

Comments on "Zeros of Discretized Continuous Systems Expressed in the Euler Operator—An Asymptotic Analysis"

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Abstract—In a recent paper, an asymptotic analysis was used to address the zero structure of discretized continuous-time systems expressed in the Euler (or forward difference) operator. Unfortunately, the analysis is flawed, so that Theorem 1 is true as stated only when the continuous-time relative degree is equal to two. In this paper, we indicate how the analysis may be rectified.

Index Terms—Discrete-time systems, poles and zeros.

I. INTRODUCTION

In a recent paper,¹ the zeros of discretized continuous-time systems expressed in terms of the Euler (or forward difference) operator $\epsilon = (z - 1)/T$ are addressed. The key idea¹ is that by considering the sampling period T as a perturbation parameter, the zero dynamics of the resulting discrete-time system expressed in the delta operator (the time-domain equivalent of the Euler operator [1]) are singularly perturbed, even though the matrices in the discrete-time state-space model are regular perturbations of their continuous-time counterparts. Theorem 1 of the above-mentioned paper states that the singularly perturbed zero dynamics leads to a separation of time scales in which the component of the zero dynamics associated with the fast time scale corresponds to the zeros introduced by the sampling process, the so-called sampling zeros (or discretization zeros [2]). Furthermore, it is argued the sampling zeros of discrete-time models expressed in the delta operator become infinite in the fast sampling limit.

Unfortunately, the analysis is flawed, with errors at several key points in the development. In particular, the asymptotic formula for the location of the sampling zeros of delta operator-based discrete-time systems presented in Theorem 1¹ is consistent with the results of [3] and [1] only when the continuous-time relative degree equals two. In this paper, we show how the analysis may be rectified.

Preliminaries: A function $f(T) = O(T)$ if $\lim_{T \rightarrow 0} f(T)/T = K$, where $0 < K < \infty$, and $f(T) = o(T)$ if $\lim_{T \rightarrow 0} f(T)/T = 0$. The set of eigenvalues of a matrix X is denoted by $\lambda(X)$.

II. DISCRETE-TIME SYSTEMS IN THE DELTA OPERATOR

The paper considers linear, single-input single-output (SISO), and minimal systems of the form

$$\begin{aligned} \dot{x} &= A_c x + b_c u \\ y &= c x, \quad x \in \mathbf{R}^n, \quad u, y \in \mathbf{R}. \end{aligned} \quad (1)$$

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¹A. Tesfaye and M. Tomizuka, *IEEE Trans. Automat. Contr.*, vol. 40, pp. 743–747, Apr. 1995.

Attention is restricted to systems whose relative degree $\gamma \geq 2$, since continuous-time systems having relative degree one do not give rise to sampling zeros. System (1) is transformed by an appropriate change of coordinates (see (3) of the paper) into the Byrnes–Isidori normal form [4] in which the relative degree (or infinite zero) structure of the system is represented by a chain of integrators, and the finite zeros are given by the eigenvalues of the companion matrix $Q_\eta \in \mathbf{R}^{(n-\gamma) \times (n-\gamma)}$ appearing in the lower right sub-block of the normal form state matrix.

With q the forward shift operator and T the sampling period, the discrete-time zero-order hold (ZOH) representation of the continuous-time system (1) can be written in terms of the δ operator ($\delta \triangleq (q - 1)/T$) as follows:

$$\begin{aligned} \delta x[k] &= A_\delta x[k] + b_\delta u[k] \\ y[k] &= c x[k] \end{aligned} \quad (2)$$

where the matrices A_δ and b_δ are regular perturbations of their continuous-time counterparts; see the above-mentioned paper for details. If the state similarity transformation $[\zeta^T \eta^T]^T = Lx$ is applied to (2), where L is defined in (6),¹ the resulting discrete-time state-space model is

$$\begin{aligned} \delta \begin{bmatrix} \zeta \\ \eta \end{bmatrix} &= L A_\delta L^{-1} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} + L b_\delta u[k] \\ y[k] &= [1 \ 0 \ \cdots \ 0] \begin{bmatrix} \zeta \\ \eta \end{bmatrix}. \end{aligned} \quad (3)$$

