993.

[42] D. Geller and F. Harary. Connectivity in digraphs. *Lecture Notes in Mathematics*, 186:105–114, 1970.

[43] J. Groos and J. Yellen (Eds.). *Handbook of Graph Theory.* CRC press, Boc Raton, 2003.

[44] Y.O. Hamidoune. A note on minimal directed graphs with given girth. *J. Combin. Theory*, 43:343–348, 1987.

[45] F. Harary. Conditional connectivity. *Networks*, 13:347–357, 1983.

[46] N. Hartsfield and G. Ringel. *Peals in Graph Theory: A Comprehensive Introduction.* Academic Press, London, 1994.

[47] C.T. Hoáng and B. Reed. A note on short cycles in digraphs. *Discrete Mathematics*, 66:103–107, 1987.

[48] A.J. Hoffman and R.R. Singleton. On moore graphs with diameter 2 and 3. *IBM J. Res. Develop*, 4:497–504, 1960.

[49] M. Imase, T. Soneoka, and K. Okada. Connectivity of regular directed graphs with small diameter. *IEEE Trans. Computers*, C-34:267–273, 1985.

[50] T. Jiang and D. Mubayi. Connectivity and separating sets of cages. *J. Graph Theory*, 29:35–44, 1998.

[51] J.L. Jolivet. Sur la connexite des graphes orientes. *C.R. Acad. Sci. Paris*, A274:148–150, 1972.

[52] L.K. Jorgensen. Diameters of cubic graphs. *Discrete Applied Mathematics*, 37/38:347–351, 1992.

[53] W.H. Kautz. *Bounds on directed (d, k)-graphs.* Theory of Cellular Logic Networks and Machines, 1968. AFCRL-68-0668 Final Report.

[54] S. Kirkpatrick, C.D.Jr. Gerlatt, and M.P. Vecchi. Optimization by simulated annealing. *Science*, 220:671–680, 1983.

[55] M. Knor. A note on radially moore digraphs. *IEEE Transactions on Computers*, 45, 1996.

[56] Y.H.H. Kwong, D.K. Garnick, and F. Lazebnik. Extremal graphs without three-cycles or four-cycles. *J. Graph Theory*, 17:633–645, 1993.

[57] F. Lazebnik, V.A. Ustimenko, and A.J. Woldar. A new series of dense graphs of high girth. *Bull. Amer. Math. Soc (N.S.)*, 32(1):73–79, 1995.

[58] F. Lazebnik and P. Wang. On the structure of extremal graphs of high girth. *J. Graph Theory*, 26, 1997.

[59] L. Lesniak. Results on the edge-connectivity of graphs. *Discrete Mathematics*, 8:351–354, 1974.

[60] X. Marcote, C. Balbuena, and I. Pelayo. Diameter, short paths and superconnectivity in digraphs. *Discrete Mathematics*, 288:113–123, 2004.

[61] X. Marcote, C. Balbuena, I. Pelayo, and J. Fàbrega. $(\delta; g)$-cages with $g \geq 10$ are 4-connected. *Discrete Mathematics*, 301:124–136, 2005.

[62] B. McKay, W. Myrvold, and J. Nadon. *Fast backtracking principles applied to find new cages*. Proceedings of the ninth annual ACM-SIAM symposium on Discrete algorithms, 1998.

[63] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller, and E. Teller. Equations of state calculations by fast computing machines. *J. Chem. Phys*, 21:1087–1092, 1958.

[64] M. Miller and I. Fris. Maximum order digraphs for diameter 2 or degree 2. *Pullman volume of Graphs and Matrices*, 139:269–278, 1992.

[65] M. Miller and J. Širáň. Digraphs of degree two which miss the moore bound by two. *Discrete Mathematics*, 226:269–274, 2001.

[66] M. Miller and J. Širáň. Moore graphs and beyond: A survery of the degree/diameter problem. *The Electronic Journal of Combinatorics*, 11, 2004.

[67] S. G. Molodtsov. Largest graphs of diameter 2 and maximum degree 6. *Lecture Notes in Computer Science*, 4123, 2006.

[68] M. Obitko. Introduction to genetic algorithms with java applet. *http://labe.felk.cvut.cz/ obitko/ga*, 1998.

[69] I. Pelayo, X. Marcote, C. Balbuena, and J. Fábrega. Using a prgressive withdrawal procedure to study superconnectivity in digraphs. *Discrete Mathematics*, 267:229–246, 2003.

[70] M. Pincus. A monte carlo method for the approximate solution of certain types of constrained optimization problems. *Oper. Res.*, 18:1225–1228, 1970.

[71] J. Plesník. Critical graphs of given diameter. *Acta Fac. Rerum Natur. Univ. Comenian, Math.*, 30:71–93, 1975.

[72] J. Plesník and Š. Znám. Strongly geodetic directed graphs. *Acta F. R. N. Univ. Comen. - Mathematica*, XXIX:29–34, 1974.

[73] G. Royle. Cubic cages. http://people.csse.uwa.edu.au/gordon/cages/index .html.

[74] N. Sauer. Extremaleigenschaften reguläre graphen gegebener taillenweite,I,II. *Osterreich. Akad. Wiss. Math. Natur. Kk. S. -B. II*, 176:27–43, 1967.

[75] J. Shen. On the girth of digraphs. *Discrete Mathematics*, 211(1-3):167–181, 2000.

[76] M. Simonovits. *Extremal Graph Theory, Selected Topics in Graph Theory 2 (L. W. Beineke and R. J. Wilson, Eds.).* Academic Press, London, 1983.

[77] Slamin. *Diregularity of Digraphs Close to Moore Bound.* Phd thesis, Department of Computer Science and Software Engineering, The University of Newcastle, 2001.

[78] T. Soneoka. Super edge-connectivity of dense digraphs and graphs. *Discrete Applied Mathematics*, 37/38:511–523, 1992.

[79] T. Soneoka. Super edge-connectivity of dense digraphs and graphs,. *Discrete Applied Mathematics*, page 37/38, 1992.

[80] T. Soneoka, H. Nakada, and M. Imase. Sufficient conditions for dense graphs to be maximally connected. *Proc. of ISCAS 85, I.E.E.E. Press*, pages 811–814, 1985.

[81] T. Soneoka, H. Nakada, M. Imase, and C. Peyrat. Suffcient conditions for maximally connected dense graphs. *Discrete Mathematics*, 63:53–66, 1987.

[82] J. Tang, Y. Lin, C. Balbuena, and M. Miller. Calculating the extremal number $\mathrm{ex}(n; \{C_3, C_4, \ldots, C_t\})$. *Discrete Applied Mathematics*, 157:2198–2206, 2009.

[83] J. Tang, M. Miller, and Y. Lin. HSAGA and its application for the construction of near-moore digraphs. *Journal of Discrete Algorithms*, 6:73–84, 2007.

[84] P. Turán. On an extremal problem in graph theory(in hungarian). *Mat. Fiz. Lapok*, pages 436–452, 1941.

[85] W.T. Tutte. A family of cubical graphs. *Proc. Cambridge Phil. Soc.*, pages 459–474, 1947.

[86] W.T. Tutte. *Connectivity in graphs*. Univ. Toronto Press, 1966.

[87] P. Wang, G.W. Dueck, and S. MacMillan. Using simulated annealing to construct extremal graphs. *Discrete Mathematics*, 235:125–135, 2001.

[88] P.K. Wong. Cages-a survey. *J. Graph Theory*, 6:1–22, 1982.

[89] B. Xu, P. Wang, and J.F. Wang. On the connectivity of $(4; g)$-cages. *Ars Combin.*, 64:181–192, 2002.