Semigroup C^* Crossed Products and Toeplitz Algebras

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

(Signed) ____________________

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Contents

Abstract

Let (G, G_+) be a quasi-lattice-ordered group with positive cone G_+. Laca and Raeburn have shown that the universal C^*-algebra $C^*(G, G_+)$ introduced by Nica is a crossed product $B_{G_+} \times_\alpha G_+$ by a semigroup of endomorphisms. Subsequent research centered on totally ordered abelian groups. We generalize the results in [**2**], [**3**] and [**5**] to extend it to the case of discrete lattice-ordered abelian groups. In particular given a hereditary subsemigroup H_+ of G_+ we introduce a closed ideal I_{H_+} of the C^*-algebra B_{G_+}. We construct an approximate identity for this ideal and show that I_{H_+} is extendibly α-invariant. It follows that there is an isomorphism between C^*-crossed products $(B_{G_+}/I_{H_+}) \times_{\widetilde{\alpha}} G_+$ and $B_{(G/H)_+} \times_\beta G_+$. This leads to one of our main results that $B_{(G/H)_+} \times_\beta G_+$ is realized as an induced C^*-algebra $\mathrm{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$. Then we use this result to show the existence of the following short exact sequence of C^*-algebras

$$0 \to I_{H_+} \times_\alpha G_+ \to B_{G_+} \times_\alpha G_+ \to \mathrm{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right) \to 0.$$

This leads to show that the ideal $I_{H_+} \times_\alpha G_+$ is generated by $\left\{ i_{B_{G_+}}(1 - 1_u) : u \in H_+ \right\}$ and therefore contained in the commutator ideal $\mathcal{C}_G$ of the C^*-algebra $B_{G_+} \times_\alpha G_+$. Moreover, we use our short exact sequence to study the primitive ideals of the C^* algebra $B_{G_+} \times_\alpha G_+$ which is isomorphic to the Toeplitz algebra $\mathcal{T}(G)$ of G.

CHAPTER 1

Introduction

The theory of crossed products of C^*-algebras by endomorphisms has been developing rapidly. This theory is a generalization of the theory of crossed products of C^*-algebras by semigroups of automorphisms, which is an interesting area of the modern theory of operator algebras. The significance of that theory have led many authors to consider more general cases of crossed products of C^*-algebras by semigroups of automorphisms such as Murphy in [**27**] and the crossed products by semigroups of endomorphisms ([**1**], [**2**], [**4**], [**9**], [**24**]).

The origins of the theory of crossed products by semigroups of endomorphisms can be traced back to Cuntz's work in which he observed that the Cuntz algebra $\mathcal{O}_n$ which is generated by certain families of isometries, can be viewed as an isometric crossed product of a UHF-algebra by a single endomorphism [**11**]. This work was later generalized and attracted many authors. It was not until Stacey's paper [**38**] that a crossed product by an endomorphism was explicitly described as an abstract object. In particular, for any C^*-algebra A and any endomorphism $\alpha : A \to A$; Stacey found a family of covariant representations for which the crossed product, which was denoted by $A \rtimes_\alpha \mathbb{N}$, is universal. Since then this idea has been improved and authors have studied isometric crossed products by actions of the semigroup $\mathbb{N}$ using Stacey's work (see [**9**], [**10**]), actions of totally ordered abelian groups ([**3**], [**4**]) and by actions of quasi-lattice ordered groups ([**21**], [**32**]). Crossed products by semigroups of endomorphisms have been used in a number of settings and provide good models for Toeplitz algebras ([**4**], [**21**]).

In this thesis, we work on crossed products by semigroups of endomorphisms, actions of general semigroups, namely the positive cone G_+ of a partially ordered discrete abelian group G and with C^*-algebras A which may not have an identity. For A unital and (G, G_+) a quasi-lattice ordered group, it was shown in [**21**, Corollary 2.4] that the C^*-algebra $C^*(G, G_+)$ is the crossed product $B_{G_+} \times_\alpha G_+$ of the dynamical system (B_{G_+}, G_+, α) consisting of the C^*-subalgebra B_{G_+} of $\ell^\infty(G_+)$ spanned by the functions $\{1_x : x \in G_+\}$, where

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise,} \end{cases}$$

and the action α given by $\alpha_x(1_y) = 1_{x+y}$ for $x, y \in G_+$. Moreover, denote by $\{\varepsilon_x : x \in G_+\}$ the usual basis for the Hilbert space $\ell^2(G_+)$. For each $x \in G_+$, there is an isometry T_x on $\ell^2(G_+)$ satisfying $T_x(\varepsilon_y) = \varepsilon_{x+y}$ for all $y \in G_+$. The *Toeplitz algebra* of G is the C^*-subalgebra $\mathcal{T}(G)$ of $B(\ell^2(G_+))$ generated by the isometries $\{T_x : x \in G_+\}$.

We are interested in the crossed product $B_{G_+} \times_\alpha G_+$ because it is generated by the elements $\{i_{G_+}(x) : x \in G_+\}$ and has the following universal property: for every covariant isometric representation V of G_+ there is a unital representation $\pi_V \times V$ of $B_{G_+} \times_\alpha G_+$ such that $\pi_V \times V(i_{G_+}(x)) = V_x$ [**21**, Proposition 2.3]. Moreover, since we are working with abelian groups, $\pi_V \times V$ is faithful if and only if $\prod_{i=1}^n (1 - V_{x_i} V_{x_i}^*) \neq 0$ whenever $x_1, x_2, ..., x_n \in G_+ \backslash \{0\}$ [**21**, Theorem 3.7]. Another interesting thing about the crossed product $B_{G_+} \times_\alpha G_+$ is that using its universal property, Laca-Raeburn have showed in [**21**, Corollary 3.8] that for any amenable quasi-lattice ordered group (G, G_+) the crossed product $B_{G_+} \times_\alpha G_+$ is isomorphic to the Toeplitz algebra of G. Thus $\mathcal{T}(G)$ is itself universal for covariant isometric representations of G_+.

To talk about crossed products of a non-unital algebra A, we need the endomorphisms $\{\alpha_x : x \in G_+\}$ to be extendible, that is each α_x has a strictly continuous extension $\overline{\alpha_x} : M(A) \to M(A)$; where $M(A)$ is the multiplier algebra of A. We need to consider algebras without identity because we will study the crossed product $I_{H_+} \times_\alpha G_+$ of an extendibly α_x-invariant ideal I_{H_+} of B_{G_+}.

We give now an outline of what we do in each chapter. We begin with a chapter of background material and some results we proved about quasi-lattice and lattice-ordered groups. The first section is about the definition of partially ordered groups. We also discuss the definitions of lattice and quasi-lattice ordered groups and we give our notion of them (G, G_+), and for that we refer to ([**18**], [**21**], [**25**]). Also we prove some results about lattice and quasi-lattice ordered groups, in particular we show that every lattice ordered group is a quasi-lattice ordered group. In the second section, we discuss some examples of quasi-lattice ordered groups and lattice-ordered groups and we also give some examples of hereditary subsets of G_+.

In Chapter 3, we introduce the crossed product of a dynamical system and we give a detailed definition of the crossed product of a C^*-algebra A by a semigroup of endomorphisms $\alpha : G_+ \to \mathrm{End}(A)$ in terms of the universal property. In the first section we give the definitions of multiplier algebras and extendible homomorphisms, and we give some important results that we need in this work. In section 2, we define dynamical systems (A, α, G_+) and crossed products for dynamical systems. Then we give our notion for crossed product $A \times_\alpha G_+$, and we remark on when a crossed product should exist (Remark 3.2.4). In section 3, we introduce the C^*-subalgebra B_{G_+} and we prove some results about it. Then we use the universal property of the crossed product $A \times_\alpha G_+$ to prove the existence of a strongly continuous action of the group $\widehat{G} := \{\gamma : G \to \mathbb{T} : \gamma \text{ is a homomorphism}\}$ in Lemma 3.4.6. This says that for a quasi-lattice ordered group (G, G_+) and a dynamical system (A, α, G_+) with A unital, there is a strongly continuous action

$$\widehat{\alpha} : \widehat{G} \to \mathrm{Aut}(A \times_\alpha G_+)$$

satisfying $\widehat{\alpha}_\gamma(i_A(a)) = i_A(a)$ and $\widehat{\alpha}_\gamma(i_{G_+}(x)) = \overline{\gamma(x)}\, i_{G_+}(x)$ for all $x \in G_+, a \in A$.

Next, in Chapter 4, we give a detailed definition of extendibly α-invariant ideals of C^*-algebras. In the first section, we fix firstly our hypothesis about the group G, which is a partially-ordered group with a positive cone G_+ such that (G, G_+) is a lattice-ordered group and H_+ is a hereditary subsemigroup of G_+. We introduce the ideal

$$I_{H_+} := \overline{\text{span}}\{1_x - 1_{x+h} : h \in H_+, x \in G_+\}$$

of the C^*-algebra B_{G_+} in Lemma 4.1.1. Then we introduce an approximate identity C_I of the ideal I_{H_+}

$$C_I = \Big\{ \mathbb{1}_{(F,h)} = \sum_{\emptyset \neq A \subset F} (-1)^{|A|+1} \prod_{x \in A} (1_x - 1_{x+h}) : (F, h) \in D \Big\}$$

in Proposition 4.1.5. At the end of the first section we prove in Corollary 4.1.6 that I_{H_+} is an extendibly α_z-invariant ideal of B_{G_+}, for all $z \in G_+$. In section 2, we discuss the relationship between the C^*-algebras $B_{(G/H)_+}$ and B_{G_+}/I_{H_+}. The main result in this section is in Proposition 4.2.4 which shows that there is an isomorphism Φ of B_{G_+}/I_{H_+} onto $B_{(G/H)_+}$.

In Chapter 5, we talk about inflated dynamical systems and induced C^*-algebras. In the first section, we talk about *irreducible representations* of C^*-algebras and state the necessary definitions and important results needed to study them. In section 2, we discuss the relationship between the crossed products $(B_{G_+}/I_{H_+}) \times_{\tilde{\alpha}} G_+$ and $B_{(G/H)_+} \times_\beta G_+$ (Lemma 5.2.2), and we prove that they are isomorphic. Then in Proposition 5.2.3 we show that there is a surjective homomorphism

$$Q : B_{(G/H)_+} \times_\beta G_+ \to B_{(G/H)_+} \times_\tau (G/H)_+.$$

In section 3, we talk about induced C^*-algebras $\operatorname{Ind}_S^K(A, \alpha)$. If K is a compact group and $\alpha : S \to \operatorname{Aut} A$ is an action of a closed subgroup S on a C^*-algebra A, the *induced C^*-algebra* $\operatorname{Ind}_S^K(A, \alpha)$ is the subalgebra of $C(K, A)$ consisting of the functions f satisfying $f(gh) = \alpha_h^{-1}(f(g))$ for $g \in K$ and $h \in S$ [**6**, page 3]. Then we give our main result about these induced algebras in Theorem 5.3.2 which is an analogue of Theorem 2.1 of [**3**] with modifications as we work here with partially ordered groups (lattice-ordered) and this has added many challenges to the proof and made it more interesting. Our theorem shows that we can realize the C^*-algebra $B_{(G/H)_+} \times_\beta G_+$ as the induced C^*-algebra $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$. To introduce that theorem we proved some lemmas and propositions (Proposition 4.2.4, Lemma

5.2.2, Proposition 5.2.3). We also used [**2**, page 2] and Remark 5.2.1 to realize that the composition $\tau \circ q$ of the action $\tau : (G/H)_+ \to \mathrm{End}\left(B_{(G/H)_+}\right)$ which satisfies $\tau_{x+H}(1_{y+H}) = 1_{x+y+H}$, with the quotient map $q : G \to G/H$ is an action of G_+ on $B_{(G/H)_+}$ by extendible endomorphisms. Although the proof of our theorem (Theorem 5.3.2) is similar in outline to that of [**6**, Theorem 2.1], problems associated to the fact that we are working with a partially ordered group and with crossed products of systems which are not Toeplitz algebras, make the argument harder and interesting. Then we prove that since the system (B_{G_+}, α, G_+) has the extendibly α_x-invariant ideal I_{H_+}, then there exists a short exact sequence of C^*-algebras

$$0 \to I_{H_+} \times_\alpha G_+ \to B_{G_+} \times_\alpha G_+ \to \mathrm{Ind}_{H^\perp}^{\widehat{G}}\left(B_{(G/H)_+} \times_\tau (G/H)_+\right) \to 0$$

in which $I_{H_+} \times_\alpha G_+$ is isomorphic to the ideal $D := \overline{\mathrm{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in I_{H_+}, x, y \in G_+\}$ of $B_{G_+} \times_\alpha G_+$ which is generated by $\left\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\right\}$. In section 4, we show that we can view the crossed product $B_{H_+} \times_\alpha H_+$ as a C^*-subalgebra of the crossed product $B_{G_+} \times_\alpha G_+$.

In Chapter 6, we talk about primitive ideals and irreducible representations of C*-algebras. We dedicate the first section to give the definitions and background results needed about irreducible representations and primitive ideals, then we prove some general lemma about irreducible representations. In section 2, we prove in Proposition 6.2.2 that there is a well-defined map $F : (H, \gamma) \mapsto \ker\left(Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H\right)$ from the disjoint union $\bigsqcup\{\widehat{H} : H \in \sum G\}$ to $\mathrm{Prim}\,(B_{G_+} \times_\alpha G_+)$. Then we show in Corollary 6.2.4 that for any irreducible representation of $B_{G_+} \times_\alpha G_+$, ρ is equivalent to $M(\gamma, \pi) \circ \Upsilon$ for some $\gamma \in \widehat{G}$.

Conventions. We use the standard conventions of our subject in this thesis. Thus, for example, homomorphisms between C^*-algebras are always $*$-preserving, ideals are always assumed to be closed and two-sided and representations of C^*-algebras are homomorphisms into the set $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ (or into some C^*-algebra A).

Background. It might be helpful to mention some non-trivial facts about C^*-algebras which we use frequently.

(i) Every C^*-algebra A has a faithful non-degenerate representation ([**35**, Theorem 24]).

(ii) Every homomorphism between C^*-algebras is norm-decreasing (and therefore continuous), and every injective homomorphism is norm preserving ([**8**, Corollary II.2.2.9]).

(iii) The range $\phi(A)$ of every homomorphism $\phi : A \to B$ between C^*-algebras is closed, and is therefore a C^*-subalgebra of B ([**35**, Corollary 23]).

CHAPTER 2

Preliminaries

In this chapter we will discuss the basic definitions of lattice and quasi-lattice ordered groups. We begin by showing that for a discrete group G under certain conditions there is a particular partial order on G which we will be using throughout this work.

2.1. Lattice and quasi-lattice ordered groups

Let G be a discrete group. A binary relation $\leq$ defined on G is a *partial order* if for $x, y, z \in G$, we have

(i) $x \leq x$ (reflexivity)
(ii) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry)
(iii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity)
(iv) $x \leq y \Rightarrow zx \leq zy$.

A non-empty group G together with a partial order $\leq$ is called a *partially ordered group*.

Definition 2.1.1. The *positive cone* of a partially ordered group G is the set of all positive elements of G ($x \in G$ is positive if $x \geq e$, where e is the identity of G) which is a semigroup.

A pair (G, G_+) consisting of a group G with identity e and a subsemigroup G_+ of G satisfying $G_+ \bigcap G_+^{-1} = \{e\}$ might be equipped with a relation "$\leq$" on G with respect to G_+ where $x \leq y$ if $x^{-1}y \in G_+$. This relation is a partial order on G which is left invariant. To see this, let $x, y, z \in G$. Note that $x^{-1}x = e \in G_+$ so the relation is reflexive. The relation is anti-symmetric, since if $x \leq y$ and $y \leq x$ then $x^{-1}y \in G_+$ and $y^{-1}x \in G_+$, which means that $x^{-1}y = (y^{-1}x)^{-1} \in G_+ \bigcap G_+^{-1} = \{e\}$

and hence $x = y$. To show the transitive condition, suppose that $x \leq y$ and $y \leq z$ then $x^{-1}y \in G_+$ and $y^{-1}z \in G_+$ and so $(x^{-1}y)(y^{-1}z) = x^{-1}z \in G_+$, since G_+ is a semigroup. About being left invariant, assume that $x \leq y$. Then $(zx)^{-1}(zy) = x^{-1}z^{-1}zy = x^{-1}y \in G_+$. Thus $zx \leq zy$.

Convention. From now on we use (G, G_+) to refer to the group G with the natural partial order $\leq$ on G determined by G_+.

Definition 2.1.2. The partially ordered group (G, G_+) is *quasi-lattice ordered* if every finite subset of G with an upper bound in G_+ has a least upper bound in G_+.

Equivalently, (G, G_+) is quasi-lattice ordered if and only if every element of G with an upper bound in G_+ has a least upper bound in G_+, and every two elements in G_+ with a common upper bound in G_+ have a least upper bound in G_+ [**32**, Section 2.1].

Definition 2.1.3. The partially ordered group (G, G_+) is said to be a *lattice-ordered* group if every two elements of G have a least upper bound in G.

We got our definition for lattice-ordered groups from [**18**] with slight changes to it because in [**18**] they insist that every two elements of G have a least upper bound and a greatest lower bound in G.

Notation. The least upper bound or *sup* of the elements x and y will be denoted by $x \vee y$.

Remark 2.1.4. There are two possible definitions for lattice-ordered groups the one we have in Definition 2.1.3 and the one mentioned in [**32**] which restricts our definition to elements from the semigroup G_+.

We now give two minor results to show that our definition implies the one in [**32**], and that every lattice-ordered group is a quasi-lattice-ordered group.

LEMMA 2.1.5. *Let (G, G_+) be a lattice-ordered group. Then every two elements of G_+ have a least upper bound in G_+.*

PROOF. If $x, y \in G_+$, then by assumption x and y have a least upper bound $x \vee y \in G$. Since $e \leq x \leq x \vee y$ then $x \vee y \in G_+$. $\square$

COROLLARY 2.1.6. *Suppose that (G, G_+) is a lattice-ordered group. Then (G, G_+) is a quasi-lattice ordered group.*

PROOF. For $x \in G$ we know that $e \leq x \vee e \in G_+$ is an upper bound for x in G_+. We claim that $x \vee e$ is a least upper bound for x in G_+. To see this suppose that $y \in G_+$ and $y \geq x$, then $y \geq e$. Hence $x \vee e \leq y$, and thus $e \vee x$ is the least upper bound of x in G_+.

To finish our proof, take $x, y \in G_+$ then Lemma 2.1.5 implies that $x \vee y \in G_+$. Thus (G, G_+) is quasi-lattice ordered group. $\square$

COROLLARY 2.1.7. *Suppose that (G, G_+) is a lattice-ordered group with G abelian. Then every two elements of G have a greatest lower bound in G.*

PROOF. Let $x, y \in G$. Then $-x, -y \in G$ and $(-x \vee -y) \in G$ (as (G, G_+) is lattice-ordered group). Since $-x, -y \leq (-x \vee -y)$ then $x \geq -(-x \vee -y)$ and $y \geq -(-x \vee -y)$. Hence $-(-x \vee -y)$ is a lower bound for x and y.

We claim that $-(-x \vee -y)$ is the greatest lower bound for x and y. To see this, suppose that there is $z \in G$ such that $z \leq x$ and $z \leq y$. Then $-z \geq -x$ and $-z \geq -y$, and so $-z \geq (-x \vee -y)$. Therefore $z \leq -(-x \vee -y)$ which means that $-(-x \vee -y)$ is the greatest lower bound for x and y. $\square$

LEMMA 2.1.8. *Let G be an abelian group. Then G is generated by its positive cone G_+ if and only if $G = G_+ - G_+$.*

PROOF. Suppose that G is generated by its positive cone G_+. Let $S = G_+ - G_+$ and notice that $G_+ \subset S$ and S is a subgroup of G. Then $G \subset S$ (as G is generated by G_+), and hence $G = S = G_+ - G_+$.

Conversely, suppose that $G = G_+ - G_+$ and take S to be a subgroup of G such that $G_+ \subset S$. Then $G_+^{-1} = -G_+ \subset S$ (as S is a subgroup of G) and so $G_+ - G_+ \subset S$. Hence $G = S$, and thus G is generated by its positive cone. $\square$

Definition 2.1.9. Let (G, G_+) be a quasi-lattice ordered group and $H \subset G_+$. Then H is said to be *hereditary* if for any $x, y \in G_+$, $e \leq x \leq y$ and $y \in H$ imply that $x \in H$ [**25**, Definition 2.3].

2.2. Examples

We now discuss some examples on the definitions of the previous section.

2.2.1. Totally ordered groups.

Suppose that (G, G_+) is totally ordered group. Then for any pair $x, y \in G$, either $x \leq y$ or $y \leq x$. Suppose that $x \leq y$. We know that $y \leq y$ therefore y is the least upper bound for x and y in G. Similarly for the case when $y \leq x$. Hence (G, G_+) is lattice-ordered group.

2.2.2. The concrete example $(\mathbb{Z}^2, \mathbb{N}^2)$.

If $G = \mathbb{Z}^2$ and $G_+ = \mathbb{N}^2$ $(0 \in \mathbb{N})$. Then we claim that the partially ordered group $(\mathbb{Z}^2, \mathbb{N}^2)$ is lattice-ordered.

To see this, fix $(m, n), (k, l) \in \mathbb{Z}^2$. Then as $\mathbb{Z}$ is totally ordered group then we know that $m \vee k$ and $n \vee l$ exists in $\mathbb{Z}$. Hence $(m \vee k, n \vee l) \in \mathbb{Z}^2$ is the least upper bound for (m, n) and (k, l) in $\mathbb{Z}^2$. Therefore $(\mathbb{Z}^2, \mathbb{N}^2)$ is a lattice-ordered group.

2.2.3. Continuous functions over an arbitrary topological space $C(X)$.

Let X be an arbitrary topological space and $C(X)$ be the additive group of all continuous functions from X to $\mathbb{R}$ (where $\mathbb{R}$ equipped with the usual topology); that is, $(f + g)(x) = f(x) + g(x)$ for all $x \in X$. Then $C(X)$ is a lattice-ordered group under the relation

$$f \leq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in X.$$

One can see that $C(X)$ is an abelian group and routine calculations show that the relation on $C(X)$ is a partial order.