In this model, the state matrix $L A_\delta L^{-1}$ is equal, to within $O(T)$, to the state matrix of the continuous-time system expressed in normal form. Using the definitions of A_δ and b_δ , together with the properties of Markov parameters of a relative degree γ system, it is possible to obtain the following asymptotic approximation to $L b_\delta$, in which the leading order term in each row has been retained:

$$L b_\delta = \begin{bmatrix} c b_\delta \\ c A_\delta b_\delta \\ c A_\delta^2 b_\delta \\ \vdots \\ c A_\delta^{\gamma-1} b_\delta \\ H b_\delta \end{bmatrix} = \begin{bmatrix} c(I + T b_1^*) b_c \\ c A_c (I + T b_1^*)^2 b_c \\ c A_c^2 (I + T b_1^*)^3 b_c \\ \vdots \\ c A_c^{\gamma-1} (I + T b_1^*)^\gamma b_c \\ H (I + T b_1^*) b_c \end{bmatrix} \quad (4)$$

$$= \begin{bmatrix} T^{\gamma-1} \alpha_0(\gamma) + O(T^\gamma) \\ T^{\gamma-2} \alpha_1(\gamma) + O(T^{\gamma-1}) \\ T^{\gamma-3} \alpha_2(\gamma) + O(T^{\gamma-2}) \\ \vdots \\ T \alpha_{\gamma-2}(\gamma) + O(T^2) \\ \alpha_{\gamma-1}(\gamma) + O(T) \\ \frac{T}{2} H A_c b_c + O(T^2) \end{bmatrix}. \quad (5)$$

The terms $\alpha_0(\gamma), \dots, \alpha_{\gamma-1}(\gamma)$ play a key role in the formula for the asymptotic location of the sampling zeros of (2), as presented in

Theorem 3.1. While it is straightforward to show that

$$\alpha_0(\gamma) = \frac{1}{\gamma!} c A_c^{\gamma-1} b_c \quad (6)$$

obtaining closed-form expressions for $\alpha_1(\gamma)$, $\alpha_2(\gamma)$, $\dots$, $\alpha_{\gamma-1}(\gamma)$ requires the evaluation of terms in the expansion of integer powers of $I + T b_1^*$. This is not, in general, a straightforward task, since b_1^* is itself a power series in T . By careful examination of the terms in the each row of $L b_\delta$ associated with $c A_c^{\gamma-1} b_c$, the first nonzero Markov parameter, it is possible to show that $\alpha_1(\gamma)$, $\dots$, $\alpha_4(\gamma)$ are given by the following expressions, which contain numerous terms absent from the original presentation:

$$\begin{aligned} \alpha_1(\gamma) &= \left(\sum_{i=1}^{\gamma-1} \frac{1}{i!(\gamma-i)!} \right) c A_c^{\gamma-1} b_c \\ &= \frac{2^\gamma - 2}{\gamma!} c A_c^{\gamma-1} b_c \end{aligned} \quad (7)$$

$$\begin{aligned} \alpha_2(\gamma) &= \left(\sum_{i=1}^{\gamma-1} \frac{2^{\gamma-i} - 2}{i!(\gamma-i)!} \right) c A_c^{\gamma-1} b_c \\ &= \frac{3^\gamma - 3 \cdot 2^\gamma + 3}{\gamma!} c A_c^{\gamma-1} b_c \end{aligned} \quad (8)$$

$$\begin{aligned} \alpha_3(\gamma) &= \left(\sum_{i=1}^{\gamma-1} \frac{3^{\gamma-i} - 3 \cdot 2^{\gamma-i} + 3}{i!(\gamma-i)!} \right) c A_c^{\gamma-1} b_c \\ &= \frac{4^\gamma - 4 \cdot 3^\gamma + 6 \cdot 2^\gamma - 4}{\gamma!} c A_c^{\gamma-1} b_c \end{aligned} \quad (9)$$

$$\begin{aligned} \alpha_4(\gamma) &= \left(\sum_{i=1}^{\gamma-1} \frac{4^{\gamma-i} - 4 \cdot 3^{\gamma-i} + 6 \cdot 2^{\gamma-i} - 4}{i!(\gamma-i)!} \right) c A_c^{\gamma-1} b_c \\ &= \frac{5^\gamma - 5 \cdot 4^\gamma + 10 \cdot 3^\gamma - 10 \cdot 2^\gamma + 5}{\gamma!} c A_c^{\gamma-1} b_c. \end{aligned} \quad (10)$$

Expressions (6)–(10) are sufficient to deal with continuous-time systems having relative degree $\gamma \leq 5$ and therefore cover the majority of practical situations. To generate the higher order terms $\alpha_5(\gamma)$, $\alpha_6(\gamma)$, $\dots$ necessary for dealing with $\gamma > 5$, we can use the binomial theorem to expand the powers of $I + T b_1^*$ in (4), in conjunction with the following recursive procedure for evaluating specified terms in integral powers of power series; see, for example ([5, p. 14])

$$\left(\sum_{k=0}^{\infty} a_k T^k \right)^n = \sum_{k=0}^{\infty} c_k T^k \quad (11)$$

where

$$c_0 = a_0^n \quad (12)$$

$$c_m = (m a_0)^{-1} \sum_{k=1}^m (k n - m + k) a_k c_{m-k}, \quad \text{for } m \geq 1. \quad (13)$$

Thus from (4) and (5)

$$\alpha_i(\gamma) = c A_c^i \left\{ \sum_{j=0}^{i+1} \binom{i+1}{j} [(T b_1^*)^j]_{\gamma-i-1} \right\} b_c, \quad i = 0, 1, \dots, \gamma - 1 \quad (14)$$

where $[(T b_1^*)^j]_k$ denotes the coefficient of T^k in the expansion of $(T b_1^*)^j$ and can be calculated using (11)–(13).

III. REVISITING THE MAIN RESULT OF THE ORIGINAL PAPER

For (3), neglecting the higher order terms in $L A_\delta L^{-1}$ and $L b_\delta$, the state feedback law

$$u = -\frac{1}{\alpha_0(\gamma) T^{\gamma-1}} \zeta_2 \quad (15)$$

zeros the output y for all time. Note that the particular form of the control law (15) differs from that presented originally. On the zero dynamics subspace $\mathcal{V}$ defined by

$$\mathcal{V} = \{x : y = c x = 0\} \quad (16)$$

$$= \{(0, \bar{\zeta}^T, \eta^T)^T : \bar{\zeta} \in \mathbf{R}^{\gamma-1}, \eta \in \mathbf{R}^{n-\gamma}\} \quad (17)$$

the dynamics of the $(\bar{\zeta}, \eta)$ variables under the state feedback law (15) are given by (18), as shown at the bottom of the page.

Remark 3.1: At the corresponding point of the analysis presented in the paper,¹ the claim is made that for $T \neq 0$, ZOH-equivalent representations of continuous-time systems expressed in the delta operator have relative degree one, regardless of the relative degree of the underlying continuous-time system; see [1, p. 97] for a counterexample to this claim.

The following is a corrected version of Theorem 1.

Theorem 3.1: Consider a continuous-time system of the form (1) and its ZOH equivalent representation (2) expressed in the delta operator. Then if system (1) has relative degree $\gamma \geq 2$, system (2) has $n - 1$ zeros which, according to their asymptotic behavior as $T \rightarrow 0$, belong to two groups.

$$\delta \begin{bmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \\ \vdots \\ \bar{\zeta}_{\gamma-2} \\ \bar{\zeta}_{\gamma-1} \\ \hline \eta \end{bmatrix} = \begin{bmatrix} -\frac{\alpha_1(\gamma)}{\alpha_0(\gamma)T} & 1 & 0 & \cdots & 0 & 0 \\ -\frac{\alpha_2(\gamma)}{\alpha_0(\gamma)T^2} & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ -\frac{\alpha_{\gamma-2}(\gamma)}{\alpha_0(\gamma)T^{\gamma-2}} & & & & 1 & 0 \\ -\frac{\alpha_{\gamma-1}(\gamma)}{\alpha_0(\gamma)T^{\gamma-1}} + r_2 & r_3 & r_4 & \cdots & r_\gamma & \bar{s} \\ \hline p_2 - \frac{H A_c b_c}{2\alpha_0(\gamma)T^{\gamma-2}} & p_3 & p_4 & \cdots & p_\gamma & Q_\eta \end{bmatrix} \begin{bmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \\ \vdots \\ \bar{\zeta}_{\gamma-2} \\ \bar{\zeta}_{\gamma-1} \\ \hline \eta \end{bmatrix} \quad (18)$$

- 1) The $\gamma - 1$ sampling zeros tend asymptotically to the set $T^{-1}\lambda(W_1)$, where

$$W_1 = \begin{bmatrix} -\frac{\alpha_1(\gamma)}{\alpha_0(\gamma)} & 1 & 0 & \cdots & 0 \\ -\frac{\alpha_2(\gamma)}{\alpha_0(\gamma)} & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ -\frac{\alpha_{\gamma-2}(\gamma)}{\alpha_0(\gamma)} & & & & 1 \\ -\frac{\alpha_{\gamma-1}(\gamma)}{\alpha_0(\gamma)} & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbf{R}^{(\gamma-1) \times (\gamma-1)}. \quad (19)$$

- 2) The remaining $n - \gamma$ zeros tend to the finite zeros of the continuous-time system (1).