To show that $C(X)$ is lattice-ordered, let $f, g \in C(X)$. Then $f(x), g(x) \in \mathbb{R}$ for all $x \in X$ and so $f(x) \vee g(x) = \max\{f(x), g(x)\}$ is in $\mathbb{R}$. Define $h(x) := f(x) \vee g(x)$ then h is a continuous function (this is a well-known fact so we skipped the details), we claim that h is the least upper bound of f, g in $C(X)$. To see this, notice that $f(x), g(x) \leq h(x)$ for all $x \in X$ and so $f, g \leq h$. Hence h is an upper bound for f, g. To show that h is the least upper bound for f, g. Suppose that there is $T \in C(X)$ such that $f, g \leq T$. Then $f(x), g(x) \leq T(x)$ and hence $f(x) \vee g(x) \leq T(x)$ for all $x \in X$. Thus $h \leq T$ and so $C(X)$ is lattice-ordered.

2.2.4. Vector spaces over ordered fields.

Let V be a vector space over an ordered field $\mathbb{S}$ with basis $\{b_i : i \in I\}$. Let $v, w \in V$; say $v = \sum_{i \in I_0} q_i b_i$ and $w = \sum_{i \in I_0} r_i b_i$ where I_0 is a finite subset of I and $q_i, r_i \in \mathbb{S}$.[1] Define

$$v \leq w \iff q_i \leq r_i \text{ for all } i \in I_0.$$

Then V is a lattice-ordered abelian group.

Since V is a vector space it is an abelian group and routine calculations show that the relation on V is a partial order. To see that V is lattice-ordered, fix $v, w \in V$. Then $v = \sum_{i \in I_0} q_i b_i$ and $w = \sum_{i \in I_0} r_i b_i$. Since $q_i, r_i \in \mathbb{S}$ for all $i \in I_0$ then $q_i \vee r_i = \max\{q_i, r_i\} \in \mathbb{S}$. Choose $z = \sum_{i \in I_0} (q_i \vee r_i) b_i \in V$. Then $v, w \leq z$. We claim that z is the least upper bound of v, w in V. To see this, suppose that $a = \sum_{i \in I_0} d_i b_i \in V$ satisfies $v, w \leq a$. Then $q_i, r_i \leq d_i$ and so $q_i \vee r_i \leq d_i$ for all $i \in I_0$. Hence $z \leq a$ and therefore V is a lattice-ordered group.

2.2.5. The continuous functions from X to $\mathbb{R} \bigcup \{\infty, -\infty\}$.

[1] The way we write v, w is possible because if $v = \sum_{j \in J_0} q_j b_j$ and $w = \sum_{k \in K_0} r_k b_k$. Then take $I_0 := J_0 \bigcup (K_0 \setminus J_0)$ and write v, w in terms of I_0.

Let X be a Hausdorff space and $D(X)$ be the set of all continuous functions from X to $\mathbb{R}\bigcup\{\infty, -\infty\}$ (with the order topology) with $\{x \in X : f(x) \notin \mathbb{R}\}$ nowhere dense.[2] Then $D(X)$ is an abelian lattice-ordered group under addition.

We know that $D(X)$ is an abelian group with partial order as in Example 2.2.3 and that for $f, g \in D(X)$, $h(x) = f(x) \vee g(x)$ exists. So we only need to show that $h \in D(X)$. Let

$$I := \{x \in X : h(x) = \pm\infty\} = \{x \in X : h(x) = \infty\} \bigcup \{x \in X : h(x) = -\infty\}.$$

The case when $h(x) = -\infty$ is straightforward by noticing that

$$\{x \in X : h(x) = -\infty\} \subseteq \{x \in X : f(x) = \infty \text{ or } -\infty\}.$$

Hence

$$\mathrm{Int}(\overline{\{x \in X : h(x) = -\infty\}}) \subseteq \mathrm{Int}(\overline{\{x \in X : f(x) = \infty \text{ or } -\infty\}}) = \emptyset.$$

Notice that with the order topology $\{x \in X : f(x) \notin \mathbb{R}\}$ is a closed set. So the closure of $\{x \in X : f(x) \notin \mathbb{R}\}$ is itself.

We now look at

$$\{x \in X : h(x) = \infty\} = \{x \in X : f(x) = \infty \text{ or } g(x) = \infty\},$$

which gives

$$\{x \in X : h(x) = \infty\} = \{x \in X : f(x) = \infty\} \bigcup \{x \in X : g(x) = \infty\}.$$

Since the union of two nowhere dense subsets is nowhere dense[3], then

$$\begin{aligned} \mathrm{Int}(\{x \in X : h(x) = \infty\}) &= \mathrm{Int}(\{x \in X : f(x) = \infty\} \bigcup \{x \in X : g(x) = \infty\}) \\ &= \emptyset, \end{aligned}$$

and hence

$$\begin{aligned} \mathrm{Int}(I) &= \mathrm{Int}(\{x \in X : h(x) = \infty\} \bigcup \{x \in X : h(x) = -\infty\}) \\ &= \emptyset. \end{aligned}$$

[2] A is nowhere dense if $\mathrm{Int}(\overline{A}) = \emptyset$.

[3] See Appendix A for the proof.

Thus $h \in D(X)$ and so $D(X)$ is lattice-ordered.

2.2.6. Direct products.

Consider the family $\{(G_i, G_{i_+}) : i \in I\}$ of quasi-lattice ordered groups. We claim that the direct product $G = \prod_{i \in I} G_i$ with a semigroup $G_+ = \prod_{i \in I} G_{i_+}$ is a quasi lattice ordered group with the order determined by G_+.

To show that this is true, we first show that $G_+ \bigcap G_+^{-1} = \{e\}$, where e is the identity of G. To see this, suppose that $x, y \in G_+$ such that $x = y^{-1}$. Then $x_i = y_i^{-1}$ for all $i \in I$ and hence $x_i = y_i = e_i$, e_i is the identity of G_i, since $G_{i_+} \bigcap G_{i_+}^{-1} = \{e_i\}$. Therefore $x = y = e$ and thus $G_+ \bigcap G_+^{-1} = \{e\}$.

Notice that for all $x, y \in G$, $(x^{-1}y)_i = x_i^{-1} y_i$. Hence $x^{-1}y \in G_+$ if and only if $x_i^{-1} y_i \in G_{i_+}$ for all $i \in I$. This implies that $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in I$.

Now we check the quasi-lattice order conditions. To start let $x \in G$ and suppose that there is $y \in G_+$ such that $x \leq y$. Then $x_i \leq y_i$ for all $i \in I$ and hence $x_i \leq \vee x_i \leq y_i$. Thus $x \leq (\vee x_i)_{i \in I} \leq y$ and $(\vee x_i)_{i \in I}$ is the least upper bound for x in G_+. Now, suppose that $z, w \in G_+$ have a common upper bound $s \in G_+$. Then we have that $z_i, w_i \leq s_i$ for all $i \in I$, and so $z_i \vee w_i \leq s_i$. Thus $z, w \leq (z_i \vee w_i)_{i \in I} \leq s$, which imply that $(z_i \vee w_i)_{i \in I}$ is the least common upper bound for z, w in G_+.

2.2.7. Examples on hereditary subsets.

Example 2.2.1. Let (G, G_+) be a quasi-lattice ordered group. For $x_0 \in G_+ \backslash \{e\}$, let $H = \{z \in G_+ : \text{there exists } n \in \mathbb{N} \text{ such that } z \leq x_0^n\}$. Then H is a hereditary subset of G_+.

To see this, notice that $H \subset G_+$ and suppose that $x, y \in G_+$, $e \leq x \leq y$ and $y \in H$. As $y \in H$ then there is $m \in \mathbb{N}$ such that $y \leq x_0^m$. Hence $x \leq x_0^m$ since $e \leq x \leq y$, thus $x \in H$.

Example 2.2.2. Let $(G, G_+) = (\mathbb{Z}^2, \mathbb{N}^2)$ be the lattice-ordered group in Example 2.2.2. The only hereditary subsemigroups of $\mathbb{N}^2$ are $\{(0,0)\}$, $\mathbb{N} \times \{0\}$, $\{0\} \times \mathbb{N}$ and $\mathbb{N}^2$.

To see this is true, notice first that all the subsemigroups mentioned are hereditary. Now suppose that H_+ is a hereditary subsemigroup of $\mathbb{N}^2$ which is different from $\{(0,0)\}$, $\mathbb{N} \times \{0\}$ and $\{0\} \times \mathbb{N}$. Then H_+ contains an element of the form (a,b) for some non-zero $a,b \in \mathbb{N}$. As H_+ is a hereditary subsemigroup we have $(1,0),(0,1) \in H_+$ and so $(m,n) \in H_+$ for all $m,n \in \mathbb{N}$. Hence $\mathbb{N}^2 \subset H_+$ and so $H_+ = \mathbb{N}^2$. The other case is when the element has the form $(b,0)$ for some $b \in \mathbb{N}$ and $b > 0$. Then as H_+ is hereditary, we have $(1,0) \in H_+$. But H_+ is a subsemigroup so it is closed under the binary operation, hence $\{(m,0) : m \in \mathbb{N}\} \subset H_+$; that is $\mathbb{N} \times \{0\} \subset H_+$. But H_+ is different from $\mathbb{N} \times \{0\}$, so H_+ should contain an element not in $\mathbb{N} \times \{0\}$, say $(0,k)$. If so then by the same argument above we deduce that $\{0\} \times \mathbb{N} \subset H_+$. Since H_+ is a subsemigroup then $(m,0)+(0,n) = (m,n) \in H_+$ for all $m,n \in \mathbb{N}$. Hence $\mathbb{N}^2 \subset H_+$ and therefore $H_+ = \mathbb{N}^2$.

CHAPTER 3

Semigroup dynamical systems and crossed products

We devote the first section of this chapter to stating the necessary definitions and required results about multiplier algebras and extendible homomorphisms. We use these results throughout this thesis. We have presented only the minimum amount of background needed to keep this thesis as self-contained as possible. More detailed treatments of the different topics can be found in the references provided.

3.1. Multiplier algebras and extendibility

DEFINITION 3.1.1. A *multiplier* (or *double centralizer*) of a C^*-algebra A is a pair (L, R) of bounded linear maps of A into A such that

$$L(a)b = L(ab), aR(b) = R(ab) \text{ and } R(a)b = aL(b).$$

LEMMA 3.1.2. *If (L, R) is a multiplier of a C^*-algebra A, then $\|L\| = \|R\|$* [**26**, Lemma 2.1.4].

If A is a C^*-algebra, we denote the set of its multipliers by $M(A)$. We define the norm of the multiplier (L, R) by $\|(L, R)\| := \|L\| = \|R\|$. If (L_1, R_1) and $(L_2, R_2) \in M(A)$, we define

$$(L_1, R_1)(L_2, R_2) := (L_1L_2, R_2R_1), \text{ and}$$
$$\lambda(L_1, R_1) + \mu(L_2, R_2) := (\lambda L_1 + \mu L_2, \lambda R_1 + \mu R_2),$$

as the product and addition respectively. Also define $(L, R)^* := (R^*, L^*)$, where $T^*(a) := (T(a^*))^*$ (for details see [**26**, §2.1]).

THEOREM 3.1.3. *Let A be a C^*-algebra, then $M(A)$ is a C^*-algebra under the multiplication, involution, norm and addition defined above* [**26**, theorem 2.1.5].

Remark 3.1.4. [**26**, § 2.1]

(i) We can identify any C^*-algebra A as a C^*-subalgebra of $M(A)$. In fact A is an ideal of $M(A)$.

(ii) $M(A)$ is unital (the multiplier $(\mathrm{id}, \mathrm{id})$ is the identity), and $A = M(A)$ if and only if A is unital.

Definition 3.1.5. Suppose that A is a C^*-algebra and $a \in A$. Let $\|.\|_a$ be the seminorm on $M(A)$ defined by $\|b\|_a = \|ab\| + \|ba\|$. The *strict topology* on $M(A)$ is the topology generated by the seminorms $\{\|.\|_a : a \in A\}$ ([**37**, Definition C.4]).

Definition 3.1.6. Let ϕ be a homomorphism from a C^*-algebra A into a C^*-algebra B. Then ϕ is *non-degenerate* if there exists an approximate identity $\{a_i\}$ in A such that $\phi(a_i)$ converges strictly to 1 in the multiplier algebra of B (i.e. $\phi(a_i)b \to b$ for all $b \in B$) [**1**, page 20]. A homomorphism $\psi : M(A) \to M(B)$ is *strictly continuous* if for each net (u_λ) in $M(A)$ converging to $u \in M(A)$ in the strict topology, we have $\psi(u_\lambda)$ converges strictly to $\psi(u)$ in $M(B)$.

Definition 3.1.7. A homomorphism $\psi : A \to M(B)$ (the multiplier algebra of B) is *extendible* if there exists an approximate identity $\{a_\lambda\}_{\lambda \in \Lambda}$ for A and a projection p_ψ in $M(B)$ such that $\psi(a_\lambda)$ converges strictly to p_ψ in $M(B)$. One can see that every non-degenerate homomorphism is extendible (by taking $p = 1_{M(B)}$). In [**1**, Proposition 3.1.1] Adji proves that the extendibility of $\psi : A \to M(B)$ is equivalent to the existence of a strictly continuous homomorphism extension $\overline{\psi}$ of $M(A)$ into $M(B)$. An extendible homomorphism satisfies the condition that $\psi(a_\lambda) \to \overline{\psi}(1_{M(B)})$ (see [**24**, §1]).

3.2. Definition of the crossed product

Let B be a unital C^*-algebra. An element $u \in B$ is said to be an *isometry* if $u^*u = 1_B$ [**26**, section 2.1].

DEFINITION 3.2.1. Let G_+ be a semigroup of a group G, B a unital C^*-algebra and V a map from G_+ to B. Then V is said to be an *isometric representation* of G_+ if it satisfies the following three conditions:

(i) $V_e = 1_B$;

(ii) $V_x^* V_x = 1_B$ for any $x \in G_+$;

(iii) $V_x V_y = V_{xy}$ for any $x, y \in G_+$.

DEFINITION 3.2.2. A *semigroup dynamical system* is a triple (A, G_+, α) where A is a C^*-algebra and α is an action of the semigroup G_+ on A by endomorphisms (i.e. $\alpha : G_+ \to \mathrm{End}(A)$ is a homomorphism such that α_x is an endomorphism of A for each $x \in G_+$). Two dynamical systems (A, G_+, α) and (B, G_+, β) are equivalent (isomorphic) if there is an isomorphism $\phi : A \to B$ such that $\phi \circ \alpha_x = \beta_x \circ \phi$ for all $x \in G_+$.

A *covariant representation* of a dynamical system (A, G_+, α) is a pair (π, V), where π is a non-degenerate representation of A on a Hilbert space $\mathcal{H}$, and V is an isometric representation of G_+ on $\mathcal{H}$ satisfying

$$\pi(\alpha_x(a)) = V_x \pi(a) V_x^* \text{ for all } x \in G_+, a \in A.$$

DEFINITION 3.2.3. A *crossed product* for a dynamical system (A, G_+, α) is a C^*-algebra B together with a non-degenerate homomorphism $i_A : A \to B$ and a homomorphism i_{G_+} of G_+ into the semigroup of isometries in $M(B)$ (the multiplier algebra of B) such that:

1. $i_A(\alpha_x(a)) = i_{G_+}(x) i_A(a) i_{G_+}(x)^*$ for $x \in G_+$ and $a \in A$;

2. for every covariant representation (π, V) of (A, G_+, α) there is a non-degenerate representation $\pi \times V$ of B such that

$$(\pi \times V) \circ i_A = \pi \text{ and } \overline{\pi \times V} \circ i_{G_+} = V;$$

3. B is generated by $\{i_A(a)\, i_{G_+}(x) : a \in A, x \in G_+\}$.

NOTATION.

- We write $A \times_\alpha G_+$ to denote the crossed product for the dynamical system (A, G_+, α).
- The homomorphisms (i_A, i_{G_+}) are the universal covariant representation.

REMARK 3.2.4.

(i) If A is unital and (A, G_+, α) has a non-trivial covariant representation, then it is shown in [**21**, Proposition 2.1] that there is a crossed product and it is unique up to isomorphism.

(ii) Let G_+ be an Ore semigroup (a cancellative semigroup which is right-reversible, in the sense that $G_+x \bigcap G_+y \neq \emptyset$ for all $x, y \in G_+$), (A, G_+, α) is a dynamical system with extendible endomorphisms and has a non-zero covariant representation. Then there exists a crossed product for the system which is unique up to isomorphism [**24**, Proposition 1.4].

REMARK 3.2.5. [**23**, page 11] If A has a unit, the representation π of Definition 3.2.2 and the homomorphism i_A of Definition 3.2.3 must be unital, and condition (2) of Definition 3.2.3 reduces to the existence of a unital representation $\pi \times V$ of B such that

$$(\pi \times V) \circ i_A = \pi \text{ and } (\pi \times V) \circ i_{G_+} = V.$$

DEFINITION 3.2.6. Let (G, G_+) be a lattice-ordered group and B a unital C^*-algebra. A *covariant isometric representation* V of G_+ by isometries in B is an isometric representation of G_+ which satisfies $V_x V_x^* V_y V_y^* = V_{x\vee y} V_{x\vee y}^*$ for all $x, y \in G_+$.

The covariant isometric representations are an important feature of research in crossed products because $B_{G_+} \times_\alpha G_+$ is universal for covariant isometric representations. That is, if W is a covariant isometric representation of G_+, then there is a representation π_W of B_{G_+} such that $\pi_W(1_x) = W_x W_x^*$, and the pair (π_W, W) is covariant for the dynamical system (B_{G_+}, G_+, α).

3.3. Covariant isometric representations of $\mathbb{N}^2$

In this section we prove two results about the covariant isometric representations of the semigroup $\mathbb{N}^2$.

LEMMA 3.3.1. *Suppose that V is an isometric representation of $\mathbb{N}^2$ on $\mathcal{H}$, then there exist isometries W, X on $\mathcal{H}$ such that $WX = XW$ and $V_{(m,n)} = W^m X^n$. Conversely, if W, X are isometries on $\mathcal{H}$ such that $WX = XW$, then $V_{(m,n)} := W^m X^n$ is an isometric representation of $\mathbb{N}^2$ on $\mathcal{H}$.*

PROOF. To see this, suppose first that V is an isometric representation of $\mathbb{N}^2$. Take $W := V_{(1,0)}$, $X := V_{(0,1)}$. Then W, X are isometries on $\mathcal{H}$ satisfying $WX = XW$ and $V_{(m,n)} = V_{(1,0)}^m V_{(0,1)}^n = W^m X^n$.

Suppose now that W, X are isometries on $\mathcal{H}$ such that $WX = XW$. Then calculations show that $V_{(m,n)} := W^m X^n$ is an isometric representation of $\mathbb{N}^2$ on $\mathcal{H}$. $\square$

COROLLARY 3.3.2. *Suppose that X, W are isometries on $\mathcal{H}$ such that $WX = XW$ and $WX^* = X^*W$. Then $V_{(m,n)} := W^m X^n$ is a covariant isometric representation of $\mathbb{N}^2$ on $\mathcal{H}$.*

PROOF. We have seen in Lemma 3.3.1 that V is an isometric representation of $\mathbb{N}^2$ on $\mathcal{H}$. So we only need to show the covariance condition here. Suppose that $(m, n), (k, l) \in \mathbb{N}^2$, then

$$V_{(m,n)} V_{(m,n)}^* V_{(k,l)} V_{(k,l)}^* = W^m X^n W^{*m} X^{*n} W^k X^l W^{*k} X^{*l}. \tag{3.3.1}$$

Now we consider the following different cases.

(i) If $m \le k$ and $n \le l$, then

$$\begin{aligned} L.H.S.\,(3.3.1) &= W^m X^n W^{*m} W^k X^{*n} X^l W^{*k} X^{*l}, \text{ since } WX^* = X^*W \\ &= W^m X^n W^{k-m} X^{l-n} W^{*k} X^{*l} \\ &= W^m W^{k-m} X^n X^{l-n} W^{*k} X^{*l} \\ &= V_{(k,l)} V_{(k,l)}^*. \end{aligned}$$

Similar argument works for $k \leq m$ and $l \leq n$.

(ii) If $m \geq k$ and $n \leq l$, then

$$\begin{aligned}
L.H.S.\,(3.3.1) &= W^m X^n W^{*m} X^{*n} X^l W^k W^{*k} X^{*l}, \text{ since } WX = XW \\
&= W^m X^n W^{*m} X^{l-n} W^k W^{*k} X^{*l} \\
&= W^m X^n X^{l-n} W^{*m} W^k W^{*k} X^{*l}, \text{ since } WX^* = X^*W \\
&= W^m X^l W^{*m-k} W^{*k} X^{*l} \\
&= W^m X^l W^{*m} X^{*l} \\
&= V_{(m,l)} V^*_{(m,l)}.
\end{aligned}$$

(iii) If $m \leq k$ and $n \geq l$, then

$$\begin{aligned}
L.H.S.\,(3.3.1) &= W^m X^n W^{*m} W^k X^{*n} X^l W^{*k} X^{*l}, \text{ since } WX^* = X^*W \\
&= W^m X^n W^{k-m} X^{*n-l} W^{*k} X^{*l} \\
&= W^m W^{k-m} X^n W^{*k} X^{*n-l} X^{*l}, \text{ since } WX = XW \\
&= W^k X^n W^{*k} X^{*n} \\
&= V_{(k,n)} V^*_{(k,n)}.
\end{aligned}$$

Thus V is a covariant isometric representation. □

3.4. The C^*-subalgebra B_{G_+} of $\ell^\infty(G_+)$

In this section we introduce the C^*-algebra B_{G_+} as in [**4**] and [**21**]. Then we prove some results which will be used later in this work.

Let (G, G_+) be a quasi-lattice ordered group. We now consider a particular C^*-subalgebra of $\ell^\infty(G_+)$. Denote by 1_x the function on G_+ defined by

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1}$$

The quasi-lattice condition gives

$$1_x 1_y = \begin{cases} 1_{x \vee y} & \text{if } x, y \text{ have a common upper bound,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.2}$$

The algebra $B_{G_+} := \overline{\operatorname{span}}\{1_x : x \in G_+\}$ is a commutative C^*-algebra with multiplication satisfying Equation (3.4.2) [**21**, page 418].

We now introduce our action α on the C^*-algebra B_{G_+} which was mentioned in [**21**, page 423].

LEMMA 3.4.1. *Let $B_{G_+} := \overline{\operatorname{span}}\{1_x : x \in G_+\}$. Then there is a left action α of G_+ on B_{G_+} by translation on $\ell^\infty(G_+)$ restricted to B_{G_+} defined by*

$$\alpha_s(f)(x) = \begin{cases} f(s^{-1}x) & \text{if } x \geq s, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.3}$$

The action α leaves B_{G_+} invariant and satisfies $\alpha_s(1_x) = 1_{sx}$ for $s, x \in G_+$.

PROOF. We first show that α is an action. To see this, fix $f, g \in \ell^\infty(G_+)$ and let $\lambda \in \mathbb{C}$. Then

(i) For $s, x \in G_+$

$$\begin{aligned} \alpha_s(\lambda f + g)(x) &= (\lambda f + g)(s^{-1}x) \\ &= \lambda f(s^{-1}x) + g(s^{-1}x) \\ &= \lambda\, \alpha_s(f)(x) + \alpha_s(x). \end{aligned}$$

Hence α_s is linear.

(ii)

$$\begin{aligned} \alpha_s(fg)(x) = fg(s^{-1}x) &= f(s^{-1}x)g(s^{-1}x) \\ &= \alpha_s(f)(x)\, \alpha_s(g)(x). \end{aligned}$$

(iii) For $y, z \in G_+$ we have

$$\begin{aligned} \alpha_y \circ \alpha_z(f)(x) = \alpha_y\big(\alpha_z(f)(x)\big) &= \alpha_z(f)(y^{-1}x) \\ &= f(z^{-1}y^{-1}x) \\ &= \alpha_{yz}(f)(x). \end{aligned}$$

Thus α is a left action. We now show that for $s, x \in G_+$, $\alpha_s(1_x) = 1_{sx}$. To see this, fix $y \in G_+$. Then

$$\begin{aligned}\alpha_s(1_x)(y) &= 1_x(s^{-1}y)\\ &= \begin{cases} 1 & \text{if } x \le s^{-1}y,\\ 0 & \text{otherwise.}\end{cases}\\ &= \begin{cases} 1 & \text{if } sx \le y,\\ 0 & \text{otherwise.}\end{cases}\\ &= 1_{sx}(y).\end{aligned}$$

Hence α leaves B_{G_+} invariant and satisfies $\alpha_s(1_x) = 1_{sx}$ for $s, x \in G_+$. □

COROLLARY 3.4.2. *Let (G, G_+) be a quasi-lattice ordered group and A a unital C^*-algebra. If $\{L_x : x \in G_+\}$ are projections in A such that*

$$L_e = 1 \text{ and } L_x L_y = L_{x\vee y} \text{ for } x, y \in G_+, \tag{3.4.4}$$

then there is a unital homomorphism $\pi_L : B_{G_+} \to A$ such that $\pi_L(1_x) = L_x$.

PROOF. We know that the C^*-algebra A has a non-degenerate faithful representation ψ on a Hilbert space $\mathcal{H}$. Then $\psi \circ L : G_+ \to B(\mathcal{H})$ satisfies the hypothesis of [**21**, Proposition 1.3] and hence there is a representation $\phi : B_{G_+} \to B(\mathcal{H})$ which satisfies $\phi(1_x) = \psi(L_x)$. To define $\psi^{-1} \circ \phi$ we need that $\operatorname{range}\phi \subset \operatorname{range}\psi$. To see this, we only need to observe that $\operatorname{range}\psi$ is a C^*-subalgebra of $B(\mathcal{H})$ and contains every $\phi(1_x)$, hence it contains $\phi(\operatorname{span}\{1_x\}) = \operatorname{span}\{\phi(1_x)\}$ and hence by continuity it contains $\phi(\overline{\operatorname{span}}\{1_x\}) = \phi(B_{G_+})$. Take $\pi_L := \psi^{-1} \circ \phi$. Then $\pi_L : B_{G+} \to A$ and $\pi_L(1_x) = \psi^{-1}(\phi(1_x)) = \psi^{-1}(\psi(L_x)) = L_x$ and $\pi_L(1_e) = L_e = 1$. □

REMARK 3.4.3. To see that the crossed product $B_{G_+} \times_\alpha G_+$ is non-trivial (non-zero C^*-algebra) we obtain a specific covariant pair for the dynamical system (B_{G_+}, G_+, α) with the group G abelian. Let each $f \in B_{G_+}$ act as the multiplication operator M_f on $\ell^2(G_+)$, and T_x be the isometry on $\ell^2(G_+)$ defined by $T_x(\delta_y) = \delta_{x+y}$, where

$$\delta_s(x) = \begin{cases} 1 & \text{if } x = s,\\ 0 & \text{otherwise.}\end{cases} \tag{3.4.5}$$

Then $M_{1_x} = T_x T_x^*$. Since

$$M_{\alpha_x(1_y)} = M_{1_{x+y}} = T_{x+y}T_{x+y}^* = T_x T_y T_y^* T_x^* = T_x M_{1_y} T_x^*,$$

the pair (M, T) is covariant. Thus the crossed product $B_{G_+} \times_\alpha G_+$ is non-trivial (for more details see [**21**, §2]).

LEMMA 3.4.4. *Let (G, G_+) be a lattice-ordered group, α be the action of G_+ in Lemma 3.4.1 and $(i_{B_{G_+}}, i_{G_+})$ denote the universal representation of the dynamical system (B_{G_+}, G_+, α). Then the homomorphism i_{G_+} is a covariant isometric representation of G_+.*

PROOF. We know by definition of i_{G_+} that it is an isometric representation of G_+. So we need only to show the condition of covariance. Fix $x, y \in G_+$, then

$$\begin{aligned} i_{G_+}(x)i_{G_+}(x)^* i_{G_+}(y)i_{G_+}(y)^* &= i_{B_{G_+}}(\alpha_x(1_e))i_{B_{G_+}}(\alpha_y(1_e)) \\ &= i_{B_{G_+}}(1_x 1_y) \\ &= i_{B_{G_+}}(1_{x\vee y}) \\ &= i_{B_{G_+}}(\alpha_{x\vee y}(1_e)) \\ &= i_{G_+}(x \vee y) i_{G_+}(x \vee y)^*. \end{aligned}$$

□

REMARK 3.4.5.

(i) [**21**, Corollary 2.4] If (G, G_+) is a quasi-lattice ordered group, then the maps $i_{B_{G_+}}$ and i_{G_+} are faithful.

(ii) Because the group G is a discrete group, its dual $\widehat{G} := \{\gamma : G \to \mathbb{T} : \gamma \text{ is a homomorphism}\}$ is a compact group under pointwise multiplication [**17**, Proposition 4.4].

LEMMA 3.4.6. *Let (G, G_+) be a quasi-lattice ordered group with G abelian, α be an action of the semigroup G_+ by endomorphisms of a unital C^*-algebra A and $\widehat{G}$*

be the compact group in Remark 3.4.5. Then there is a continuous action

$$\widehat{\alpha} : \widehat{G} \to \mathrm{Aut}(A \times_\alpha G_+)$$

satisfying $\widehat{\alpha}_\gamma(i_A(a)) = i_A(a)$ *and* $\widehat{\alpha}_\gamma(i_{G_+}(x)) = \overline{\gamma(x)}\, i_{G_+}(x)$ *for all* $x \in G_+, a \in A$. *Further, if* $\gamma_n \to \gamma$ *pointwise in* $\widehat{G}$, *then* $\widehat{\alpha}_{\gamma_n}(b) \to \widehat{\alpha}_\gamma(b)$ *for all* $b \in A \times_\alpha G_+$*; the action* $\widehat{\alpha}$ *is called the dual action of* $\widehat{G}$ *on* $A \times_\alpha G_+$.

Remark. In the proof of [**4**, Theorem 1.2] they mentioned the existence of the continuous action $\widehat{\alpha}$ in the case of totally ordered abelian groups, here we prove in detail that this action exists for any partially ordered group G under certain conditions.

Proof. Fix $\gamma \in \widehat{G}$ and define $(\overline{\gamma}\, i_{G_+})(x) = \overline{\gamma(x)}\, i_{G_+}(x)$. We claim that $(A \times_\alpha G_+, i_A, \overline{\gamma}\, i_{G_+})$ is a crossed product for the dynamical system (A, G_+, α). To see this, we need to check the followings.

(i) Let $x \in G_+$, then

$$\begin{aligned}\big((\overline{\gamma}\, i_{G_+})(x)\big)^*\big((\overline{\gamma}\, i_{G_+})(x)\big) &= \overline{\overline{\gamma(x)}}\, i_{G_+}(x)^* \overline{\gamma(x)}\, i_{G_+}(x)\\ &= |\gamma(x)|^2\, i_{G_+}(x)^* i_{G_+}(x)\\ &= 1_{A\times_\alpha G_+}.\end{aligned}$$

Hence $\overline{\gamma}\, i_{G_+}$ is an isometric representation.

(ii) For $a \in A, x \in G_+$, then

$$\begin{aligned} i_A\big(\alpha_x(a)\big) &= i_{G_+}(x) i_A(a) i_{G_+}(x)^*\\ &= |\gamma(x)|^2 i_{G_+}(x) i_A(a) i_{G_+}(x)^*\\ &= \overline{\gamma(x)}\, i_{G_+}(x) i_A(a) \overline{\overline{\gamma(x)}}\, i_{G_+}(x)^*\\ &= \big(\overline{\gamma(x)}\, i_{G_+}(x)\big) i_A(a) \big(\overline{\gamma(x)}\, i_{G_+}(x)\big)^*.\end{aligned}$$

(iii) Suppose that (π, V) is a covariant representation of the dynamical system (A, G_+, α) on a Hilbert space $\mathcal{H}$. Then $(\pi, \gamma V)$ is also a covariant representation for (A, G_+, α). Hence there is a unital representation

$$\rho : A \times_\alpha G_+ \to B(\mathcal{H}) \text{ such that } \rho \circ i_A = \pi \text{ and } \rho \circ i_{G_+} = \gamma V.$$

For $x \in G_+$ we have

$$\begin{aligned}\rho \circ \big(\overline{\gamma}\, i_{G_+}\big)(x) &= \overline{\gamma(x)}\rho\big(i_{G_+}(x)\big)\\ &= \overline{\gamma(x)}\gamma(x)V(x)\\ &= V(x).\end{aligned}$$

Thus ρ is a unital representation of $A \times_\alpha G_+$ on $\mathcal{H}$ satisfying $\rho \circ i_A = \pi$ and $\rho \circ \overline{\gamma}\, i_{G_+} = V$.

(iv) Since the C^*-algebra A is unital, then [**21**, Proposition 2.1] implies that $\big\{i_A(a) : a \in A\big\} \bigcup \big\{i_{G_+}(x) : x \in G_+\big\}$ generates $A \times_\alpha G_+$. Since $\gamma(x) \in \mathbb{T}$ for all $x \in G_+$, $\big\{i_A(a) : a \in A\big\} \bigcup \big\{\overline{\gamma(x)} i_{G_+}(x) : x \in G_+\big\}$ generates $A \times_\alpha G_+$.

By uniqueness of the crossed product there exists an isomorphism, say $\widehat{\alpha}_\gamma : A \times_\alpha G_+ \to A \times_\alpha G_+$ such that $\widehat{\alpha}_\gamma(i_A(a)) = i_A(a)$ and $\widehat{\alpha}_\gamma(i_{G_+}(x)) = \overline{\gamma(x)}\, i_{G_+}(x)$ for all $x \in G_+, a \in A$.

Our next step is to check that the map $\gamma \mapsto \widehat{\alpha}_\gamma$ is a homomorphism. To do so we will check that $\big(\widehat{\alpha}_\gamma \circ \widehat{\alpha}_\chi\big)(i_A(a)) = \widehat{\alpha}_{\gamma\chi}(i_A(a))$ and $\big(\widehat{\alpha}_\gamma \circ \widehat{\alpha}_\chi\big)(i_{G_+}(x)) = \widehat{\alpha}_{\gamma\chi}(i_{G_+}(x))$. On one hand, both $\widehat{\alpha}_\gamma \circ \widehat{\alpha}_\chi\big(i_A(a)\big)$ and $\widehat{\alpha}_{\gamma\chi}(i_A(a))$ equal $i_A(a)$. On the other hand,

$$\begin{aligned}\big(\widehat{\alpha}_\gamma \circ \widehat{\alpha}_\chi\big)(i_{G_+}(x)) = \widehat{\alpha}_\gamma\big(\widehat{\alpha}_\chi(i_{G_+}(x))\big) &= \widehat{\alpha}_\gamma(\overline{\chi(x)}\, i_{G_+}(x))\\ &= \overline{\chi(x)}\widehat{\alpha}_\gamma(i_{G_+}(x)), \widehat{\alpha}_\gamma \text{ is a homomorphism}\\ &= \overline{\chi(x)}\ \overline{\gamma(x)} i_{G_+}(x)\\ &= \widehat{\alpha}_{\gamma\chi}(i_{G_+}(x)).\end{aligned}$$

So the map $\gamma \mapsto \widehat{\alpha}_\gamma$ is a homomorphism.

To establish continuity, fix $\gamma \in \widehat{G}, b \in A \times_\alpha G_+$ and $\varepsilon > 0$. Choose a finite sum

$$c := \sum_{x,y} \lambda_{x,y} i_{G_+}(x)^* i_A(a_{x,y}) i_{G_+}(y) \text{ such that } \|b - c\| < \varepsilon/3.$$

The map $\gamma \mapsto \gamma(x)\overline{\gamma(y)}$ is continuous, because it is the the product of two continuous functions. As scalar multiplication is also continuous then so is the map

$$\gamma \mapsto \widehat{\alpha}_\gamma(c) = \sum_{x,\,y} \lambda_{x,y}\gamma(x)\overline{\gamma(y)} i_{G_+}(x)^* i_A(a_{x,y}) i_{G_+}(y).$$

Now choose a neighborhood N of γ such that if $\sigma \in N$, then $\|\widehat{\alpha}_\sigma(c) - \widehat{\alpha}_\gamma(c)\| < \varepsilon/3$. Then for $\chi \in N$, we have

$$\begin{aligned}
\|\widehat{\alpha}_\chi(b) - \widehat{\alpha}_\gamma(b)\| &= \|\widehat{\alpha}_\chi(b) - \widehat{\alpha}_\chi(c) + \widehat{\alpha}_\chi(c) - \widehat{\alpha}_\gamma(c) + \widehat{\alpha}_\gamma(c) - \widehat{\alpha}_\gamma(b)\| \\
&\leq \|\widehat{\alpha}_\chi(b) - \widehat{\alpha}_\chi(c)\| + \|\widehat{\alpha}_\chi(c) - \widehat{\alpha}_\gamma(c)\| + \|\widehat{\alpha}_\gamma(c) - \widehat{\alpha}_\gamma(b)\| \\
&= \|\widehat{\alpha}_\chi(b-c)\| + \|\widehat{\alpha}_\gamma(b-c)\| + \|\widehat{\alpha}_\chi(c) - \widehat{\alpha}_\gamma(c)\| \\
&= \|b-c\| + \|b-c\| + \|\widehat{\alpha}_\chi(c) - \widehat{\alpha}_\gamma(c)\|\,, \text{ each } \widehat{\alpha}_\gamma \text{ is an automorphism} \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Thus $\widehat{\alpha}$ is continuous.

Finally, suppose that $\gamma_n \to \gamma$ pointwise in $\widehat{G}$. To prove that $\widehat{\alpha}_{\gamma_n}(b) \to \widehat{\alpha}_\gamma(b)$ for all $b \in A \times_\alpha G_+$. We will check that on generators. On one hand

$$\widehat{\alpha}_{\gamma_n}(i_A(a)) = i_A(a) = \widehat{\alpha}_\gamma(i_A(a)).$$

On the other hand,

$$\widehat{\alpha}_{\gamma_n}(i_{G_+}(x)) = \overline{\gamma_n(x)}\, i_{G_+}(x) \to \overline{\gamma(x)}\, i_{G_+}(x) = \widehat{\alpha}_\gamma(i_{G_+}(x)).$$

□

CHAPTER 4

Extendibly invariant Ideals

In this chapter we discuss the definition of extendible α-invariant ideals of C^*-algebras. Then we introduce a specific extendibly α-invariant ideal of the C^*-algebra B_{G_+}.

DEFINITION 4.0.7. Suppose that α is an extendible endomorphism of a C^*-algebra A and I is an ideal of A. Let $\psi : A \to M(I)$ denote the canonical non-degenerate homomorphism defined by $\psi(a)b = ab, a \in A, b \in I$. Let $\overline{\psi}$ be the strictly continuous extension of $M(A)$ into $M(I)$. Then I is called *extendibly α-invariant* if it is α-invariant, in the sense that $\alpha(I) \subset I$, and there exists an approximate identity (i_λ) for I such that $\alpha(i_\lambda)$ converges strictly to $\overline{\psi}(\overline{\alpha}(1_{M(A)}))$ in $M(I)$ [**24**, Definition 1.6].

REMARK. The homomorphism $\psi : A \to M(I)$ in the above definition is non-degenerate because if $\{a_\lambda\}$ is an approximate identity of A, then for any $b \in I$ we have

$$\psi(a_\lambda)b = a_\lambda b \to b = 1_{M(I)}b\,.$$

4.1. Construction of the ideal I_{H_+}

Henceforth we assume that G_+ is the positive cone of a partially ordered discrete abelian group G, (G, G_+) is a lattice-ordered group and H_+ is a hereditary subsemigroup of G_+. Moreover, we change to additive notation rather than multiplicative for the group operation. We now introduce the ideal I_{H_+} of B_{G_+} and prove that it is extendibly α-invariant.

Lemma 4.1.1. *Let (G, G_+) be a lattice-ordered group with G abelian and let H_+ be a hereditary subsemigroup of G_+. Then*

$$I_{H_+} = \overline{\operatorname{span}}\big\{1_x - 1_{x+h} : h \in H_+,\, x \in G_+\big\}$$

is a closed ideal in B_{G_+}.

Proof. Let $x \in G_+$, $h \in H_+$ and take $1_y \in B_{G_+}$. Then

$$1_y(1_x - 1_{x+h}) = 1_y 1_x - 1_y 1_{x+h} = 1_{y\vee x} - 1_{y\vee(x+h)}. \tag{4.1.1}$$

To show that $1_y(1_x - 1_{x+h}) \in I_{H_+}$, we show first that $y \vee (x+h) \le (y \vee x) + h$. To see this, note that $(y \vee x) + h \ge y \vee x$ and $y \vee x \ge y$. Therefore $(y \vee x) + h \ge y$. Now as $y \vee x \ge x$ then $(y \vee x) + h \ge x + h$. Thus $y \vee (x + h) \le (y \vee x) + h$. Observe that $y \vee (x + h) - (y \vee x) \le h$ (left invariant property of the order $\le$ on G) also $y \vee (x+h) - (y \vee x) \in G_+$. As H_+ is hereditary, then $y \vee (x+h) - (y \vee x) \in H_+$. Hence by (4.1.1), $1_y(1_x - 1_{x+h})$ is an element of I_{H_+} and by continuity of multiplication in B_{G_+} we conclude that I_{H_+} is a closed ideal in B_{G_+}. $\square$

Remark 4.1.2. Let (G, G_+) be a lattice-ordered group and α be the action of G_+ in Lemma 3.4.1. Since B_{G_+} is a unital C^*-algebra with unit $1_{B_{G_+}} = 1_e$, then $M(B_{G_+}) = B_{G_+}$ and so each α_x is an extendible endomorphism of B_{G_+}.

Lemma 4.1.3. *Let (G, G_+) be a lattice-ordered group with G abelian and let H_+ be a hereditary subsemigroup of G_+. Then the set*

$$D = \big\{(F, h) : F \text{ is a finite subset of } G_+,\, h \in H_+\big\}$$

is a directed set when $(F, h) \le (F', h') \Longleftrightarrow F \subset F'$ and $h \le h'$.

Proof. To show that the given relation directs D we need to show that it is reflexive, transitive and any two elements of D have an upper bound. To see this, consider $(F, h), (F', h'), (F'', h'') \in D$. This relation is reflexive, since $F \subset F$ and $h \le h$. It is transitive, since if $(F, h) \le (F', h')$ and $(F', h') \le (F'', h'')$ we have $F \subset F'$, $h \le h'$ and $F' \subset F''$, $h' \le h''$ and hence $F \subset F''$ and $h \le h''$. For the last condition, take any (F, h), $(F', h') \in D$. Choose $F'' = F \cup F'$ and $h'' = h \vee h'$.

Then $F'' \in D$ and because $h'' \leq h + h'$ and H_+ is hereditary then $h'' \in H_+$. Thus $(F'', h'') \geq (F, h)$ and $(F'', h'') \geq (F', h')$. $\square$

LEMMA 4.1.4. *Let (G, G_+) be a lattice-ordered group with G abelian, H_+ be a hereditary subsemigroup of G_+, D denote the directed set in Lemma 4.1.3 and let $E(F, h) := \{y \in G_+ : \exists\, x \in F$ such that $y \geq x$ and $y \not\geq x + h\}$. Then*

$$\mathbb{1}_{(F,\,h)} := \sum_{\emptyset \neq A \subset F} (-1)^{|A|+1} \prod_{x \in A} (1_x - 1_{x+h}) = \chi_{E(F,\,h)}$$

for all $(F, h) \in D$.

PROOF. Fix $(F,\, h) \in D$ and $y \in G_+$. For $x \in G_+$ and $h \in H_+$ we have

$$(1_x - 1_{x+h})(y) = \begin{cases} 1 & \text{if } y \geq x \text{ and } y \not\geq x + h, \\ 0 & \text{if } y \geq x + h \text{ or } y \not\geq x. \end{cases}$$

Define $E(x, h) := \{y : \, y \geq x \text{ and } y \not\geq x + h\}$, then we can write

$$1_x - 1_{x+h} = \chi_{\{y :\, y \geq x \text{ and } y \not\geq x+h\}} = \chi_{E(x,\,h)} \tag{4.1.2}$$

Using equation (4.1.2) we have

$$\begin{aligned} \mathbb{1}_{(F,\,h)}(y) &= \sum_{\emptyset \neq A \subset F} (-1)^{|A|+1} \prod_{x \in A} (1_x - 1_{x+h})(y) \\ &= \sum_{\emptyset \neq A \subset F} (-1)^{|A|+1} \prod_{x \in A} \chi_{E(x,\,h)}(y) \\ &= \sum_{\emptyset \neq A \subset F} (-1)^{|A|+1} \chi_{\bigcap_{x \in A} E(x,\,h)}(y). \end{aligned}$$

Suppose first that $B = \{x \in F : y \in E(x, h)\}$ is non empty. Then $y \in \bigcap_{x \in B} E(x, h)$ and $y \notin \bigcap_{x \in A} E(x, h)$ for $A \not\subseteq B$, and hence

$$
\begin{aligned}
\mathbb{1}_{(F,\,h)}(y) &= \sum_{\emptyset \neq A \subset B} (-1)^{|A|+1} \\
&= \sum_{k=1}^{|B|} (-1)^{k+1} \binom{|B|}{k} \\
&= -\Bigg(\sum_{k=1}^{|B|} \binom{|B|}{k} (-1)^k \Bigg) \\
&= -\Bigg(\Big(\sum_{k=0}^{|B|} \binom{|B|}{k} (-1)^k \Big) - 1 \Bigg) \\
&= -\Big(\big(1 + (-1)\big)^{|B|} - 1 \Big), \text{ by the binomial theorem} \\
&= 1.
\end{aligned}
$$

Now suppose $B = \{x \in F : y \in E(x, h)\} = \emptyset$ then $\mathbb{1}_{(F,\,h)}(y) = 0$. Thus

$$
\mathbb{1}_{(F,\,h)} = \chi_{\{y\,:\,\exists\, x \in F \text{ such that } y \geq x \text{ and } y \not\geq x+h\}} = \chi_{E(F,\,h)}.
$$

$\square$

PROPOSITION 4.1.5. *Let (G, G_+) be a lattice-ordered group with G abelian, H_+ be a hereditary subsemigroup of G_+, I_{H_+} be the ideal in Lemma 4.1.1 and D denote the directed set in Lemma 4.1.3. Then the set*

$$
C_I = \Big\{ \mathbb{1}_{(F,\,h)} = \sum_{\emptyset \neq A \subset F} (-1)^{|A|+1} \prod_{x \in A} (1_x - 1_{x+h}) : (F, h) \in D \Big\}
$$

is an approximate identity for I_{H_+}.