Proof: Following the same line of reasoning as in the paper,¹ the dynamics of the $(\bar{\zeta}, \eta)$ variables on the subspace $\mathcal{V}$ are cast into a standard singular perturbation form. This can be accomplished by rescaling the $(\bar{\zeta}, \eta)$ variables in (18) as follows:

$$\begin{bmatrix} z \\ \eta \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} \bar{\zeta} \\ \eta \end{bmatrix}$$

where $M_1 = \text{diag}[1, T, T^2, \dots, T^{\gamma-2}]$, M_2 , and M_3 are zero matrices of appropriate sizes, and $M_4 = I_{(n-\gamma) \times (n-\gamma)}$. Note that the choice of M_1 here differs from that used in the original paper. It follows that the zero dynamics satisfy

$$\delta \begin{bmatrix} Tz \\ \eta \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ p^T M_1^{-1} & Q_\eta \end{bmatrix} \begin{bmatrix} z \\ \eta \end{bmatrix} + O(T) \quad (20)$$

where

$$p^T = \left[p_2 - \frac{H A_c b_c}{2\alpha_0(\gamma)T^{\gamma-2}} \quad p_3 \quad \cdots \quad p_\gamma \right].$$

In (20), W_1 is given by (19), and Q_η is the companion matrix whose eigenvalues are the finite zeros of the continuous-time system (1). Equation (20) is in the standard two time-scale form [6], and since the eigenvalues of the matrix on the right-hand side are asymptotically given by $\lambda(W_1) \cup \lambda(Q_\eta)$, the dynamics of the $(\bar{\zeta}, \eta)$ variables on the subspace $\mathcal{V}$ are those of $\text{diag}[T^{-1}W_1, Q_\eta]$ and the result follows. $\square$

Remark 3.2: Since $\delta = (q-1)/T$, it would follow that the sampling zeros tend to $-\infty$ asymptotically if it could be established that the sampling zeros of discrete-time models expressed in the shift operator tend asymptotically to negative real values. While such behavior has been conjectured by Hagiwara *et al.* [2, p. 1333], it remains to be shown.

Remark 3.3: The quantification of discrete-time zeros in Theorem 3.1 is much less direct than simply using the change of variables $\delta = (q-1)/T$ in conjunction with known results for the limiting zeros of shift operator models; see, for example, [2], [3], and [7]. Nevertheless, the method proposed¹ has the distinct advantage of using a state-space framework. As a consequence, it is possible to use Theorem 3.1 as the basis for studying the mapping of finite and infinite zeros of a large class of multivariable systems under ZOH sampling [8].

REFERENCES

- [1] R. H. Middleton and G. C. Goodwin, *Digital Control and Estimation: A Unified Approach*. Englewood Cliffs, NJ: Prentice-Hall, 1990.
- [2] T. Hagiwara, T. Yuasa, and M. Araki, "Stability of the limiting zeros of sampled-data systems with zero- and first-order holds," *Int. J. Contr.*, vol. 58, no. 6, pp. 1325–1346, 1993.
- [3] K. J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica*, vol. 20, no. 1, pp. 21–38, 1984.
- [4] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Berlin: Springer-Verlag, 1995.
- [5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, corrected and enlarged ed. San Diego, CA: Academic, 1980.
- [6] P. Koktović, H. Khalil, and J. O'Reilly, *Singular Perturbations in Control: Analysis and Design*. New York: Academic, 1986.
- [7] Y. Hayakawa, S. Hosoe, and M. Ito, "On the limiting zeros of sampled multivariable systems," *Syst. Contr. Lett.*, vol. 2, no. 5, pp. 292–300, Feb. 1983.
- [8] S. R. Weller, "Limiting zeros of decouplable MIMO systems," *IEEE Trans. Automat. Control*, to be published.

Authors' Reply

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We thank S. Weller for his interest in our paper, detecting an error, and providing a correction to it. Our initial mistake occurred as we sought to rewrite the expressions for each row of $Lb\delta$ in the set of equations preceding (7) so as to obtain an asymptotic approximation to $Lb\delta$ without sacrificing its structure. In going from the initially correct expression to the asymptotic approximation mistakes were made and terms were lost which propagated to the main result of our paper expressed in the form of Theorem 1. Weller, by using a corrected expression for $Lb\delta$ and following the same analysis method used in our paper, obtains the correct result. As pointed out by Weller, the location of discrete-time zeros in the fast sampling limit still remains unproved.

Finally, we appreciate his view that the important contribution of our paper "... is the method rather than the result." The methodology has the distinct advantage of using a state-space framework which provides a good basis for the study of finite and infinite zeros of multivariable control systems.

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