PROOF. To prove this proposition we need to check that C_I satisfies the conditions of approximate identity. Firstly, if $\mathbb{1}_{(F,\,h)} \in C_I$ then $\mathbb{1}_{(F,\,h)} = \chi^*_{E(F,\,h)} \chi_{E(F,\,h)} \geq 0$ and $\|\mathbb{1}_{(F,\,h)}\| = \sup |\mathbb{1}_{(F,\,h)}(y)| \leq 1$.

Secondly, suppose that $(F,\, h) \leq (F',h')$. For $y \in G_+$, we know from Lemma 4.1.4 that $\mathbb{1}_{(F,\,h)}(y) = 1$ if and only if there exists $x \in F$ such that $y \geq x$ and $y \not\geq x + h$. As $F \subset F'$ then $x \in F'$ and so $x + h \leq x + h'$ (since $h \leq h'$). Therefore if $y \in E(F,\, h)$ then $y \in E(F',\, h')$. Hence, $\mathbb{1}_{(F,\,h)} \leq \mathbb{1}_{(F',\,h')}$.

We now show that $\mathbb{1}_{(F,h)} f \to f$ for all $f \in I_{H_+}$. To do so we will show first that it is true for $1_x - 1_{x+k} \in I_{H_+}$ and then we show it for any $f \in I_{H_+}$. For $1_x - 1_{x+k} \in I_{H_+}$, take $F = \{x\} \subset G_+$ and choose $h = k \in H_+$. Then

$$\begin{aligned}
\mathbb{1}_{(F,k)}(1_x - 1_{x+k}) &= \chi_{\{y:\, \exists z \in F \text{ such that } y \geq z \text{ and } y \not\geq z+k\}} \chi_{\{w:\, w \geq x \text{ and } w \not\geq x+k\}} \\
&= \chi_{\{w:\, w \geq x \text{ and } w \not\geq x+k\}}, \text{ as } F = \{x\} \\
&= 1_x - 1_{x+k}.
\end{aligned}$$

For $f \in I_{H+}$ we know that f is a limit of finite sums of elements in the spanning set of I_{H_+}. Take $\varepsilon > 0$ and choose $f_0 \in \operatorname{span}\{1_x - 1_{x+h} : x \in G_+, h \in H_+\}$ such that $\|f - f_0\| < \varepsilon/3$. Choose $(F, h_0) \in D$ such that for $(F', k) \geq (F, h_0)$ then $\mathbb{1}_{(F',k)} f_0 = f_0$. Therefore, for $(F'', k) \geq (F, h_0)$ we have

$$\begin{aligned}
\|\mathbb{1}_{(F'',k)} f - f\| &= \|(\mathbb{1}_{(F'',k)}(f - f_0) + \mathbb{1}_{(F'',k)} f_0 - f_0 + f_0 - f\| \\
&\leq \|\mathbb{1}_{(F'',k)}(f - f_0)\| + \|\mathbb{1}_{(F'',k)} f_0 - f_0\| + \|f - f_0\| \\
&< \|\mathbb{1}_{(F'',k)}\| \, \|f - f_0\| + \varepsilon/3 + \varepsilon/3 \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

Thus C_I is an approximate identity for I_{H_+}. □

Corollary 4.1.6. *Let (G, G_+) be a lattice-ordered group with G abelian, H_+ be a hereditary subsemigroup of G_+, I_{H_+} be the ideal in Lemma 4.1.1 and α denote the action in Remark 4.1.2. Then I_{H_+} is an extendibly α_z-invariant ideal of B_{G_+} for all $z \in G_+$.*

Proof. For $(1_x - 1_{x+k})$ in the spanning set of I_{H_+} we have

$$\begin{aligned}
\alpha_z(1_x - 1_{x+k}) &= \alpha_z(1_x) - \alpha_z(1_{x+k}), \text{ since } \alpha_z \text{ is a homomorphism} \\
&= 1_{z+x} - 1_{(z+x)+k} \in I_{H_+}.
\end{aligned}$$

By linearity and continuity of α_z we conclude $\alpha_z(f) \in I_{H_+}$, for all $f \in I_{H_+}$. Hence I_{H_+} is α_z-invariant. Our next step is to show that for $(1_x - 1_{x+k}) \in I_{H_+}$, the approximate identity $\{\mathbb{1}_{(F,h)}\}$ in Lemma 4.1.5 satisfies

$$\alpha_z(\mathbb{1}_{(F,h)})(1_x - 1_{x+k}) \to \psi\big(\alpha_z(1_{B_{G_+}})\big)(1_x - 1_{x+k}).$$

Since $\alpha_z(1_{B_{G_+}}) = 1_z$, this is equivalent to

$$\alpha_z(\mathbb{1}_{(F,h)})(1_x - 1_{x+k}) \to \psi(1_z)(1_x - 1_{x+k}) = 1_z(1_x - 1_{x+k})\,. \tag{4.1.3}$$

To prove (4.1.3) it is enough to find $(F,h) \in D$ (directed set in Lemma 4.1.3) such that if $(F', h') \geq (F, h)$, then

$$\alpha_z(\mathbb{1}_{(F',h')})(1_x - 1_{x+k}) = 1_z(1_x - 1_{x+k})\,. \tag{4.1.4}$$

We claim that $(F, h) = \big(\{(x \vee z) - z\}, k\big)$ suffices. Notice that

$$1_x - 1_{x+k} = \chi_{\{y\colon y \geq x \text{ and } y \ngeq x+k\}}.$$

Fix $(F', h') \geq (F, h)$ and $y \in G_+$. If $y \ngeq x$ or $y \geq x + k$, then

$$\alpha_z(\mathbb{1}_{(F',h')})(1_x - 1_{x+k})(y) = 0 = 1_z(1_x - 1_{x+k})(y)\,.$$

So we are left to consider y which satisfies $y \geq x$ and $y \ngeq x+k$. Then $(1_x - 1_{x+k})(y) = 1$, and so we need to check that $\alpha_z(\mathbb{1}_{(F',h')})(y) = 1_z(y)$. Using Lemma 4.1.4 we have

$$\begin{aligned}\alpha_z(\mathbb{1}_{(F',h')})(y) &= \mathbb{1}_{(F',h')}(y - z)\\ &= \begin{cases} 1 & \text{if there is } x' \in F' \text{ such that } y - z \geq x' \text{ and } y - z \ngeq x' + h', \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now we check what happens when $y \ngeq z$ and when $y \geq z$. If $y \ngeq z$ then $y \ngeq x' + z$ for all $x' \in F'$ and therefore $1_z(y) = 0 = \alpha_z(\mathbb{1}_{(F',h')})(y)$. On the other hand, if $y \geq z$ then $1_z(y) = 1$. As $y \geq x$ then $y \geq x \vee z$. So choose $x' = (x \vee z) - z \in F \subset F'$ then $y \geq x' + z$ and since $y \ngeq x + k$ then $y \ngeq (x \vee z) + k$. Therefore $y \ngeq (x \vee z) + h' = x' + z + h'$. So $\alpha_z(\mathbb{1}_{(F',h')})(y) = 1$ and hence the functions in (4.1.4) agree at every $y \in G_+$.

To finish off, let $b \in I_{H+}$ we know that b is a limit of finite sums of elements in the spanning set of I_{H_+}. Take $\varepsilon > 0$ and choose $b_0 \in \operatorname{span}\big\{1_x - 1_{x+h} : x \in G_+, h \in H_+\big\}$ such that $\|b - b_0\| < \varepsilon/3$. Choose $(F, h_0) \in D$ such that for $(F', k) \geq (F, h_0)$ then

$\alpha_z(\mathbb{1}_{(F', k)})b_0 = 1_z\, b_0$ (this is true by (4.1.4)). Therefore, for $(F', k) \geq (F, h_0)$ we have

$$\begin{aligned}\|\alpha_z(\mathbb{1}_{(F', k)})b - 1_z\, b\| &= \|\alpha_z(\mathbb{1}_{(F', k)})(b - b_0) + \alpha_z(\mathbb{1}_{(F', k)})b_0 - 1_z\, b_0 + 1_z\, b_0 - 1_z\, b\| \\ &\leq \|\alpha_z(\mathbb{1}_{(F', k)})(b - b_0)\| + \|\alpha_z(\mathbb{1}_{(F', k)})b_0 - 1_z\, b_0\| + \|1_z(b - b_0)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon.\end{aligned}$$

Thus I_{H_+} is an extendibly α_z-invariant ideal of B_{G_+}. □

4.2. The C^*-algebras $B_{(G/H)_+}$ and B_{G_+}/I_{H_+}

In this section we will introduce the C^*-algebra $B_{(G/H)_+}$ and to do so we need the group (G/H) to be lattice-ordered. Let $H = H_+ - H_+$, which is a subgroup of G, $q : G \to G/H$ be the quotient map of G onto G/H and define the positive cone of (G/H) to be $(G/H)_+ := \{q(x) : x \in G_+\}$.

LEMMA 4.2.1. *For $x, y \in G$, we have $q(y) - q(x) \in (G/H)_+$ if and only if there exists $h \in H$ such that $x \leq y + h$.*

PROOF. Suppose that $x, y \in G$ and $q(y) - q(x) \in (G/H)_+$. Then there is $z \in G_+$ such that $q(y) - q(x) = q(z)$, and so $(y - x) + H = z + H$. Then $y - x - z \in H$, and hence there is $h \in H$ such that $y - x - z = h$. Therefore $y - h = x + z \geq x$. Thus $x \leq y + k$, for $k = -h \in H$.

Conversely, suppose that $x, y \in G$ and there is $h \in H$ such that $x \leq y + h$. Then $y + h - x \geq 0$ (i.e $\in G_+$), and hence there is $w \in G_+$ such that $y - x + h = w$. Which implies $(y - x) - w = -h \in H$, and so $(y - x) + H = w + H$. Therefore $q(y) - q(x) = q(w)$, and thus $q(y) - q(x) \in (G/H)_+$. □

LEMMA 4.2.2. *The quotient group G/H is a lattice-ordered abelian group with order*

$$q(x) \leq q(y) \iff \text{there exists } h \in H \text{ such that } x \leq y + h.$$

Moreover, $q(x \vee y) = q(x) \vee q(y)$.

PROOF. The group G/H is abelian because G is abelian. We now show that the relation on G/H is a partial order, and to do so we need only to show that $(G/H)_+ \bigcap (G/H)_+^{-1} = H$. Suppose that $t \in (G/H)_+ \bigcap (G/H)_+^{-1}$. Then there are $z, w \in G_+$ such that $t = q(z)$ and $t = q(-w)$, which imply that $q(z) = q(-w)$. So $z + w \in H$, which means that there are $h_1, h_2 \in H_+$ such that $z + w = h_1 - h_2$ (since $H = H_+ - H_+$). Hence $z + w + h_2 = h_1 \in H_+ \subset H$. Noticing that $0 \leq z \leq z + w + h_2$ and H_+ is hereditary, then $z \in H_+ \subset H$. Therefore $q(z) = q(0) = H$ (that is, $t = H$).

Now we show that every two elements of G/H have a least upper bound in G/H. Fix $x, y \in G$. We know that $x \leq x \vee y$ ($x \vee y$ exists because G is lattice-ordered), and so $q(x) \leq q(x \vee y)$. The same is true for y, therefore $q(x \vee y)$ is an upper bound for $q(x)$ and $q(y)$. Suppose that for $z \in G$, $q(z)$ is an upper bound for $q(x)$ and $q(y)$. Then there exists $h_1, h_2 \in H$ such that $x \leq z + h_1$ and $y \leq z + h_2$. Without loss of generality we may suppose that $h_1, h_2 \in H_+$ (this is true because $h_1 = t - t'$ for some $t, t' \in H_+$ so $z + t - t' - x \geq 0$ and then $z + t - x \geq t' \geq 0$ and a similar argument works for h_2). Then $x, y \leq z + h_1 + h_2$ and hence $x \vee y \leq z + h_1 + h_2$. Thus $q(x \vee y) \leq q(z)$, and it follows that $q(x \vee y)$ is the least upper bound of $q(x)$ and $q(y)$ in G/H. Hence G/H is lattice-ordered. $\square$

REMARK 4.2.3. Since $(G/H, (G/H)_+)$ is a lattice-ordered group, then by [**21**] applied to G/H we have $B_{(G/H)_+} := \overline{\text{span}}\{1_{q(x)} : x \in G_+\}$ is an abelian C^*-algebra with unit $1_{q(0)} = 1_H$.

PROPOSITION 4.2.4. *Let I_{H_+} be the ideal of B_{G_+} defined in Lemma 4.1.1 and q be the quotient map of the group G onto the group G/H. Then there is an isomorphism Φ of B_{G_+}/I_{H_+} onto $B_{(G/H)_+}$ such that $\Phi(1_x + I_{H_+}) = 1_{q(x)}$ for all $x \in G_+$.*

We will now state and prove some lemmas and corollaries which will lead to the proof of this proposition.

Lemma 4.2.5. *Let q be the quotient map of the group G onto the group G/H. Then there is a surjective unital homomorphism $\phi : B_{G_+} \to B_{(G/H)_+}$ such that $\phi(1_x) = 1_{q(x)}$ for $x \in G_+$.*

Proof. Take the set $\{L_x := 1_{q(x)} : x \in G_+\}$ in the C^*-algebra $B_{(G/H)_+}$. Then for $x, y \in G_+$ we have

$$1_{q(x)}1_{q(y)} = 1_{q(x)\vee q(y)} = 1_{q(x\vee y)} \text{ and } 1_{q(0)}1_{q(x)} = 1_{q(0\vee x)} = 1_{q(x\vee 0)} = 1_{q(x)},$$

and this is true in $B_{(G/H)_+}$ because G/H is lattice-ordered group and $q(x) \vee q(y) = q(x \vee y)$ (Lemma 4.2.2). Hence $L_0 = 1_{q(0)} = 1_{B_{(G/H)_+}}$ and $L_x L_y = L_{x\vee y}$. Thus by Corollary 3.4.2 there exists a unital homomorphism $\phi : B_{G_+} \to B_{(G/H)_+}$ such that $\phi(1_x) = 1_{q(x)}$ and since the range of ϕ contains all the generators of $B_{(G/H)_+}$, ϕ is surjective. $\square$

Corollary 4.2.6. *Let q be the quotient map of G onto G/H and I_{H_+} be the ideal of B_{G_+} in Lemma 4.1.1. Then there exists a unital homomorphism $\widetilde{\phi} : B_{G_+}/I_{H_+} \to B_{(G/H)_+}$ satisfying $\widetilde{\phi}(1_x + I_{H_+}) = 1_{q(x)}$.*

Proof. To prove this corollary we need first to show that $I_{H_+} \subset \ker(\phi)$ (ϕ is the unital homomorphism of Lemma 4.2.5). To see this, take $x \in G_+$, $h \in H_+$. Then

$$\begin{aligned}
\phi(1_x - 1_{x+h}) &= \phi(1_x) - \phi(1_{x+h}), \text{ since } \phi \text{ is a homomorphism} \\
&= 1_{q(x)} - 1_{q(x+h)} \\
&= 1_{q(x)} - 1_{q(x)}, \text{ since } h \in H_+ \\
&= 0.
\end{aligned}$$

Now, for any $f \in I_{H_+}$ we know that f is a limit of elements in the spanning set of I_{H_+}, so by linearity and continuity of ϕ we have $\phi(f) = 0$. Hence $I_{H_+} \subset \ker(\phi)$ and as I_{H_+} is a closed ideal in B_{G_+}, there exists a unital homomorphism $\widetilde{\phi} : B_{G_+}/I_{H_+} \to B_{(G/H)_+}$ with the desired properties. $\square$

Lemma 4.2.7. *Let I_{H_+} be the ideal of B_{G_+} in Lemma 4.1.1 and q be the quotient map of G onto G/H. Then there is a unital homomorphism $\psi : B_{(G/H)_+} \to B_{G_+}/I_{H_+}$ satisfying $\psi(1_{q(x)}) = 1_x + I_{H_+}$ for $x \in G_+$.*

Proof. Take the set $\{L_{q(x)} := 1_x + I_{H_+}, x \in G_+\}$ in the C^*-algebra B_{G_+}/I_{H_+}. We want to apply Corollary 3.4.2, so we need first to show that $L_{q(x)}$ is well-defined. To see this, suppose that $x, y \in G_+$ satisfy $q(x) = q(y)$. Then $x - y \in H$, and since $H = H_+ - H_+$, we have $x - y = h_1 - h_2$ for some $h_1, h_2 \in H_+$. Notice that $1_x = 1_{x+h_2} + (1_x - 1_{x+h_2})$ and so $1_x \in 1_{x+h_2} + I_{H_+}$. But as $x + h_2 = y + h_1$, then $1_x \in 1_{y+h_1} + I_{H_+}$. Now $1_{y+h_1} + I_{H_+} = 1_y - (1_y - 1_{y+h_1}) + I_{H_+}$, therefore $1_{y+h_1} + I_{H_+} = 1_y + I_{H_+}$. Hence $1_x \in 1_y + I_{H_+}$. Thus $1_x + I_{H_+} = 1_y + I_{H_+}$, which means that $L_{q(x)} = L_{q(y)}$. Our next step is to check the conditions of Corollary 3.4.2. For $f \in B_{G_+}$ we have

$$(1_0 + I_{H_+})(f + I_{H_+}) = (1_0\, f) + I_{H_+} = f + I_{H_+}, \text{ since } 1_0 = 1_{B_{G_+}} \in B_{G_+}$$

and since B_{G_+} is commutative, $(f + I_{H_+})(1_0 + I_{H_+}) = f + I_{H_+}$. Hence $L_{q(0)} = 1$ $\in B_{G_+}/I_{H_+}$. Moreover,

$$\begin{aligned} L_{q(x)}L_{q(y)} = (1_x + I_{H_+})(1_y + I_{H_+}) &= 1_x 1_y + I_{H_+} \\ &= 1_{x\vee y} + I_{H_+} \\ &= L_{q(x\vee y)} \\ &= L_{q(x)\vee q(y)}, \text{ by Lemma 4.2.2.} \end{aligned}$$

Since G/H is a lattice-ordered group, then Corollary 3.4.2 gives a unital homomorphism $\psi : B_{(G/H)_+} \to B_{G_+}/I_{H_+}$ such that $\psi(1_{q(x)}) = 1_x + I_{H_+}$. $\square$

Proof of Proposition 4.2.4. To show that the two C^*-algebras $B_{(G/H)_+}$ and B_{G_+}/I_{H_+} are isomorphic, consider the homomorphisms $\widetilde{\phi}$ and ψ of Corollary 4.2.6 and Lemma 4.2.7 respectively. It is enough to show that $\psi\circ\widetilde{\phi}$ and $\widetilde{\phi}\circ\psi$ are the respective identity maps. Since ψ and $\widetilde{\phi}$ are both continuous and linear it suffices to check that $\psi \circ \widetilde{\phi}(1_{q(x)}) = 1_{q(x)}$ and $\widetilde{\phi} \circ \psi(1_x + I_{H_+}) = 1_x + I_{H_+}$. By the definitions of $\widetilde{\phi}$ and ψ in Corollary 4.2.6 and Lemma 4.2.7 respectively. The conditions $\psi \circ \widetilde{\phi}(1_{q(x)}) = 1_{q(x)}$

and $\widetilde{\phi} \circ \psi(1_x + I_{H_+}) = 1_x + I_{H_+}$ for $x \in G_+$, follow immediately. Thus take the isomorphism Φ to be the unital homomorphism $\widetilde{\phi}$ in Corollary 4.2.6 which has the desired properties. $\square$

CHAPTER 5

Inflated dynamical systems

We use the first section of this chapter to give the necessary definitions and results on the subject of irreducible representations of C^*-algebras. We use these results throughout the last two chapters.

5.1. Irreducible representations of C^*-algebras

Definition 5.1.1. A non-zero representation π of a C^*-algebra A on a Hilbert space $\mathcal{H}$ is called *irreducible* if the only closed subspaces $\mathcal{K}$ of $\mathcal{H}$ such that $\pi(a)\mathcal{K} \subset \mathcal{K}$ are $\mathcal{K} = \{0\}$ and $\mathcal{K} = \mathcal{H}$ [**37**, A.1].

The following lemma gives a necessary and sufficient conditions for a representation π of a C^*-algebra A to be irreducible. The proof can be found in [Lemma A.1][**37**].

Lemma 5.1.2. *A representation π of a C^*-algebra A is irreducible if and only if the only operators commuting with $\pi(A)$ are multiples of the identity operator $1_{\mathcal{H}}$.*

The following theorem says that for any element a of a C^*-algebra A, there is an irreducible representation π that preserves the norm of a. For the proof see [**37**, Theorem A.14].

Theorem 5.1.3. *Let A be a C^*-algebra. Then for each $a \in A$ there is an irreducible representation π of A with $\|\pi(a)\| = \|a\|$.*

Remark 5.1.4. We can read the previous theorem in the following way. For any C^* algebra A and any non-zero element $a \in A$, there is an irreducible representation π of A such that $\pi(a) \neq 0$. This means that if $\pi(a) = 0$ for every irreducible representation π of A, then $a = 0$.

We now prove a minor corollary which we use later in section §5.3.

COROLLARY 5.1.5. *Suppose that $\phi : A \to B$ is a homomorphism of C^*-algebras and that for every irreducible representation π of A there is a representation ρ of B such that $\pi = \rho \circ \phi$. Then ϕ is injective.*

PROOF. Let $a \in A$ and suppose that $\phi(a) = 0$. Then $\rho \circ \phi(a) = 0$ for any representation of B. Therefore, $\pi(a) = 0$ for all irreducible representations of A. Hence Remark 5.1.4 implies that $a = 0$ and thus ϕ is injective. $\square$

5.2. The relationship between inflated systems

Let I_{H_+} be the ideal of B_{G_+} in Lemma 3.4.2 and α be the action of G_+ by endomorphisms of B_{G_+} in Lemma 3.4.1. By Remark 4.1.2, each α_x is extendible. Since I_{H_+} is an α_x-invariant ideal of B_{G_+} (Corollary 4.1.6), then as in [**2**, page 2] it follows that each α_x induces an endomorphism $\widetilde{\alpha}_x$ of the quotient C^*-algebra B_{G_+}/I_{H_+} characterised by $\widetilde{\alpha}_x(1_y + I_{H_+}) = \alpha_x(1_y) + I_{H_+}$. Because $\widetilde{\alpha}_x \circ \widetilde{\alpha}_y = \widetilde{\alpha}_{x+y}$, $\widetilde{\alpha}$ is an action of G_+ on B_{G_+}/I_{H_+}. Since the C^*-algebra B_{G_+}/I_{H_+} is unital (because I_{H_+} is a closed ideal of B_{G_+}), then each $\widetilde{\alpha}_x$ is extendible.

REMARK 5.2.1.

(i) Since $(G/H, (G/H)_+)$ is a lattice-ordered group, then there is an action $\tau : (G/H)_+ \to \operatorname{End}\left(B_{(G/H)_+}\right)$ such that $\tau_{x+H}(1_{y+H}) = 1_{x+y+H}$ and every τ_{x+H} is extendible because $B_{(G/H)_+}$ is unital. If we define $\beta := \tau \circ q$, then β is an action of G_+ on $B_{(G/H)_+}$ by extendible endomorphisms.

(ii) Isomorphic dynamical systems give isomorphic crossed products. That is, suppose (A, P, ξ), (B, P, ζ) are two dynamical systems and ϕ is an isomorphism of A onto B satisfying $\phi \circ \xi_t = \zeta_t \circ \phi$ for all $t \in P$. Then there is an isomorphism ϑ of $A \times_\xi P$ onto $B \times_\zeta P$ satisfying

$$\vartheta(i_A(a)) = i_B(\phi(a)) \text{ and } \vartheta(i_P(t)) = i_P(t) \text{ for all } a \in A, t \in P.$$

LEMMA 5.2.2. *There is an isomorphism* Ω *of the crossed product* $(B_{G_+}/I_{H_+})\times_{\widetilde{\alpha}} G_+$ *onto the crossed product* $B_{(G/H)_+}\times_\beta G_+$.

PROOF. Proposition 4.2.4 gives an isomorphism $\Phi : B_{G_+}/I_{H_+} \to B_{(G/H)_+}$ satisfying $\Phi(1_x + I_{H_+}) = 1_{q(x)}$. So to prove this lemma it is enough to show that $\Phi\circ\widetilde{\alpha}_x = \beta_x\circ\Phi$ for all $x\in G_+$. Let $y\in G_+$, then

$$\begin{aligned}
\Phi\circ\widetilde{\alpha}_x(1_y + I_{H_+}) = \Phi\big(1_{x+y} + I_{H_+}\big) &= 1_{q(x+y)} \\
&= \tau_{x+H}(1_{y+H}) \\
&= \beta_x(1_{q(y)}) \\
&= \beta_x\circ\Phi(1_y + I_{H_+}).
\end{aligned} \tag{5.2.1}$$

Since Φ, β_x and $\widetilde{\alpha}_x$ are all continuous and linear then $\Phi\circ\widetilde{\alpha}_x(b) = \beta_x\circ\Phi(b)$ for every $b\in B_{G_+}/I_{H_+}$. Thus $(B_{G_+}/I_{H_+})\times_{\widetilde{\alpha}} G_+$ and $B_{(G/H)_+}\times_\beta G_+$ are isomorphic. □

PROPOSITION 5.2.3. *Let* $\big(i_{B_{(G/H)_+}}, i_{(G/H)_+}\big)$ *and* $\big(j_{B_{(G/H)_+}}, j_{G_+}\big)$ *denote the universal representations of the dynamical systems* $\big(B_{(G/H)_+}, (G/H)_+, \tau\big)$ *and* $\big(B_{(G/H)_+}, G_+, \beta\big)$ *respectively and let* q *be the quotient map of* G *onto* G/H. *Then there exists a surjective homomorphism*

$$Q : B_{(G/H)_+}\times_\beta G_+ \to B_{(G/H)_+}\times_\tau (G/H)_+,$$

such that $Q\circ j_{B_{(G/H)_+}} = i_{B_{(G/H)_+}}$ *and* $Q\circ j_{G_+} = i_{(G/H)_+}\circ q$.

PROOF. The map $i_{(G/H)_+}\circ q$ is a homomorphism of the semigroup G_+ into the semigroup of isometries of $B_{(G/H)_+}\times_\tau (G/H)_+$ and the map $i_{B_{(G/H)_+}}$ is a unital homomorphism such that, for $x\in G_+$ and $a\in B_{(G/H)_+}$ we have

$$i_{B_{(G/H)_+}}\big(\beta_x(a)\big) = i_{B_{(G/H)_+}}\big(\tau_{q(x)}(a)\big) = i_{(G/H)_+}(q(x))\, i_{B_{(G/H)_+}}(a)\, i_{(G/H)_+}(q(x))^*.$$

Therefore the pair $\big(i_{B_{(G/H)_+}}, i_{(G/H)_+}\circ q\big)$ is a covariant representation of the dynamical system $\big(B_{(G/H)_+}, G_+, \beta\big)$ in the C^*-algebra $B_{(G/H)_+}\times_\tau (G/H)_+$. Thus there exists a unital homomorphism

$$Q : B_{(G/H)_+}\times_\beta G_+ \to B_{(G/H)_+}\times_\tau (G/H)_+$$

satisfying $Q \circ j_{B_{(G/H)_+}} = i_{B_{(G/H)_+}}$ and $Q \circ j_{G_+} = i_{(G/H)_+} \circ q$. And since the range of Q is a C^*-subalgebra of $B_{(G/H)_+} \times_\tau (G/H)_+$ containing all the generators, Q is surjective. $\square$

REMARK 5.2.4. Since H is a subgroup of G, then $(G/H)^\wedge$ is isomorphic to $H^\perp = \{\xi \in \widehat{G} : \xi(x) = 1 \text{ for all } x \in H\}$ and $\widehat{G}/H^\perp$ is isomorphic to $\widehat{H}$ [**17**, Theorem 4.39].

5.3. Crossed products of inflated systems

DEFINITION 5.3.1. Suppose that K is a compact group and $\alpha : S \to \operatorname{Aut} A$ is an action of a closed subgroup S of K on a C^*-algebra A. Then the *induced C^*-algebra* $\operatorname{Ind}_S^K(A, \alpha)$ is the subalgebra of $C(K, A)$ consisting of the functions f satisfying $f(gh) = \alpha_h^{-1}(f(g))$ for $g \in K$ and $h \in S$ [**6**, page 3].

To proceed, let us review our assumptions. Suppose that (G, G_+) is a lattice-ordered group with G abelian , H_+ is a hereditary subsemigroup of G_+ and q is the quotient map of the group G onto the group G/H. We now introduce one of our main results in this chapter which shows that we can realize the C^*-algebra $B_{(G/H)_+} \times_\beta G_+$ as the induced C^*-algebra $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$. This theorem is an analogue of Theorem 2.1 in [**3**]. The argument of the proof of our theorem is similar in outline to the one of Theorem 2.1 in [**6**] with extra work to be done. However, under our assumptions especially that we are working with lattice-ordered not totally ordered groups more challenges to the proof have been added, for example there are no analogues of claims 2, 5, 6 and 7 in [**6**].

THEOREM 5.3.2. *Let $\widehat{\beta}$ be the dual action as in Lemma 3.4.6, τ be the action in Remark 5.2.1 and Q be the surjective homomorphism of Proposition 5.2.3. Then there is an isomorphism Ψ of the crossed product $B_{(G/H)_+} \times_\beta G_+$ onto the induced C^*-algebra $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ such that $\Psi(a)(\gamma) = Q\big(\widehat{\beta}_\gamma^{-1}(a)\big)$ for $a \in B_{(G/H)_+} \times_\beta G_+$ and $\gamma \in \widehat{G}$.*

PROOF. Given $\mu \in H^\perp = (G/H)^\wedge$ (Remark 5.2.4). We claim that

$$Q \circ \widehat{\beta}_\mu^{-1} = \widehat{\tau}_\mu^{-1} \circ Q, (\widehat{\tau} \text{ is the dual action as in Lemma 3.4.6}). \tag{5.3.1}$$

For $b \in B_{(G/H)_+}$, we have

$$\begin{aligned}
Q \circ \widehat{\beta}_\mu^{-1}(j_{B_{(G/H)_+}}(b)) = Q\big(\widehat{\beta}_\mu^{-1}(j_{B_{(G/H)_+}}(b))\big) &= Q(j_{B_{(G/H)_+}}(b)) \\
&= i_{B_{(G/H)_+}}(b) \\
&= \widehat{\tau}_\mu^{-1}\big(i_{B_{(G/H)_+}}(b)\big) \\
&= \widehat{\tau}_\mu^{-1} \circ Q(j_{B_{(G/H)_+}}(b))
\end{aligned}$$

and for $x \in G_+$ we have

$$\begin{aligned}
Q \circ \widehat{\beta}_\mu^{-1}(j_{G_+}(x)) = Q\big(\widehat{\beta}_\mu^{-1}(j_{G_+}(x))\big) &= Q\big(\mu(x) j_{G_+}(x)\big) \\
&= \mu(x)\big(i_{(G/H)_+} \circ q(x)\big) \\
&= \widehat{\tau}_\mu^{-1}\big((i_{(G/H)_+} \circ q)(x)\big) \\
&= \widehat{\tau}_\mu^{-1} \circ Q(j_{G_+}(x)).
\end{aligned}$$

Thus the homomorphisms $Q \circ \widehat{\beta}_\mu^{-1}$ and $\widehat{\tau}_\mu^{-1} \circ Q$ agree on generators and hence on all of the C^*-algebra $B_{(G/H)_+} \times_\beta G_+$.

Because $\widehat{\beta}$ is continuous, by Lemma 3.4.6, then $\Psi(a) : \widehat{G} \to B_{(G/H)_+} \times_\tau (G/H)_+$ is continuous. For $\gamma \in \widehat{G}$, $\mu \in H^\perp$ we have

$$\begin{aligned}
\Psi(a)(\gamma\mu) = Q\big(\widehat{\beta}_{\gamma\mu}^{-1}(a)\big) &= Q\big(\widehat{\beta}_\mu^{-1} \circ \widehat{\beta}_\gamma^{-1}(a)\big) \\
&= \widehat{\tau}_\mu^{-1}\big(Q(\widehat{\beta}_\gamma^{-1}(a))\big) \\
&= \widehat{\tau}_\mu^{-1}\big(\Psi(a)(\gamma)\big).
\end{aligned}$$

Hence $\Psi(a) \in \operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$. Further routine calculations show that Ψ is a homomorphism of C^*-algebras. For details, see appendix A.

Our next step is to show that Ψ is injective and to do so we will apply Theorem A.14 of [**37**] and Corollary 5.1.5. Given that π is an irreducible representation of $B_{(G/H)_+} \times_\beta G_+$, we need to find a representation σ of $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ such that $\pi = \sigma \circ \Psi$.

Claim 1. $\pi \circ j_{G_+}$ is an isometric representation of G_+ which is unitary on H_+.

PROOF. We know that the composition of any isometric representation with a unital homomorphism will give an isometric representation. So we only need to show that $\pi \circ j_{G_+}$ is unitary on H_+. For $y \in H_+$, we have

$$\begin{aligned}\big(\pi \circ j_{G_+}(y)\big)\big(\pi \circ j_{G_+}(y)^*\big) &= \pi\big(j_{G_+}(y) j_{B_{(G/H)_+}}(1_{B_{(G/H)_+}}) j_{G_+}(y)^*\big) \\ &= \pi\Big(j_{B_{(G/H)_+}}\big(\beta_y(1_{B_{(G/H)_+}})\big)\Big) \\ &= \pi\big(j_{B_{(G/H)_+}}(1_{B_{(G/H)_+}})\big) \\ &= \pi\big(1_{B_{(G/H)_+} \times_\beta G_+}\big) \\ &= 1_{\mathcal{H}}\,.\end{aligned}$$

□

CLAIM 2. For $x \in H_+$, $\pi \circ j_{G_+}(x)$ commutes with every $\pi \circ j_{G_+}(y)$, $\pi \circ j_{G_+}(y)^*$ and $\pi \circ j_{B(G/H)_+}(f)$, for $y \in G_+$ and $f \in B_{(G/H)_+}$.

PROOF. Let $x \in H_+$ and $y \in G_+$, then

$$\begin{aligned}\big(\pi \circ j_{G_+}(x)\big)\big(\pi \circ j_{G_+}(y)\big) &= \pi\big(j_{G_+}(x+y)\big) \\ &= \pi\big(j_{G_+}(y+x)\big) \\ &= \big(\pi \circ j_{G_+}(y)\big)\big(\pi \circ j_{G_+}(x)\big),\end{aligned} \tag{1}$$

and

$$\begin{aligned}\big(\pi \circ j_{G_+}(x)\big)\big(\pi \circ j_{G_+}(y)^*\big) &= \pi\big(j_{G_+}(x)\big)\pi\big(j_{G_+}(y)^*\big) \\ &= \pi\big(j_{G_+}(x) j_{G_+}(y)^*\big) \\ &= \pi\big(j_{G_+}(x) j_{G_+}(y)^* j_{G_+}(x)^* j_{G_+}(x)\big) \\ &= \pi\big(j_{G_+}(x) j_{G_+}(x)^* j_{G_+}(y)^* j_{G_+}(x)\big), \text{ by (1)} \\ &= \pi\big(j_{G_+}(x) j_{G_+}(x)^*\big)\pi\big(j_{G_+}(y)^* j_{G_+}(x)\big) \\ &= 1_{\mathcal{H}} \cdot \pi\big(j_{G_+}(y)^*\big)\pi\big(j_{G_+}(x)\big), \text{ by Claim 1} \\ &= \big(\pi \circ j_{G_+}(y)^*\big)\big(\pi \circ j_{G_+}(x)\big).\end{aligned} \tag{2}$$

Finally, for $y \in G_+$ we have

$$\begin{aligned}
\big(\pi \circ j_{G_+}(x)\big)\big(\pi \circ j_{B_{(G/H)_+}}(1_{q(y)})\big) &= \pi\Big(j_{B_{(G/H)_+}}\big(\beta_x(1_{q(y)})\big)j_{G_+}(x)\Big)\\
&= \pi\Big(j_{B_{(G/H)_+}}\big(1_{x+y+H}\big)j_{G_+}(x)\Big)\\
&= \pi\Big(j_{B_{(G/H)_+}}\big(1_{y+H}\big)j_{G_+}(x)\Big), \text{ as } x \in H_+ \subset H\\
&= \big(\pi \circ j_{B_{(G/H)_+}}(1_{q(y)})\big)\big(\pi \circ j_{G_+}(x)\big).
\end{aligned}$$

Since both π and $j_{B_{(G/H)_+}}$ are linear and continuous and by limit properties we have $\big(\pi \circ j_{G_+}(x)\big)$ and $\big(\pi \circ j_{B_{(G/H)_+}}(f)\big)$ commute for every $f \in B_{(G/H)_+}$. ☐

By Claim 2 we have $\big(\pi \circ j_{G_+}(x)\big) \in \pi\Big(B_{(G/H)_+} \times_\beta G_+\Big)' = \mathbb{C}\,1_{\mathcal{H}}$ (since π is irreducible) and so there is a character γ of H_+ such that $\big(\pi \circ j_{G_+}(x)\big) = \gamma(x)\,1_{\mathcal{H}}$.

Claim 3. The character $\gamma : H_+ \to \mathbb{T}$ has an extension χ to G.

proof. Since $\pi \circ j_{G_+}$ is unitary on H_+, $\gamma(x)$ has $|\gamma(x)| = 1$ for all $x \in H_+$. Notice that the abelian subgroup $H = H_+ - H_+$ of G is generated by H_+, which is a normal subsemigroup of H. So by [**20**, Lemma 1.1] there exists a unique group homomorphism γ' extending γ to all of H. Since H is a closed subgroup of G, then [**17**, Corollary 4.41] implies that there exists a character χ of G such that $\chi|_H = \gamma'$. ☐

Consider the isometric representation $U : x \mapsto \overline{\chi(x)}\,\pi(j_{G_+}(x))$ of G_+. Then $U_x = 1$ for all $x \in H_+$ since $\chi|_{H_+} = \gamma$.

Claim 4. U is constant on H cosets.

proof. suppose that $x, y \in G_+$ satisfy $x + H = y + H$. Then $x - y \in H$ and as $H = H_+ - H_+$ we have $x - y = h_1 - h_2$ for $h_1, h_2 \in H_+$. Therefore $x + h_2 = y + h_1$, and as $U_{h_1} = 1 = U_{h_2}$ we have

$$U_x = U_x U_{h_2} = U_{x+h_2} = U_{y+h_1} = U_y U_{h_1} = U_y\,.$$

Thus U is constant on H cosets. ☐

To construct σ, let q be the quotient map of G onto G/H and define $W : (G/H)_+ \to B(\mathcal{H})$ by $W_{q(x)} = U_x$, then W is an isometric representation of $(G/H)_+$. We claim that W is covariant. To see this let $x, y \in G_+$, since $0 \leq x \leq x \vee y$ and (G, G_+) is lattice-ordered then $x \vee y \in G_+$. Lemma 3.4.4 says that since (G, G_+) is lattice-ordered group then j_{G_+} is covariant isometric representation of the semigroup G_+ into the semigroup of isometries in $B_{(G/H)_+} \times_\beta G_+$. Then

$$\begin{aligned}
W_{q(x)}W^*_{q(x)}W_{q(y)}W^*_{q(y)} &= |\chi(x)|^2\pi\big(j_{G_+}(x)\big)\pi\big(j_{G_+}(x)\big)^*|\chi(y)|^2\pi\big(j_{G_+}(y)\big)\pi\big(j_{G_+}(y)\big)^* \\
&= \pi\big(j_{G_+}(x)j_{G_+}(x)^*j_{G_+}(y)j_{G_+}(y)^*\big) \\
&= \pi\big(j_{G_+}(x \vee y)j_{G_+}(x \vee y)^*\big) \\
&= U_{x\vee y}U^*_{x\vee y} \\
&= W_{q(x\vee y)}W^*_{q(x\vee y)} \\
&= W_{q(x)\vee q(y)}W^*_{q(x)\vee q(y)}, \text{ by Lemma 4.2.2.}
\end{aligned}$$

Thus W is covariant isometric representation of $(G/H)_+$ as claimed. We know from [3] that $B_{(G/H)_+} \times_\tau (G/H)_+$ is universal for covariant isometric representations of $(G/H)_+$, hence there is a representation ρ_W of $B_{(G/H)_+} \times_\tau (G/H)_+$ such that $\rho_W\big(i_{(G/H)_+}(q(x))\big) = W_{q(x)}$. Notice that ρ_W is an irreducible representation since it has the same range as π. We know from [**37**, Proposition 6.16] that every irreducible representation of $\mathrm{Ind}_{H^\perp}^{\widehat{G}}\big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ has the form $M(\gamma, \rho) : f \to \rho(f(\gamma))$ for some $\gamma \in \widehat{G}$ and some irreducible representation ρ of $B_{(G/H)_+} \times_\tau (G/H)_+$. So we take for σ the representation $M(\chi, \rho_W)$ and check that $\pi = M(\chi, \rho_W) \circ \Psi$.

Fix $y \in G_+$, then

$$(\star) \qquad \begin{aligned} M(\chi, \rho_W) \circ \Psi\big(j_{G_+}(y)\big) &= \rho_W\Big(\Psi\big(j_{G_+}(y)(\chi)\big)\Big) \\ &= \rho_W\Big(Q\big(\widehat{\beta}_\chi^{-1}(j_{G_+}(y))\big)\Big) \\ &= \rho_W\Big(Q\big(\chi(y) j_{G_+}(y)\big)\Big) \\ &= \chi(y)\rho_W\big(i_{(G/H)+} \circ q(y)\big) \\ &= \chi(y) W_{q(y)} \\ &= \chi(y) U_y \\ &= \pi\big(j_{G_+}(y)\big). \end{aligned}$$

Since $\big(j_{B_{(G/H)_+}}, j_{G_+}\big)$ is the universal representation of the dynamical system $\big(B_{(G/H)+}, G_+, \beta\big)$ then $j_{B_{(G/H)_+}}(1_{y+H}) = j_{G_+}(y) j_{G_+}(y)^*$ and so the elements $j_{G_+}(y)$ generate the C^*-algebra $B_{(G/H)_+} \times_\beta G_+$. Therefore $(\star)$ implies that $M(\chi, \rho_W) \circ \Psi = \pi$ and thus Ψ is injective.

We still need to show that Ψ is surjective. To do so we make the following claims.

Claim 5. For fixed $\gamma \in \widehat{G}$ the set $\big\{\Psi(b)(\gamma) : b \in B_{(G/H)_+} \times_\beta G_+\big\} = B_{(G/H)_+} \times_\tau (G/H)_+$.

proof. Let $a \in B_{(G/H)_+} \times_\tau (G/H)_+$. Proposition 5.2.3 implies that there exists $c \in B_{(G/H)_+} \times_\beta G_+$ such that $Q(c) = a$. Then

$$\Psi\big(\widehat{\beta}_\gamma(c)\big)(\gamma) = Q\big(\widehat{\beta}_\gamma^{-1}(\widehat{\beta}_\gamma(c))\big) = Q(c) = a.$$

$\square$

Claim 6. range$(\Psi) \supset C\big(\widehat{G}/H^\perp\big) \cong C(\widehat{H})$ (Remark 5.2.4).

PROOF. For $x \in H_+$ and $\gamma \in \widehat{G}$ we have

$$\begin{aligned}
\Psi\big(j_{G_+}(x)\big)(\gamma) &= Q\Big(\widehat{\beta}_\gamma^{-1}\big(j_{G_+}(x)\big)\Big) \\
&= Q\Big(\gamma(x) j_{G_+}(x)\Big) \\
&= \gamma(x)\, i_{(G/H)_+} \circ q(x) \\
&= \gamma(x)\, i_{(G/H)_+}(x+H) \\
&= \gamma(x)\, i_{(G/H)_+}(H),\ x \in H_+ \subset H \\
&= \gamma(x)\, 1_{B_{(G/H)_+} \times_\tau (G/H)_+}, \text{ since } i_{(G/H)_+} \text{ is an isometric representation.}
\end{aligned}$$

Moreover, for $x \in H_+$ we claim that $\Psi\big(j_{G_+}(x)\big)$ is constant on $H^\perp$-orbits. To see this, suppose that $\gamma, \mu \in \widehat{G}$ satisfy $\gamma H^\perp = \mu H^\perp$. Then $\mu\gamma^{-1} = \mu\overline{\gamma} \in H^\perp$ and

$$\begin{aligned}
\Psi\big(j_{G_+}(x)\big)(\gamma) &= \gamma(x)\, 1_{B_{(G/H)_+} \times_\tau (G/H)_+} \\
&= \gamma(x)\big(\mu\overline{\gamma}\big)(x)\, 1_{B_{(G/H)_+} \times_\tau (G/H)_+} \\
&= \gamma(x)\big(\overline{\gamma(x)}\mu(x)\big)\, 1_{B_{(G/H)_+} \times_\tau (G/H)_+} \\
&= |\gamma(x)|^2 \mu(x)\, 1_{B_{(G/H)_+} \times_\tau (G/H)_+} \\
&= \Psi\big(j_{G_+}(x)\big)(\mu).
\end{aligned}$$

So $\Psi\big(j_{G_+}(x)\big)$ is really a function on $\widehat{G}/H^\perp$. We now show that the set $S = \operatorname{span}\big\{e_x : \gamma H^\perp \mapsto \gamma(x) : x \in H\big\}$ is dense in $C(\widehat{G}/H^\perp)$. To do this we apply the Stone-Weierstrass Theorem, so we need to show that S is a $*$-subalgebra of $C(\widehat{G}/H^\perp)$ which contains the constant functions on $\widehat{G}/H^\perp$ and separates points of $\widehat{G}/H^\perp$. To begin, fix $x, y \in H$, $\gamma H^\perp \in \widehat{G}/H^\perp$ and $\lambda \in \mathbb{C}$. Then

$$e_x e_y(\gamma H^\perp) = e_x(\gamma H^\perp) e_y(\gamma H^\perp) = \gamma(x)\gamma(y) = \gamma(x+y) = e_{x+y}(\gamma H^\perp).$$

So $e_x e_y \in S$ and hence S is a subalgebra of $C(\widehat{G}/H^\perp)$. Furthermore,

$$\overline{e_x(\gamma H^\perp)} = \overline{\gamma(x)} = \gamma(-x) = e_{-x}(\gamma H^\perp).$$

Hence S is a $*$-subalgebra. Now, λe_0 is the constant function of value λ, so S contains the constant functions on $\widehat{G}/H^\perp$. Moreover, let $\gamma H^\perp, \chi H^\perp \in \widehat{G}/H^\perp$ such that $\gamma H^\perp \neq \chi H^\perp$. Then $\gamma\chi^{-1} \notin H^\perp$ which implies that there is $x \in H$ such that

$\gamma(x)\chi^{-1}(x) \neq 1$, and hence $\gamma(x) \neq \chi(x)$. So $e_x(\gamma H^\perp) \neq e_x(\chi H^\perp)$ and therefore S separates points of $\widehat{G}/H^\perp$. Hence by the Stone-Weierstrass Theorem S is dense in $C(\widehat{G}/H^\perp)$.

Notice that every $h \in H$ has the form $h_1 - h_2$ for some $h_1, h_2 \in H_+$, and so $e_h = e_{h_1} e_{h_2}^*$; then the elements $\{e_x : x \in H_+\}$ generate $C(\widehat{G}/H^\perp)$ as a C^*-algebra.

There is an obvious embedding Θ of $C(\widehat{G}/H^\perp)$ into $\mathrm{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$ given by $\Theta(f)(\gamma) = f(\gamma H^\perp) 1_{B_{(G/H)_+} \times_\tau (G/H)_+}$, and so $\Theta(e_x) = \Psi(j_{G_+}(x))$. Therefore the set $\{\Psi(j_{G_+}(x)) : x \in H_+\}$ generates $C(\widehat{G}/H^\perp)$, and hence the range of Ψ -which is a C^*-algebra- contains $C(\widehat{G}/H^\perp)$. $\square$

Claim 7. Ψ is surjective.

proof. Given $f \in \mathrm{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$. Since $\mathrm{rang}(\Psi)$ is a C^*-subalgebra of $\mathrm{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$, to show that $f \in \mathrm{range}(\Psi)$ it is enough to approximate f by elements of $\mathrm{rang}(\Psi)$. Fix $\varepsilon > 0$. For $\gamma \in \widehat{G}$, Claim 5 implies that there exist $a_\gamma \in B_{(G/H)_+} \times_\beta G_+$ such that $\Psi(a_\gamma)(\gamma) = f(\gamma)$. Since both $\Psi(a_\gamma)$ and f are continuous on $\widehat{G}$, there exists an open neighborhood V_γ of γ such that

$$\chi \in V_\gamma \Rightarrow \|\Psi(a_\gamma)(\chi) - f(\chi)\| < \varepsilon. \tag{5.3.2}$$

Hence, for $\mu \in H^\perp$ we have

$$\begin{aligned} f(\gamma\mu) = \widehat{\tau}_\mu^{-1}(f(\gamma)) &= \widehat{\tau}_\mu^{-1}(\Psi(a_\gamma)(\gamma)) \\ &= \widehat{\tau}_\mu^{-1}(Q(\widehat{\beta}_\gamma^{-1}(a_\gamma))) \\ &= Q \circ \widehat{\beta}_\mu^{-1}(\widehat{\beta}_\gamma^{-1}(a_\gamma)), \text{ by Equation (5.3.1)} \\ &= Q(\widehat{\beta}_{\gamma\mu}^{-1}(a_\gamma)) \\ &= \Psi(a_\gamma)(\gamma\mu). \end{aligned}$$

Suppose that $\sigma : \widehat{G} \to \widehat{G}/H^\perp$ is the quotient map of the compact group $\widehat{G}$ onto $\widehat{G}/H^\perp$. Since the quotient maps of topological groups are open, then $\sigma(V_\gamma) = V_\gamma H^\perp$ is an open neighborhood of $\gamma H^\perp$. Therefore, we have

$$\chi H^\perp \in V_\gamma H^\perp \Rightarrow \|\Psi(a_\gamma)(\chi) - f(\chi)\| < \varepsilon.$$

Now choose a finite subset $\{V_{\gamma_i}H^\perp\}_{i=1}^n$ of $\{V_\gamma H^\perp\}$ such that $\widehat{G}/H^\perp = \bigcup_{i=1}^n V_{\gamma_i}H^\perp$. By [**37**, Lemma 4.34] we have a partition of unity $\phi_i : \widehat{G}/H^\perp \to [0,1]$, $i = 1, ..., n$ (continuous functions) subordinate to $\{V_{\gamma_i}H^\perp\}_{i=1}^n$ such that $\sum_{i=1}^n \phi_i(\gamma H^\perp) = 1$ for all $\gamma H^\perp \in \widehat{G}/H^\perp$ and $\operatorname{supp}\phi_i \subset V_{\gamma_i}H^\perp$. Let $a_i = a_{\gamma_i}$ and notice that $\phi_i \in C(\widehat{G}/H^\perp)$. Claim 6 says that $C(\widehat{G}/H^\perp)$ is contained in range$(\Psi) \subset \operatorname{Ind}_{H^\perp}^{\widehat{G}}\big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$. Therefore, for each i there exists b_i such that $\Psi(b_i) = \phi_i$. Hence $\sum_{i=1}^n \phi_i\Psi(a_i) = \Psi\big(\sum_{i=1}^n b_i a_i\big)$ belongs to the range of Ψ. Now for $\chi \in \widehat{G}$ we have

$$\begin{aligned}
\Big\|\Big(\sum_{i=1}^n \phi_i\Psi(a_i)\Big)(\chi) - f(\chi)\Big\| &= \Big\|\Big(\sum_{i=1}^n \phi_i\Psi(a_i)\Big)(\chi) - \sum_{i=1}^n \phi_i(\chi H^\perp)f(\chi)\Big\| \\
&= \Big\|\sum_{i=1}^n \phi_i(\chi H^\perp)\big(\Psi(a_i)(\chi) - f(\chi)\big)\Big\| \\
&\le \sum_{i=1}^n \phi_i(\chi H^\perp)\|\Psi(a_i)(\chi) - f(\chi)\| \\
&\le \sum_{i=1}^n \phi_i(\chi H^\perp)\,\varepsilon \\
&= \varepsilon, \text{ since } \sum_{i=1}^n \phi_i(\gamma H^\perp) = 1 \text{ for all } \gamma H^\perp \in \widehat{G}/H^\perp.
\end{aligned}$$

Hence Ψ is surjective. $\square$

Thus Ψ is an isomorphism of $B_{(G/H)_+} \times_\beta G_+$ onto $\operatorname{Ind}_{H^\perp}^{\widehat{G}}\big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ and this completes the proof of Theorem 5.3.2. $\square$

LEMMA 5.3.3. *Let* $\big(i_{B_{G_+}}, i_{G_+}\big)$ *and* $\big(j_{B_{(G/H)_+}}, j_{G_+}\big)$ *denote the universal representations of the dynamical systems* $\big(B_{G_+}, G_+, \alpha\big)$ *and* $\big(B_{(G/H)_+}, G_+, \beta\big)$ *respectively and* q *be the quotient map of* G *onto* G/H*. Then there exists a surjective homomorphism*

$$\theta_H : B_{G_+} \times_\alpha G_+ \to B_{(G/H)_+} \times_\beta G_+,$$

such that $\theta_H \circ i_{B_{G_+}}(1_x) = j_{B_{(G/H)_+}}(1_{q(x)})$ *and* $\theta_H \circ i_{G_+}(y) = j_{G_+}(y)$ *for all* $x, y \in G_+$.

PROOF. Lemma 4.2.5 says that there is a surjective homomorphism $\phi : B_{G_+} \to B_{(G/H)_+}$ satisfying $\phi(1_x) = 1_{q(x)}$ for $x \in G_+$, so the map $j_{B_{(G/H)_+}} \circ \phi : B_{G_+} \to B_{(G/H)+} \times_\beta G_+$ is a unital homomorphism. The map j_{G_+} is a covariant isometric representation of G_+ into the semigroup of isometries of $B_{(G/H)+} \times_\beta G_+$. For $x, y \in G_+$ we have

$$j_{B_{(G/H)_+}} \circ \phi\big(\alpha_x(1_y)\big) = j_{B_{(G/H)_+}}(1_{q(x+y)}) \tag{5.3.3}$$
$$= j_{B_{(G/H)_+}}\big(\beta_x(1_{q(y)})\big)$$
$$= j_{G_+}(x) j_{B_{(G/H)_+}}(1_{q(y)}) j_{G_+}(x)^*$$
$$= j_{G_+}(x) j_{B_{(G/H)_+}}\big(\phi(1_y)\big) j_{G_+}(x)^*.$$

Hence by linearity and continuity of $j_{B_{(G/H)_+}}, \phi$ and α_x, the calculations in (5.3.3) are true for every $a \in B_{G_+}$. Therefore the pair $\big(j_{B_{(G/H)_+}} \circ \phi, j_{G_+}\big)$ is a covariant representation of the dynamical system $\big(B_{G_+}, G_+, \alpha\big)$ in the C^*-algebra $B_{(G/H)_+} \times_\beta G_+$. Thus there exists a unital homomorphism

$$\theta_H : B_{G_+} \times_\alpha G_+ \to B_{(G/H)_+} \times_\beta G_+,$$

such that $\theta_H \circ i_{G_+}(y) = j_{G_+}(y)$ and $\theta_H \circ i_{B_{G_+}}(1_x) = j_{B_{(G/H)_+}}\big(\phi(1_x)\big) = j_{B_{(G/H)_+}}(1_{q(x)})$ for all $x, y \in G_+$. Moreover, since the range of θ_H is a C^*-subalgebra of $B_{(G/H)_+} \times_\beta G_+$ containing all the generators, θ_H is surjective. $\square$

THEOREM 5.3.4. *Let I_{H_+} be the extendibly α_x-invariant ideal of B_{G_+} in Corollary 4.1.6, Ψ be the isomorphism of Theorem 5.3.2, $\big(i_{B_{G_+}}, i_{G_+}\big)$ and $\big(j_{B_{(G/H)_+}}, j_{G_+}\big)$ denote the universal homomorphisms of the crossed products $B_{G_+} \times_\alpha G_+$ and $B_{(G/H)_+}$ $\times_\beta G_+$ respectively and θ_H be the homomorphism of Lemma 5.3.3. Define $\Upsilon = \Psi \circ \theta_H$. Then the following is a short exact sequence of C^*-algebras*

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\Upsilon} \mathrm{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big) \to 0 \tag{5.3.4}$$

in which ϕ is an isomorphism of $I_{H_+} \times_\alpha G_+$ onto the ideal $D := \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a)$ $i_{G_+}(y) : a \in I_{H_+}, x, y \in G_+\}$.

PROOF. We will apply Theorem 1.7 of [**24**] and to do so we need first to check that G_+ is an Ore-semigroup of G. Since G_+ is a subset of G then it is cancellative. We still need G_+ to be right-reversible, so for $y, z \in G_+$, we have $y + G_+ \bigcap z + G_+ \neq \emptyset$ since $y + z \in y + G_+$ and $z + y \in z + G_+$ therefore $z + y \in y + G_+ \bigcap z + G_+$. Hence

G_+ is Ore-semigroup of G. Therefore [**24**, Theorem 1.7] implies that there is a short exact sequence

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\varphi} B_{G_+}/I_{H_+} \times_{\widetilde{\alpha}} G_+ \to 0$$

in which

$$\varphi \circ i_{B_{G_+}}(1_x) = j_{B_{G_+}/I_{H_+}}(1_x + I_{H_+}) \text{ and } \varphi \circ i_{G_+}(y) = j_{G_+}(y),$$

and $I_{H_+} \times_\alpha G_+$ is isomorphic to the ideal $D := \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in I_{H_+}, x, y \in G_+\}$ in $B_{G_+} \times_\alpha G_+$. But Lemma 5.2.2 says that $B_{(G/H)_+} \times_\beta G_+$ is isomorphic to $B_{G_+}/I_{H_+} \times_{\widetilde{\alpha}} G_+$. Therefore there is a short exact sequence

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\theta_H} B_{(G/H)_+} \times_\beta G_+ \to 0 \tag{5.3.5}$$

in which

$$\theta_H \circ i_{B_{G_+}}(1_x) = j_{B_{(G/H)_+}}(1_{q(x)}) \text{ and } \theta_H \circ i_{G_+}(y) = j_{G_+}(y).$$

Now as $B_{(G/H)_+} \times_\beta G_+$ is isomorphic to $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$, then $\Upsilon = \Psi \circ \theta_H$ is a map from $B_{G_+} \times_\alpha G_+$ onto $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$ with kernel $I_{H_+} \times_\alpha G_+$ (this is true by exactness of (5.3.5) and because Ψ is an isomorphism of $B_{(G/H)_+} \times_\beta G_+$ onto $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$). Thus we have the following short exact sequence

$$0 \to I_{H_+} \times_\alpha G_+ \xrightarrow{\phi} B_{G_+} \times_\alpha G_+ \xrightarrow{\Upsilon} \operatorname{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right) \to 0.$$

$\square$

COROLLARY 5.3.5. *Let $\left(i_{B_{G_+}}, i_{G_+}\right)$ be the universal homomorphisms of the crossed product $B_{G_+} \times_\alpha G_+$. Then the ideal $D = \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in I_{H_+}, x, y \in G_+\}$ of $B_{G_+} \times_\alpha G_+$ in Theorem 5.3.4 is generated by $\left\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\right\}$.*

PROOF. Since $i_{G_+}(x)^*, i_{G_+}(y) \in B_{G_+} \times_\alpha G_+$, D is generated by $\left\{i_{B_{G_+}}(a) : a \in I_{H_+}\right\}$. So to prove this corollary it suffices to show that for $a \in I_{H_+}$, $i_{B_{G_+}}(a)$ is in the ideal generated by $\left\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\right\}$. To see this, fix $x \in G_+$ and $h \in H_+$.

Then

$$\begin{aligned}
i_{B_{G_+}}(1_x - 1_{x+h}) &= i_{B_{G_+}}(1_x) - i_{B_{G_+}}(1_{x+h}) \\
&= i_{G_+}(x) i_{G_+}(x)^* - i_{G_+}(x+h)\, i_{G_+}(x+h)^* \\
&= i_{G_+}(x)(1 - i_{G_+}(h)\, i_{G_+}(h)^*) i_{G_+}(x)^* \\
&= i_{G_+}(x)\, i_{B_{G_+}}(1 - 1_h)\, i_{G_+}(x)^*.
\end{aligned}$$

Hence $i_{B_{G_+}}(1_x - 1_{x+h})$ is in the ideal generated by $\big\{ i_{B_{G_+}}(1 - 1_u) : u \in H_+ \big\}$. Therefore by continuity of $i_{B_{G_+}}$ we have that $i_{B_{G_+}}(a)$ is in the ideal generated by $\big\{ i_{B_{G_+}}(1 - 1_u) : u \in H_+ \big\}$ for all $a \in I_{H_+}$. $\square$

REMARK 5.3.6. Let $\big(i_{B_{G_+}}, i_{G_+}\big)$ be the universal covariant representation of the dynamical system (B_{G_+}, G_+, α). Then $i_{B_{G_+}}(1_x) = i_{G_+}(x) i_{G_+}(x)^*$ and from [**21**, Corollary 2.4] we know that the map $i_{B_{G_+}}$ is injective, so for simplicity we write 1_x for $i_{G_+}(x) i_{G_+}(x)^*$. Hence one can say that the crossed product $I_{H_+} \times_\alpha G_+$ in (5.3.5) is generated by the set $\{1 - 1_u : u \in H_+\}$.

5.4. The crossed product $B_{H_+} \times_\alpha H_+$ and its commutator ideal

The following proposition is interesting as it allows us to view the crossed product $B_{H_+} \times_\alpha H_+$ as a C^*-subalgebra of the crossed product $B_{G_+} \times_\alpha G_+$.

PROPOSITION 5.4.1. *Let (G, G_+) be a lattice-ordered group with G abelian, H_+ be a hereditary subsemigroup of G_+ and $\big(i_{B_{G_+}}, i_{G_+}\big)$ denote the universal representation of the dynamical systems $\big(B_{G_+}, G_+, \alpha\big)$ in which α is the action in Lemma 3.4.1. Then there is an isomorphism ι of $B_{H_+} \times_\alpha H_+$ into $B_{G_+} \times_\alpha G_+$.*

PROOF. The existence of the crossed product $B_{H_+} \times_\alpha H_+$ follows directly from Remark 3.2.4 and Remark 3.4.3. Let $V := i_{G_+}|_{H_+}$. Then V is a covariant isometric representation of H_+. Since $B_{H_+} \times_\alpha H_+$ is universal for covariant isometric representations, there is a unital representation $\pi_V : B_{H_+} \to B_{G_+} \times_\alpha G_+$ such that $\pi_V(1_x) = V_x V_x^*$ for all $x \in H_+$. Hence, there is a unital representation $\pi_V \times V : B_{H_+} \times_\alpha H_+ \to B_{G_+} \times_\alpha G_+$ such that $(\pi_V \times V) \circ i_{B_{H_+}} = \pi_V$ and $(\pi_V \times V) \circ i_{H_+} = V$.

Notice that

$$\begin{aligned}\pi_V(1_x) = V_x V_x^* &= i_{G_+}(x) i_{G_+}(x)^* \\ &= i_{B_{G_+}}(1_x), \text{ since } \big(i_{B_{G_+}}, i_{G_+}\big) \text{ is the universal representation.}\end{aligned}$$

Then π_V and $i_{B_{G_+}}$ agree on the generators of B_{H_+}. Therefore $\pi_V = i_{B_{G_+}}|_{B_{H_+}}$ and so π_V is faithful (see Remark 3.4.5). By Proposition 3.1 and Theorem 3.7 of [**21**] $\pi_V \times_\alpha V$ is faithful. Taking $\iota := \pi_V \times_\alpha V$ we obtain the desired result. □

Definition 5.4.2. Let A be a C^*-algebra. The *commutator* ideal $\mathcal{C}$ of A is the closed ideal generated by $\{ab - ba : a, b \in A\}$.

Remark 5.4.3. The commutator ideal of a C^*-algebra A is the smallest closed ideal $\mathcal{C}$ in A such that $A/\mathcal{C}$ is commutative [**26**, §3.5].

The following results will lead us to identify the commutator ideal of the C^*-algebra $B_{H_+} \times_\alpha H_+$. We first introduce the algebra

$$B_{H_+,\infty} := \{f \in B_{H_+} : \lim_{h\to\infty} f(h) = 0\}. \tag{5.4.1}$$

Adji in [**1**] has talked about the commutator ideal in the case of totally ordered groups. Here, we are generalizing her results to more general cases (lattice-ordered groups) so extra work need to be done and more challenges to the proofs have been added.

Lemma 5.4.4. *Suppose that (G, G_+) is a lattice-ordered group with G abelian and suppose that H_+ is a hereditary subsemigroup of G_+. Then the algebra $B_{H_+,\infty}$ is the closed span of $\{1 - 1_h : h \in H_+\}$.*

Proof. Let A be the closed span of $\{1 - 1_h : h \in H_+\}$. Fix $h \in H_+$, for $u \geq h$ we have

$$(1 - 1_h)(u) = 1(u) - 1_h(u) = 0.$$

Therefore $\lim_{u\to\infty}(1 - 1_h)(u) = 0$ and so $1 - 1_h \in B_{H_+,\infty}$.

For any $f \in A$, $f = \lim_{n\to\infty} f_n$ where $f_n = \sum_{h_i \in F_n} \lambda_i(1 - 1_{h_i})$ and F_n is a finite subset of H_+. Fix $\varepsilon > 0$ then there exists $n \in \mathbb{N}$ such that $\|f - f_n\| < \varepsilon$. Let $h_n = \vee F_n$, then for $u \geq h_n$ we have

$$\begin{aligned} |f(u)| &= |f(u) - f_n(u) + f_n(u)| \\ &\leq |f(u) - f_n(u)| + |f_n(u)| \\ &< \varepsilon + 0 = \varepsilon, \text{ since } |f(u) - f_n(u)| \leq \|f - f_n\|. \end{aligned}$$

Hence $f \in B_{H_+,\infty}$ and so $A \subset B_{H_+,\infty}$.

To show that $B_{H_+,\infty} \subset A$, we need first to show that for any $f \in B_{H_+}$, $\lim_{u\to\infty} f(u)$ exists. To see this, suppose that $f \in B_{H_+}$. Then $f = \lim_{n\to\infty} f_n$ where $f_n = \sum_{h_i \in F_n} \lambda_i 1_{h_i}$ and F_n is a finite subset of H_+.

Claim. Suppose that $x_n := \lim_{u\to\infty} f_n(u)$, then $\{x_n\}$ converges.

Proof. Notice that every $x_n \in \mathbb{C}$ so it is enough to show that $\{x_n\}$ is a Cauchy sequence (this is true since $\mathbb{C}$ is a Hilbert space). But $\{f_n\}$ is a Cauchy sequence in B_{H_+}, therefore $\{x_n\}$ is a Cauchy sequence. To see this, fix $\varepsilon > 0$ then there exists N such that

$$\|f_n - f_m\| < \varepsilon \text{ for all } n, m > N,$$

where $\|f_n - f_m\| = \sup_{x \in H_+} |f_n(x) - f_m(x)|$. Now

$$\begin{aligned} |x_n - x_m| &= |\lim_{u\to\infty} f_n(u) - \lim_{u\to\infty} f_m(u)| \\ &= |\lim_{u\to\infty} (f_n(u) - f_m(u))| \\ &= \lim_{u\to\infty} |f_n(u) - f_m(u)| \\ &\leq \|f_n - f_m\| \\ &< \varepsilon. \end{aligned}$$

$\square$

Fix $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\|f - f_m\| < \varepsilon/2$ and $|\lim_{n\to\infty} x_n - x_m| < \varepsilon/2$. Let $h_n = \vee F_n$. Then for $u \geq h_n$ we have

$$\begin{aligned}
|f(u) - \lim_{n\to\infty} x_n| &= |f(u) - f_m(u) + f_m(u) - \lim_{n\to\infty} x_n| \\
&\leq |f(u) - f_m(u)| + |f_m(u) - \lim_{n\to\infty} x_n| \\
&< \varepsilon/2 + |x_m - \lim_{n\to\infty} x_n|, \text{ as } u \geq h_n \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Hence $\lim_{u\to\infty} f(u)$ exists.

To complete the proof, suppose that $f \in B_{H_+}$ such that $\lim_{u\to\infty} f(u) = 0$. Then there exists $\{f_n\}$ such that $f_n \to f$ where $f_n = \sum_{h_i \in F_n} \lambda_i 1_{h_i}$ and F_n is a finite subset of H_+. Let $x_n = \lim_{u\to\infty} f_n(u)$, then $\lim_{n\to\infty} x_n = 0$ (by the previous part of this proof). Define $g_n := f_n - x_n 1$ then $g_n = \sum_{h_i \in F_n} -\lambda_i(1 - 1_{h_i}) \in A$ and

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} (f_n - x_n 1) = f.$$

Therefore $B_{H_{+,\infty}} \subset A$. Consequently, $A = B_{H_{+,\infty}}$. □

LEMMA 5.4.5. *Suppose that (G, G_+) is a lattice-ordered group with G abelian, H_+ is a hereditary subsemigroup of G_+ and α is the action in Lemma 3.4.1. Then the algebra $B_{H_{+,\infty}}$ is an extendibly α-invariant ideal of B_{H_+}.*

PROOF. To see that $B_{H_{+,\infty}}$ is a closed ideal, fix $t, u \in H_+$. Then

$$1_t(1 - 1_u) = 1_t - 1_{t\vee u} = (1 - 1_{t\vee u}) - (1 - 1_t) \in B_{H_{+,\infty}},$$

and by continuity of multiplication in B_{H_+} we conclude that $B_{H_{+,\infty}}$ is a closed ideal of B_{H_+}.

Calculations show that the set $S = \{1 - 1_u : u \in H_+\}$ is an approximate identity for $B_{H_{+,\infty}}$. For details, see appendix A.

For $z \in H_+$, α_z is linear and continuous so routine calculations show that $B_{H_{+,\infty}}$ is α-invariant. Another routine calculation shows that for $(1 - 1_t) \in B_{H_{+,\infty}}$ the approximate identity S satisfies

$$\alpha_z(1 - 1_u)(1 - 1_t) \to \psi(\alpha_z(1))(1 - 1_u), \tag{5.4.2}$$

where ψ is the canonical map in Definition 4.0.7. For any $b \in B_{H_+,\infty}$, standard $\varepsilon/3$ argument (see the argument in the proof of Corollary 4.1.6) shows that it satisfies Equation 5.4.2 with $(1-1_t)$ replaced by b. Thus this completes the proof that $B_{H_+,\infty}$ is an extendibly α-invariant ideal of B_{H_+}. $\square$

REMARK 5.4.6. In [**3**, §3] Adji shows that for a totally-ordered group Γ with positive cone Γ^+, there is a short exact sequence

$$0 \to B_{\Gamma^+,\infty} \xrightarrow{\iota} B_{\Gamma^+} \xrightarrow{\delta} \mathbb{C} \to 0,$$

where $\delta : B_{\Gamma^+} \to \mathbb{C}$ defined by $\delta(f) = \lim_{x\to\infty} f(x)$. This result still holds for a lattice-ordered group (G, G_+).

COROLLARY 5.4.7. *Suppose that (G, G_+) is a lattice-ordered group with G abelian, H_+ is a hereditary subsemigroup of G_+, α is the action in Lemma 3.4.1, $(i_{B_{H_+}}, i_{H_+})$ is the universal covariant representation of (B_{H_+}, H_+, α) and $B_{H_+,\infty}$ is the extendibly α-invariant ideal in Lemma 5.4.5. Then there is a short exact sequence of C^*-algebras*

$$0 \to B_{H_+,\infty} \times_\alpha H_+ \xrightarrow{\phi} B_{H_+} \times_\alpha H_+ \to C(\widehat{H}) \to 0,$$

in which ϕ is an isomorphism of $B_{H_+,\infty} \times_\alpha H_+$ onto the ideal $D = \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in B_{H_+,\infty},\ x, y \in H_+\}$ of $B_{H_+} \times_\alpha H_+$. Moreover, $B_{H_+,\infty} \times_\alpha H_+$ is the commutator ideal of $B_{H_+} \times_\alpha H_+$.

PROOF. Since H_+ is an Ore-semigroup of H (this is true, because in the proof of Theorem 5.3.4 we showed that G_+ is an Ore-semigroup of G and as H_+ is a subset of H). Then [**24**, Theorem 1.7] implies that there exists the following short exact sequence

$$0 \to B_{H_+,\infty} \times_\alpha H_+ \to B_{H_+} \times_\alpha H_+ \to (B_{H_+}/B_{H_+,\infty}) \times_{\tilde{\alpha}} H_+ \to 0 \tag{5.4.3}$$

with $B_{H_+,\infty} \times_\alpha H_+$ isomorphic to the ideal $D = \overline{\operatorname{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in B_{H_+,\infty},\ x, y \in H_+\}$ of $B_{H_+} \times_\alpha H_+$.

We know from Remark 5.4.6 that $B_{H_+}/B_{H_+,\infty}$ is isomorphic to $\mathbb{C}$. Moreover, notice that $\mathbb{C}$ has only the trivial action, i.e. id, so the crossed product $B_{H_+}/B_{H_+,\infty} \times_{\tilde{\alpha}}$

H_+ will be isomorphic to $\mathbb{C} \times_{\mathrm{id}} H_+$. Since $\mathbb{C}$ has only the unital representation $z \mapsto z\,1$, then the covariance condition gives that the system $(\mathbb{C}, H_+, \mathrm{id})$ consists of unitaries. Moreover, since $H = H_+ - H_+$ then [**29**] gives that $\mathbb{C} \times_{\mathrm{id}} H_+$ is isomorphic to $C^*(H)$ and as H is abelian then $C^*(H)$ is isomorphic to $C(\widehat{H})$. Thus we have the desired short exact sequence.

We know from Corollary 5.3.5 that the ideal $D = \overline{\mathrm{span}}\{i_{G_+}(x)^* i_{B_{G_+}}(a) i_{G_+}(y) : a \in B_{H_+,\infty},\ x, y \in H_+\}$ of $B_{H_+} \times_\alpha H_+$ is generated by $\{1 - 1_u : u \in H_+\}$. For $u \in H_+$, $1 - 1_u = i_{H_+}(u)^* i_{H_+}(u) - i_{H_+}(u) i_{H_+}(u)^* \in \mathcal{C}_H$ (the commutator ideal) of $B_{H_+} \times_\alpha H_{H_+}$, which means that $B_{H_+,\infty} \times_\alpha H_+ \subset \mathcal{C}_H$. Moreover, since $(B_{H_+} \times_\alpha H_+ / B_{H_+,\infty} \times_\alpha H_+) \simeq C(\widehat{H})$ is commutative then $\mathcal{C}_H \subset B_{H_+,\infty} \times_\alpha H_+$. Thus $B_{H_+,\infty} \times_\alpha H_+$ is the commutator ideal of $B_{H_+} \times_\alpha H_+$. $\square$

CHAPTER 6

Primitive ideals in the crossed product $B_{G_+} \times_\alpha G_+$

We start this chapter with a section on the necessary definitions and results about primitive ideals in a C^*-algebra A.

6.1. Definitions and background material

DEFINITION 6.1.1. A non-zero representation π of a C*-algebra A on a Hilbert space $\mathcal{H}$ is called *irreducible* if the only closed subspaces $\mathcal{K}$ of $\mathcal{H}$ such that $\pi(a)\mathcal{K} \subset \mathcal{K}$ are $\mathcal{K} = \{0\}$ and $\mathcal{K} = \mathcal{H}$ for all $a \in A$ [**37**, A.1].

DEFINITION 6.1.2. A *primitive* ideal of a C*-algebra A is an ideal which is the kernel of an irreducible representation of A [**8**, Definition II.6.5.1].

We now prove a minor lemma in order to use it later in some results.

LEMMA 6.1.3. *Suppose that A, C are two C^*-algebras, $\phi : A \to C$ is surjective homomorphism and $\pi : C \to B(\mathcal{H})$ is an irreducible representation of C. Then $\pi \circ \phi$ is an irreducible representation of A.*

PROOF. Suppose that $\mathcal{K} \subset \mathcal{H}$ is a closed subspace of $\mathcal{H}$ such that $\pi \circ \phi(a)\mathcal{K} \subset \mathcal{K}$ for all $a \in A$. Since ϕ is surjective, then for each $c \in C$ there is $a \in A$ such that $\phi(a) = c$. Notice that

$$\pi(c) = \pi(\phi(a)) = \pi \circ \phi(a),$$

and hence

$$\pi \circ \phi(a)\mathcal{K} = \pi(c)\mathcal{K} \subset \mathcal{K}.$$

But π is an irreducible representation, therefore either $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$. Thus $\pi \circ \phi$ is an irreducible representation of A. □

The following proposition talks about primitive ideals and their relation to other ideals in a C^*-algebra A. The proof can be found in [**37**, proposition A.17].

PROPOSITION 6.1.4. *Let A be a C^*-algebra. Then*

(i) *every closed ideal I in A is the intersection of the primitive ideals containing it;*

(ii) *if I is a primitive ideal in A, and J, K are two ideals such that $J \bigcap K \subset I$, then either $J \subset I$ or $K \subset I$.*

6.2. The composition $Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H$ and primitive ideals

In this section we use our construction theorem (Theorem 5.3.4) to study the primitive ideals in the crossed product $B_{G_+} \times_\alpha G_+$.

REMARK 6.2.1. Given a lattice-ordered abelian group (G, G_+), the set $\sum(G)$ of all subgroups $H := H_+ - H_+$, where H_+ is any hereditary subsemigroup of G_+, is partially ordered by inclusion.

The following proposition is a version of Adji-Raeburn theorem [**5**, Theorem 3.1] about primitive ideals in the crossed product $B_{G_+} \times_\alpha G_+$. Our proposition is interesting and has more challenges since we are working under the assumption that the group G is lattice-ordered, which extends their result to more general cases.

PROPOSITION 6.2.2. *Let (G, G_+) be a lattice-ordered abelian group, $\sum(G)$ be the chain of subgroups H in G as in Remark 6.2.1, Q the homomorphism in Proposition 5.2.3, $\widehat{\beta}_\gamma^{-1}$ the dual action in Lemma 3.4.6 and θ_H be the surjective homomorphism in Lemma 5.3.3. Then for $H \in \sum(G)$ and $\gamma \in \widehat{G}$, $\ker(Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H)$ is a primitive ideal of $B_{G_+} \times_\alpha G_+$ which depends only on $\gamma|_H$ and the map $(H, \gamma) \mapsto \ker(Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H)$ induces a well-defined map F from the disjoint union $\bigsqcup\{\widehat{H} : H \in \sum G\}$ to $\mathrm{Prim}\,(B_{G_+} \times_\alpha G_+)$.*

Remark 6.2.3.

(i) We write (H,γ) for the element of the disjoint union $\bigsqcup \widehat{H}$ corresponding to $\gamma \in \widehat{H}$.

(ii) Since H is a closed subgroup of G and G is discrete, then by [**17**, Corollary 4.41] for any $\gamma \in \widehat{H}$ there exists a character χ of G such that $\chi|_H = \gamma$.

(iii) Given $\gamma \in \widehat{H}$, then we choose $\chi \in \widehat{G}$ such that $\chi|_H = \gamma$ in order to realize $F(H,\gamma)$ as a kernel. Thus

$$F(H,\gamma) := \ker(Q \circ \widehat{\beta}_\chi^{-1} \circ \theta_H) \text{ in which } \chi \in \widehat{G} \text{ satisfies } \chi|_H = \gamma.$$

Proof of proposition 6.2.2. To prove this proposition we are going to use two facts, one is the short exact sequence in Corollary 5.3.4 and the other is from [**37**, Proposition 6.16] which proves that every irreducible representation of $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ has the form $M(\gamma,\rho) : f \to \rho(f(\gamma))$ for some $\gamma \in \widehat{G}$ and some irreducible representation ρ of $B_{(G/H)_+} \times_\tau (G/H)_+$.

Since the identity representation ι of $B_{(G/H)_+} \times_\tau (G/H)_+$ is irreducible (see [**30**, Theorem 3.13]), then

$$\begin{aligned}
\ker(Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H) &= \big\{a \in B_{G_+} \times_\alpha G_+ : Q(\widehat{\beta}_\gamma^{-1}(\theta_H(a))) = 0\big\} \\
&= \big\{a \in B_{G_+} \times_\alpha G_+ : \iota(Q(\widehat{\beta}_\gamma^{-1}(\theta_H(a)))) = 0\big\}, \text{ since } \iota \text{ is faithful} \\
&= \big\{a \in B_{G_+} \times_\alpha G_+ : \iota(\Psi(\theta_H(a))(\gamma)) = 0\big\}, \Psi = Q \circ \widehat{\beta}_\gamma^{-1} \\
&= \big\{a \in B_{G_+} \times_\alpha G_+ : M(\gamma,\iota) \circ \Psi(\theta_H(a)) = 0\big\} \\
&= \big\{a \in B_{G_+} \times_\alpha G_+ : M(\gamma,\iota) \circ \Upsilon(a) = 0\big\}, \Upsilon = \Psi \circ \theta_H \\
&= \ker M(\gamma,\iota) \circ \Upsilon.
\end{aligned}$$

The map $M(\gamma,\iota)$ is an irreducible representation of $\operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ and Υ is a surjective homomorphism, therefore Lemma 6.1.3 implies that $M(\gamma,\iota) \circ \Upsilon$ is an irreducible representation of $B_{G_+} \times_\alpha G_+$. Hence $\ker(Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H)$ is a primitive ideal.

To show F well-defined recall Equation (5.2.1) which says

$$Q \circ \widehat{\beta}_\mu^{-1} = \widehat{\tau}_\mu^{-1} \circ Q, \text{ for } \mu \in H^\perp = (G/H)^\wedge.$$

Suppose that $\gamma, \mu \in \widehat{G}$ such that $\gamma H^{\perp} = \mu H^{\perp}$, then $\gamma\mu^{-1} \in H^{\perp}$. Therefore

$$\begin{aligned}
\ker(Q \circ \widehat{\beta}_{\gamma}^{-1} \circ \theta_H) &= \ker(Q \circ \widehat{\beta}_{\gamma\mu\mu^{-1}}^{-1} \circ \theta_H), \text{ since } \mu\mu^{-1} = \mathrm{id} \\
&= \ker(Q \circ \widehat{\beta}_{\mu\gamma\mu^{-1}}^{-1} \circ \theta_H) \\
&= \ker(Q \circ \widehat{\beta}_{\gamma\mu^{-1}}^{-1} \circ \widehat{\beta}_{\mu}^{-1} \circ \theta_H), \ \widehat{\beta}^{-1} \text{is a homomorphism} \\
&= \ker(\widehat{\tau}_{\gamma\mu^{-1}}^{-1} \circ Q \circ \widehat{\beta}_{\mu}^{-1} \circ \theta_H), \text{ since } \gamma\mu^{-1} \in H^{\perp} \\
&= \ker(Q \circ \widehat{\beta}_{\mu}^{-1} \circ \theta_H), \text{ since } \widehat{\tau}_{\gamma\mu^{-1}}^{-1} \text{ is injective.}
\end{aligned}$$

Thus F is well-defined. □

Corollary 6.2.4. *Suppose that ρ is an irreducible representation of $B_{G_+} \times_\alpha G_+$. Then there are a subgroup $H \in \sum(G)$, a character $\gamma \in \widehat{G}$ and an irreducible representation π of $B_{(G/H)_+} \times_\tau (G/H)_+$ such that ρ is equivalent to $M(\gamma, \pi) \circ \Upsilon$.*

Proof. Let $V := \rho \circ i_{G_+}$. Then V is a covariant isometric representation of G_+. To see this, fix $x \in G_+$. Then

$$\begin{aligned}
V_x^* V_x &= (\rho \circ i_{G_+}(x))^* (\rho \circ i_{G_+}(x)) \\
&= \rho(i_{G_+}(x)^* i_{G_+}(x)) \\
&= \rho(1_{B_{G_+} \times_\alpha G_+}), \text{ since } i_{G_+} \text{ is an isometric representation} \\
&= 1_{B(\mathcal{H})}.
\end{aligned}$$

So V is an isometric representation of G_+ and since i_{G_+} is covariant (Lemma 3.4.4), then V is a covariant isometric representation of G_+. Hence we have $\rho = \rho_V$ for $V = \rho \circ i_{G_+}$ (this is true because if δ was the map associated to V, then $\delta(i_{G_+}(x)) = V_x$ for all $x \in G_+$. But $V_x = \rho(i_{G_+}(x))$, hence $\delta(i_{G_+}(x)) = \rho(i_{G_+}(x))$. Since $\{i_{G_+}(x) : x \in G_+\}$ generates $B_{G_+} \times_\alpha G_+$, then δ and ρ agree on generators and thus $\delta = \rho$).

We now construct the subgroup H. Let $H_+ := \{x \in G_+ : V_x V_x^* = 1\}$, then H_+ is a hereditary subsemigroup of G_+. To see this, suppose that $y \in H_+$ and $0 \leq x \leq y$. Then

$$1 = V_y V_y^* = V_{y-x} V_x V_x^* V_{y-x}^*,$$

and conjugating by V_{y-x}^* implies that $V_xV_x^* = 1$, so that $x \in H_+$. Hence take $H := H_+ - H_+$.

We know that the short exact sequence (5.3.4) has an ideal $I_{H_+} \times_\alpha G_+$ generated by $\{1 - 1_h : h \in H_+\}$. So for $h \in H_+$, we have

$$\begin{aligned}\rho_V(1 - 1_h) &= 1 - \rho_V(1_h)\\ &= 1 - \rho_V(i_{G_+}(h)i_{G_+}(h)^*)\\ &= 1 - \rho_V(i_{G_+}(h))\rho_V(i_{G_+}(h))^*\\ &= 0,\end{aligned}$$

and hence ρ_V vanishes on $I_{H_+} \times_\alpha G_+$. Thus it follows from the exactness of (5.3.4) that there is a representation σ of $\mathrm{Ind}_{H^\perp}^{\widehat{G}}\left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$ such that $\rho_V = \sigma \circ \Upsilon$ and since σ and ρ_V have the same range we deduce that σ is irreducible. Therefore by [**37**, Proposition 6.16] we have σ is equivalent to an irreducible representation $M(\gamma, \pi)$ for some $\gamma \in \widehat{G}$ and some irreducible representation ρ of $B_{(G/H)_+} \times_\tau (G/H)_+$. Thus ρ_V is equivalent to $M(\gamma, \pi) \circ \Upsilon$. □

6.3. Examples

EXAMPLE 6.3.1. We know that $(\mathbb{Z}, \mathbb{N})$ is a lattice-ordered abelian group. By [**30**] we have a short exact sequence of C^*-algebras

$$0 \to \mathcal{K} \to \mathcal{T}(\mathbb{Z}) \xrightarrow{\Phi} C(\widehat{\mathbb{Z}}) \to 0,$$

in which $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ is the set of compact operators on $\ell^2(\mathbb{N})$, a C^*-subalgebra of $B(\ell^2(\mathbb{N})))$. We also know that the primitive ideals of $\mathcal{T}(\mathbb{Z})$ (the Toeplitz algebra of $\mathbb{Z}$, for details see the introduction page 10) are of the form $\{0\}$ and $\Phi^{-1}(I_\gamma)$ where γ is an element of $\widehat{\mathbb{Z}}$ and $I_\gamma = \{f \in C(\widehat{\mathbb{Z}}) : f(\gamma) = 0\}$. We want to compare those primitive ideals with the ones we get from our construction theorem (Theorem 5.3.4). First of all notice that we have only two hereditary subsemigroups H_+ of $\mathbb{N}$ namely $\{0\}$ and $\mathbb{N}$ itself.

For $H = \{0\}$, $G/H = \mathbb{Z}/\{0\} \cong \mathbb{Z}$ and so the ideal $I_{H_+} \times_\alpha G_+$ will be the zero ideal. Moreover, θ_H and Q will be both isomorphisms of $B_\mathbb{N} \times_\alpha \mathbb{N} \cong \mathcal{T}(\mathbb{Z})$ onto itself.

For any $\gamma \in \widehat{H}$, we can extend γ to $\mathrm{id} \in \widehat{\mathbb{Z}}$ so $\widehat{\beta}_\gamma^{-1}$ is the identity map and therefore $\ker(Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H) = \{0\}$. We know from [**30**, Theorem 3.13] that $\{0\}$ is a primitive ideal of $\mathcal{T}(\mathbb{Z})$.

If $H_+ = \mathbb{N}$, $H = \mathbb{N} - \mathbb{N} = \mathbb{Z}$ and $G/H = \mathbb{Z}/\mathbb{Z} \cong \{0\}$. Therefore

$$\mathrm{Ind}_{\mathbb{Z}^\perp}^{\widehat{\mathbb{Z}}}(B_{\{0\}} \times_\tau \{0\}) \cong \mathrm{Ind}_{\mathbb{Z}^\perp}^{\widehat{\mathbb{Z}}}(\mathbb{C}), \text{ since } B_{\{0\}} \times_\tau \{0\} \cong \mathcal{T}(\{0\}) \cong \mathbb{C}.$$

Now $\mathbb{Z}^\perp$ contains the identity character only, hence

$$\mathrm{Ind}_{\mathbb{Z}^\perp}^{\widehat{\mathbb{Z}}}(\mathbb{C}) \cong C(\widehat{\mathbb{Z}}) \cong C(\mathbb{T}).$$

Now for $\gamma \in \widehat{\mathbb{Z}}$ we have

$$\begin{aligned}
\Upsilon^{-1}(I_\gamma) &= \{\Upsilon^{-1}(f) : f(\gamma) = 0\} \\
&= \{a \in \mathcal{T}(\mathbb{Z}) : a = \Upsilon^{-1}(f) \text{ and } f(\gamma) = 0\} \\
&= \{a \in \mathcal{T}(\mathbb{Z}) : \Upsilon(a)(\gamma) = 0\} \\
&= \{a \in \mathcal{T}(\mathbb{Z}) : Q(\widehat{\beta}_\gamma^{-1}(\theta_H(a))) = 0\} \\
&= \ker(Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H).
\end{aligned}$$

EXAMPLE 6.3.2. We know that $(\mathbb{Z}^2, \mathbb{N}^2)$ is a lattice-ordered group with $\mathbb{Z}^2$ abelian. We will compare the primitive ideals in the crossed product $B_{\mathbb{N}^2} \times_\alpha \mathbb{N}^2$ with those we can get from our decomposition theorem (Theorem 5.3.4). Recall that we have four different hereditary subsemigroups of $\mathbb{N}^2$, $\mathcal{T}(\mathbb{Z}) \cong C^*(\mathbb{N})$ and $\mathcal{T}(\mathbb{Z}^2) \cong \mathcal{T}(\mathbb{Z}) \otimes \mathcal{T}(\mathbb{Z}) \cong B_{\mathbb{N}^2} \times_\alpha \mathbb{N}^2$.

For H=$\{(0,0)\}$, $G/H = \mathbb{Z}^2/\{(0,0)\} \cong \mathbb{Z}^2$. The ideal $I_{H_+} \times_\alpha G_+$ is generated by $\{1 - 1_h : h \in H_+\} = \{0\}$, so $I_{H_+} \times_\alpha G_+ = \{0\}$. Hence, $\{0\}$ is a primitive ideal of $B_{\mathbb{N}^2} \times_\alpha \mathbb{N}^2$ since the identity representation of $B_{\mathbb{N}^2} \times_\alpha \mathbb{N}^2$ is irreducible (see [**30**, Theorem 3.13]), and $B_{\mathbb{N}^2} \times_\alpha \mathbb{N}^2 \cong \mathrm{Ind}_{H^\perp}^{\widehat{\mathbb{Z}}^2}(B_{\mathbb{N}^2} \times_\tau \mathbb{N}^2)$.

For $H = \mathbb{N} \times \{0\} - \mathbb{N} \times \{0\} = \mathbb{Z} \times \{0\} \cong \mathbb{Z} \cong \{0\} \times \mathbb{Z}$, we have $G/H = \mathbb{Z}^2/\mathbb{Z} \cong \mathbb{Z}$ and $I_{H_+} \times \mathbb{N}^2$ is generated by $\{1 - 1_{(m,0)} : m \in \mathbb{N}\}$ (one can see that $I_{H_+} \times \mathbb{N}^2$ is generated by one element $1 - 1_{(1,0)}$). The set $\{\Theta_{m,n} : m, n \in \mathbb{N}\}$ in which

$$\Theta_{m,n} := i_{G_+}(m, 0)(1 - i_{G_+}(1, 0) i_{G_+}(1, 0)^*) i_{G_+}(n, 0)^*,$$

is a set of matrix units (a set of non-zero elements $\{e_{i\,j}\}$ satisfying the relations $e_{i\,j}\,e_{k\,\ell} = \begin{Bmatrix} e_{i\,\ell} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{Bmatrix}$ and $e_{i\,j}^* = e_{j\,i}$) in the ideal $I_{H_+} \times_\alpha \mathbb{N}^2$, which we will now show. First we write $S_{(m,0)}$ for $i_{G_+}(m,0)$ and notice that

(i) Each $\Theta_{m,n} \neq 0$ since $1 - S_{(1,0)}S^*_{(1,0)} \neq 0$. (Remember that we write $S_{(m,0)}S^*_{(m,0)}$ for $1_{(m,0)}$)

(ii) $\Theta^*_{m,n} = S_{(n,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(m,0)} = \Theta_{n,m}$.

(iii) $\Theta_{m,n}\Theta_{p,l} = S_{(m,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(n,0)}S_{(p,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(l,0)}$,

if $n = p$ then

$$\Theta_{m,n}\Theta_{p,l} = S_{(m,0)}(1 - S_{(1,0)}S^*_{(1,0)})S_{(l,0)} = \Theta_{m,l}.$$

If $n \neq p$ then either $n \geq p$ or $n \leq p$. Suppose first that $n \geq p$ then

$$\begin{aligned}
\Theta_{m,n}\Theta_{p,l} &= S_{(m,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(n-p,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(l,0)} \\
&= S_{(m,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(n-p-1,0)}S^*_{(1,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(l,0)} \\
&= 0, \text{ since } S^*_{(1,0)}(1 - S_{(1,0)}S^*_{(1,0)}) = 0.
\end{aligned}$$

For $n \leq p$ we have

$$\begin{aligned}
\Theta_{m,n}\Theta_{p,l} &= S_{(m,0)}(1 - S_{(1,0)}S^*_{(1,0)})S_{(p-n,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(l,0)} \\
&= S_{(m,0)}(1 - S_{(1,0)}S^*_{(1,0)})S_{(1,0)}S_{(p-n-1,0)}(1 - S_{(1,0)}S^*_{(1,0)})S^*_{(l,0)} \\
&= 0, \text{ since } (1 - S_{(1,0)}S^*_{(1,0)})S_{(1,0)} = 0.
\end{aligned}$$

Hence $\{\Theta_{m,n} : m, n \in \mathbb{N}\}$ is a set of matrix units in the ideal $I_{H_+} \times_\alpha \mathbb{N}^2$ which generates a copy of the compact operators $\mathcal{K}(\ell^2(\mathbb{N}^2))$. Notice that we are dealing here with nuclear C*-algebras and so further calculations show that the ideal $I_{H_+} \times_\alpha \mathbb{N}^2 \cong \mathcal{K} \otimes \mathcal{T}(\mathbb{Z})$. Hence the primitive ideals in $I_{H_+} \times_\alpha \mathbb{N}^2$ will be $\mathcal{K} \otimes J$ for some $J \in \mathrm{Prim}(A)$ and they are primitive ideals in $B_{\mathbb{N}^2} \times_\alpha \mathbb{N}^2$. Moreover, using our short exact sequence we have

$$\mathrm{Ind}^{\widehat{\mathbb{Z}}^2}_{(\mathbb{Z}\times\{0\})^\perp}(B_{\mathbb{N}} \times \mathbb{N}) \cong \big(\mathcal{T}(\mathbb{Z}) \otimes \mathcal{T}(\mathbb{Z})\big)/\big(\mathcal{K} \otimes \mathcal{T}(\mathbb{Z})\big) \cong C(\mathbb{T}) \otimes \mathcal{T}(\mathbb{Z}),$$

so the primitive ideals in $\operatorname{Ind}_{(\mathbb{Z}\times\{0\})^\perp}^{\widehat{\mathbb{Z}}^2}(B_{\mathbb{N}}\times\mathbb{N})$ are

$$I\otimes\mathcal{T}(\mathbb{Z})+C(\mathbb{T})\otimes J \text{ for some } I\in\operatorname{Prim}(C(\mathbb{T})) \text{ and } J\in\operatorname{Prim}(\mathcal{T}(\mathbb{Z})).$$

Hence

$$\Upsilon^{-1}(L) \text{ where } L=I\otimes\mathcal{T}(\mathbb{Z})+C(\mathbb{T})\otimes J \text{ for } I\in\operatorname{Prim}(C(\mathbb{T})) \text{ and } J\in\operatorname{Prim}(\mathcal{T}(\mathbb{Z}))$$

are the primitive ideals in $B_{\mathbb{N}^2}\times_\alpha\mathbb{N}^2$ (Υ is the surjective homomorphism in the short exact sequence (5.3.4)).

For $H=\{0\}\times\mathbb{Z}$, $G/H=\mathbb{Z}^2/\mathbb{Z}\cong\mathbb{Z}$. This case is similar to the previous case with minor differences. Here the ideal $I_{H_+}\times_\alpha\mathbb{N}^2$ will be isomorphic to $\mathcal{T}(\mathbb{Z})\otimes\mathcal{K}$, and therefore the primitive ideals will have the form $J\otimes\mathcal{K}$ for some $J\in\operatorname{Prim}(A)$ and they are primitive ideals in $B_{\mathbb{N}^2}\times_\alpha\mathbb{N}^2$. For the induced algebra we will have

$$\operatorname{Ind}_{(\mathbb{Z}\times\{0\})^\perp}^{\widehat{\mathbb{Z}}^2}(B_{\mathbb{N}}\times\mathbb{N})\cong\big(\mathcal{T}(\mathbb{Z})\otimes\mathcal{T}(\mathbb{Z})\big)/\big(\mathcal{T}(\mathbb{Z})\otimes\mathcal{K}\big)\cong\mathcal{T}(\mathbb{Z})\otimes C(\mathbb{T}),$$

and the primitive ideals in the induced algebra will be

$$I\otimes C(\mathbb{T})+\mathcal{T}(\mathbb{Z})\otimes J \text{ where } I\in\operatorname{Prim}(\mathcal{T}(\mathbb{Z})) \text{ and } J\in\operatorname{Prim}(C(\mathbb{T})).$$

Hence this will give the primitive ideals

$$\Upsilon^{-1}(D) \text{ where } D=I\otimes C(\mathbb{T})+\mathcal{T}(\mathbb{Z})\otimes J \text{ for } I\in\operatorname{Prim}(\mathcal{T}(\mathbb{Z})) \text{ and } J\in\operatorname{Prim}(C(\mathbb{T}))$$

in $B_{\mathbb{N}^2}\times_\alpha\mathbb{N}^2$.

For $H=\mathbb{N}^2-\mathbb{N}^2=\mathbb{Z}^2$, $G/H=\mathbb{Z}^2/\mathbb{Z}^2\cong\{0\}$, and

$$\operatorname{Ind}_{H^\perp}^{\widehat{\mathbb{Z}}^2}(B_{\{0\}}\times_\tau\{0\})\cong\operatorname{Ind}_{(\mathbb{Z}^2)^\perp}^{\widehat{\mathbb{Z}}^2}(\mathbb{C});\ \text{since } B_{\{0\}}\times_\tau\{0\}\cong\mathcal{T}(\{0\})\cong\mathbb{C}.$$

We know that $(\mathbb{Z}^2)^\perp$ has only one character, namely id. So for $f\in\operatorname{Ind}_{(\mathbb{Z}^2)^\perp}^{\widehat{\mathbb{Z}}^2}(\mathbb{C})$, we have $f(\gamma\,\mathrm{id})=\widehat{\tau}_{\mathrm{id}}^{-1}(f(\gamma))=f(\gamma)$ and hence $\operatorname{Ind}_{(\mathbb{Z}^2)^\perp}^{\widehat{\mathbb{Z}}^2}(\mathbb{C})\cong C(\widehat{\mathbb{Z}}^2)\cong C(\mathbb{T}^2)$. Therefore each element of the set $\{\Upsilon^{-1}(I_\gamma):\gamma\in\widehat{\mathbb{Z}}^2\}$, where Υ is the surjective homomorphism in the short exact sequence (5.3.4) and $I_\gamma:=\{f\in C(\widehat{\mathbb{Z}}^2):f(\gamma)=0\}$, is a primitive ideal of $B_{\mathbb{N}^2}\times_\alpha\mathbb{N}^2$. The ideal $I_{\mathbb{N}^2}\times\mathbb{N}^2$ is generated by $\{1-1_{(m,n)}:(m,n)\in\mathbb{N}^2\}$. Write V_m for $i_{G_+}(m)$ for $m\in\mathbb{N}^2$, and define

$$\Theta_{m,n}:=V_m(1-V_{(1,0)}V_{(1,0)}^*)(1-V_{(0,1)}V_{(0,1)}^*)V_n^*$$

then calculations show the set $\{\Theta_{m,n} : m, n \in \mathbb{N}^2\}$ is a set of matrix units and that the ideal $I_{H_+} \times_\alpha \mathbb{N}^2 \cong \mathcal{K} \otimes \mathcal{T}(\mathbb{Z}) + \mathcal{T}(\mathbb{Z}) \otimes \mathcal{K}$. To identify the primitive ideals in $I_{H_+} \times_\alpha \mathbb{N}^2$ notice that $\mathcal{K} \otimes \mathcal{T}(\mathbb{Z})$ is an ideal in $\mathcal{K} \otimes \mathcal{T}(\mathbb{Z}) + \mathcal{T}(\mathbb{Z}) \otimes \mathcal{K}$ and so we will have the following short sequence

$$0 \to \mathcal{K} \otimes \mathcal{T}(\mathbb{Z}) \to \mathcal{K} \otimes \mathcal{T}(\mathbb{Z}) + \mathcal{T}(\mathbb{Z}) \otimes \mathcal{K} \xrightarrow{\Phi} C(\mathbb{T}) \otimes \mathcal{K} \to 0$$

in which Φ is the quotient map. Hence the primitive ideals in $I_{H_+} \times_\alpha \mathbb{N}^2$ will be the ones in $\mathcal{K} \otimes \mathcal{T}(\mathbb{Z})$ which we already know, and $\Phi^{-1}(D)$ where $D = I_\gamma \otimes \mathcal{K}$ for some $I_\gamma \in \mathrm{Prim}(C(\mathbb{T}))\}$.

CHAPTER 7

Some concluding remarks

The study of crossed products of C^*-algebras by endomorphisms and their tractable use as models for Toeplitz algebras has attracted many authors. In this thesis we are generalizing the work in [**2**], [**3**] and [**5**] to cover the crossed product of C^*-algebras by semigroups of endomorphisms and actions of the positive cone G_+ of a lattice-ordered discrete abelian group G.

One of our early results is the existence of a strongly continuous action

$$\widehat{\alpha} : \widehat{G} \to \mathrm{Aut}(A \times_\alpha G_+)$$

for any lattice-ordered abelian group (G, G_+).

The next main result is producing an extendibly α-invariant ideal of the crossed product $B_{G_+} \times_\alpha G_+$ which was important for the rest of this thesis.

The realization of the C^*-algebra $B_{(G/H)_+} \times_\beta G_+$ as the induced C^*-algebra $\mathrm{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ was a goal of this thesis and is used to show the existence of a short exact sequence of C^*-algebras involving $B_{G_+} \times G_+$ which is a generalization of the work in [**6**, Theorem 2.1].

Then we show the existence of an isomorphism of $B_{H_+} \times H_+$ into $B_{G_+} \times_\alpha G_+$ (which allows us to realize $B_{H_+} \times H_+$ as a C^*-subalgebra of $B_{G_+} \times_\alpha G_+$) and identify the commutator ideal of $B_{H_+} \times H_+$.

Later in this work, we use our short exact sequence and give some results about primitive ideals of the C^* algebra $B_{G_+} \times_\alpha G_+$ which is isomorphic to the Toeplitz algebra $\mathcal{T}(G)$ of G.

APPENDIX A

We give here the proof that the map Ψ of Theorem 5.3.2 is a homomorphism of C^*-algebras.

Proof. To see that $\Psi : B_{(G/H)_+} \times_\beta G_+ \to \operatorname{Ind}_{H^\perp}^{\widehat{G}} \big(B_{(G/H)_+} \times_\tau (G/H)_+\big)$ is a homomorphism. Fix $a, b \in B_{(G/H)_+} \times_\beta G_+$, $\gamma \in \widehat{G}$ and $t \in \mathbb{C}$, then

(i)

$$\begin{aligned}\Psi(ta)(\gamma) &= Q\big(\widehat{\beta}_\gamma^{-1}(ta)\big)\\ &= Q\big(t\widehat{\beta}_\gamma^{-1}(a)\big), \widehat{\beta}_\gamma^{-1} \text{ is linear}\\ &= tQ\big(\widehat{\beta}_\gamma^{-1}(a)\big), Q \text{ is linear}\\ &= t\Psi(a)(\gamma).\end{aligned}$$

(ii)

$$\begin{aligned}\Psi(a+b)(\gamma) &= Q\big(\widehat{\beta}_\gamma^{-1}(a+b)\big)\\ &= Q\big(\widehat{\beta}_\gamma^{-1}(a) + \widehat{\beta}_\gamma^{-1}(b)\big), \widehat{\beta}_\gamma^{-1} \text{ is linear}\\ &= Q\big(\widehat{\beta}_\gamma^{-1}(a)\big) + Q\big(\widehat{\beta}_\gamma^{-1}(b)\big), Q \text{ is linear}\\ &= \Psi(a)(\gamma) + \Psi(b)(\gamma).\end{aligned}$$

(iii)

$$\begin{aligned}\Psi(ab)(\gamma) &= Q\big(\widehat{\beta}_\gamma^{-1}(ab)\big)\\ &= Q\big(\widehat{\beta}_\gamma^{-1}(a)\widehat{\beta}_\gamma^{-1}(b)\big), \widehat{\beta}_\gamma^{-1} \text{ is a homomorphism}\\ &= Q\big(\widehat{\beta}_\gamma^{-1}(a)\big)Q\big(\widehat{\beta}_\gamma^{-1}(b)\big), Q \text{ is a homomorphism}\\ &= \Psi(a)(\gamma)\Psi(b)(\gamma).\end{aligned}$$

(iv)

$$
\begin{aligned}
\Psi(a)^*(\gamma) &= Q\big(\widehat{\beta}_\gamma^{-1}(a)\big)^* \\
&= Q\big(\widehat{\beta}_\gamma^{-1}(a)^*\big), Q \text{ is a homomorphism} \\
&= Q\big(\widehat{\beta}_\gamma^{-1}(a^*)\big), \widehat{\beta}_\gamma^{-1} \text{ is a homomorphism} \\
&= \Psi(a^*)(\gamma).
\end{aligned}
$$

Thus Ψ is a homomorphism of C^*-algebras. □

LEMMA A.0.3. *Suppose that (X, τ) is a topological space and A, B are subsets of X such that A is nowhere dense and* $\mathrm{Int}(B) = \emptyset$. *Then* $\mathrm{Int}(A \cup B) = \emptyset$.

PROOF. Take $x \in A \cup B$. Then for any open subset V containing x, there is $y \in V$ such that $y \notin \overline{A}$ (this is true since $\mathrm{Int}(\overline{A}) = \emptyset$, ie. $\overline{A}$ contains no open sets). Now we have $y \in V \setminus \overline{A} = V \cap (\overline{A})^{\mathcal{C}}$ which an open subset. So there is $z \in V \setminus \overline{A}$ such that $z \notin B$. Hence, $z \in V \setminus (A \cup B)$ (which means that $A \cup B$ does not contain any non-empty open subsets). Thus $\mathrm{Int}(A \cup B) = \emptyset$. □

LEMMA A.0.4. *Suppose that (X, τ) is a topologigal space and A, B are nowhere dense subsets of X. Then $A \cup B$ is a nowhere dense subset of X.*

PROOF. We know that $\overline{\overline{A}} = \overline{A}$, therefore $\mathrm{Int}(\overline{\overline{A}}) = \emptyset$ and hence $\overline{A}$ itself is nowhere dense (this is true because A is nowhere dense). Since B is nowhere dense, then by Lemma A.0.3

$$\mathrm{Int}(\overline{A \cup B}) = \mathrm{Int}(\overline{A} \cup \overline{B}) = \emptyset.$$

Thus $A \cup B$ is nowhere dense. □

Proof related to Lemma 5.4.5. To show that the set $S = \{1 - 1_u : u \in H_+\}$ in Lemma 5.4.5 is an approximate identity for $B_{H_+,\infty}$, we show that S satisfies the conditions of approximate identity. Firstly, for any $1 - 1_u \in S$, $1 - 1_u$ is a projection and therefore $\|1_e - 1_h\| \leq 1$.

Secondly, we show that $(1 - 1_u)f \to f$ for all $f \in B_{H_+,\infty}$. To do so we start by showing that it is true for any $1 - 1_t \in B_{H_+,\infty}$ and then we show it for any $f \in B_{H_+,\infty}$.

For $1 - 1_t \in B_{H_+,\infty}$ we have

$$\begin{aligned}\|(1-1_u)(1-1_t)-(1-1_t)\| &= \|1-1_t-1_u+1_{u\vee t}-1+1_t\| \\ &= \|1_{u\vee t}-1_u\| \\ &= 0, \text{ if } u \text{ is large enough such that } u \geq t.\end{aligned}$$

Similar calculations show that this is true for $(1-1_t)(1-1_u)$. For $f \in I_{H+}$ we know that f is a limit of finite sums of elements in the spanning set of I_{H_+}. Fix $\varepsilon > 0$ and choose $f_0 \in \operatorname{span}\left\{1-1_t : t \in H_+\right\}$ such that $\|f - f_\circ\| < \varepsilon/3$. Choose $h_0 \in H_+$ such that if $t \geq h_0$ then $\|(1-1_t)f_0 - f_0\| < \varepsilon/3$. Hence,

$$\begin{aligned}\|(1-1_t)f-f\| &= \|(1-1_t)(f-f_0)+(1-1_t)f_0-f_0+f_0-f\| \\ &\leq \|(1-1_t)(f-f_0)\| + \|(1-1_t)f_0-f_0\| + \|f-f_0\| \\ &\leq \|1-1_t\|\,\|f-f_0\| + \|(1-1_t)f_0-f_0\| + \|f-f_0\| \\ &< \varepsilon/3+\varepsilon/3+\varepsilon/3 = \varepsilon.\end{aligned}$$

Thus $\left\{1-1_u : u \in H_+\right\}$ is an approximate identity for $B_{H_+,\infty}$.

□

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