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A Method for Switching Controller Design for Discrete Time Hybrid Systems *

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Abstract

In this paper we propose a systematic switching control design method applicable to a class of piecewise linear hybrid systems. We consider a class of systems controlled by a finite state actuator (i.e. switching controller). For the class of systems considered, precise conditions for stabilizability are unknown. However, by considering the same systems with unknown but bounded exogenous disturbances, we are able to give finitely computable conditions, sufficient for stabilizability without disturbances, yet necessary for stabilizability with disturbances.

1 Introduction

Despite some recent advances [1, 2, 5, 6], only a few control design methods applicable to particular classes of hybrid systems have been suggested [4, 9, 10, 11, 15]. The difficulties in this area are serious and stem mainly from the lack of computationally tractable and non-conservative stability tests. The results on hybrid systems stability date back to the sixties [7] when some results on Lyapunov stability were obtained. Some recent and more general results for hybrid systems have been suggested in [2, 8, 3] using the concept of "multiple Lyapunov functions" and piecewise quadratic Lyapunov functions. The problem of quadratic stabilization has been analysed recently in a number of papers (see, for example, [11, 12]). In [14] a method is proposed for a system consisting of two linear vector fields. The so-called *min-switch strategy* is suggested in [4]. The direct application of stability theorems was studied in [9]. In [15] an approach based on the idea of multiple polyhedral Lyapunov functions was developed. However, difficulty in assessing the degree of conservatism is a feature common to the approaches mentioned above. The importance of this becomes even more obvious if we recall some recent results (see [13]) showing undecidability of the stability problem in the context of LTV systems.

The objective of this paper is to present a systematic approach to stabilizability analysis and switching control applicable to a class of piecewise linear hybrid systems. The analysis is based on the notions of a cell transition and a cell trajectory whose properties play the major role in determining stability/instability of a controlled system. A desirable feature of the proposed approach is the possibility of a trade-off between the conservativeness of the results and computational burden.

2 Problem Statement

We consider the following class of nonlinear discrete time hybrid systems.

High Level Dynamics:

$$\mathbf{S} = \{Q,\ \Sigma,\ \delta,\ \Delta T,\ \Omega\} \tag{1}$$

Low Level Dynamics:

$$x(t+1) = A(i(x), \xi(t))x(t) + d(t) \tag{2}$$

where $\mathbf{S}$ is a finite state machine with Q and Σ being finite sets of symbolic states and events, respectively; $\Sigma = \Sigma_c \bigcup \Sigma_{un} \neq \varnothing$, Σ_c and Σ_{un} denote the sets of controlled and uncontrolled events, respectively; $\delta : Q \times \Sigma \to Q$ is a partial transition function. Each state $\xi \in Q$ of the automaton $\mathbf{S}$ is interpreted as an activity (phase) of length $\Delta t(\xi) \in \Delta T(\xi) \subset \mathbf{N}_+$ while a transition $\xi_i \to \xi_j$, $\forall \xi_i, \xi_j \in Q$ enabled by the event $\sigma_{ij} \in \Sigma$ is instantaneous. A triple $e_{ij} = \{\xi_i, \sigma_{ij}, \xi_j\}$ such that $\xi_i, \xi_j \in Q$ and $\delta(\xi_i, \sigma_{ij})!$ will be referred to as a transition from ξ_i to ξ_j enabled by σ_{ij}. The *normal operating domain* $\Omega \subset \mathbf{R}^n$ reflects continuous specifications. Equation (2) describes the low level dynamics (LLD) with $i(x)$ and $\xi(t)$ being the uncontrolled and controlled (respectively) switching indices affecting the evolution of the LLD. We will use the notion of a *cell partition* [3] defined on a compact set $\Omega \subset \mathbf{R}^n$. For each $\xi \in Q$ the set Ω is presented as a finite union of convex polytopes $\Omega = \bigcup_{i=1}^{L} \Omega_i$, $\Omega_i \subset \mathbf{R}^n$, $\forall i = 1, 2, ..., L$. Within each cell (assuming that i is fixed) the dynamics are $x(t+1) = A(\xi(t))x(t)$. Thus, the indices i and ξ are responsible for the changes in the dynamics caused by the evolution of the continuous

and discrete states, respectively. One example of such systems is the multiswitch system depicted in Figure 1 where $\mathbf{W}_i$ represent electrical components, and $\mathbf{K}_i$ represent switches. For every combinations of the binary switches the system evolves as a cell partition based one. The high level switching controller is de-
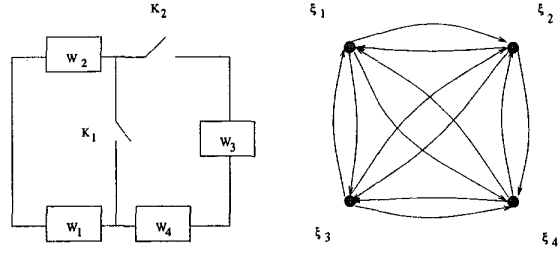

Figure 1: Multiswitch Hybrid System

fined as $\Gamma(\xi, x) : Q \times \mathbf{R}^n \to 2^{\Sigma_c}$, that is, to each pair $\{\xi, x\}$ we relate a particular combination of the positions of the switches (enabled and disabled events). The following assumptions are used: **(A1)** $0 \in \text{int } \Omega$; **(A2)** Each Ω_i, $i = 1, 2, ..., L$ is a convex polytope in $\mathbf{R}^n$; **(A3)** The cells are non-overlapping, that is, $\{\Omega_i - \partial\Omega_i\} \bigcap \{\Omega_j - \partial\Omega_j\} = \varnothing$, $\forall i \neq j$, $\forall i, j = 1, ..., L$; **(A4)** For any $x \in \Omega_i \bigcap \Omega_j \neq \varnothing$, $\forall i, j = 1, ..., L$ and $\xi \in Q$ the LLD are defined by $x(t+1) \in \{x(t+1) : x(t+1) = A(l, \xi)x(t), l \in \{i, j\}\}$; **(A5)** $\Sigma = \Sigma_c = \Sigma_{fc}$ where Σ_{fc} is a set of forced transitions; **(A6)** For every $\{\xi_i, \xi_j\}$ there exists a transition $e_{ij} = \{\xi_i, \sigma_{ij}, \xi_j\}$; **(A7)** $0 \in \text{int } \Omega_i$ for at least one $\Omega_i \in \Omega$. In the sequel we will refer to that cell as Ω_1.

Assumptions (A1)-(A4) and (A7) are transparent, while Assumptions (A5),(A6) define every transition as admissible, controlled and forcible. Since our objective is a switching con-

trol method the description of the system explicitly showing only controlled transitions appears convenient.

Definition 2.1 *An exogenous disturbance $d(t)$ is said to belong to the class $\mathcal{G}$ if $\|d(t)\| \leq k' \max_j \{\text{diam } (\Omega_j)\}$ with k' being a positive constant and diam $(\Omega_j) = \max_{x,y \in \Omega_j} \|x - y\|$.*

Definition 2.2 *A hybrid system (1),(2) controlled by $\Gamma(\xi, x)$ is said to be asymptotically stable with $d(t) = 0$ if (1) $x(t) \in \Omega$ for all $t \geq t_0$, $t_0 \in \mathbf{N}_+$; (2) $\lim_{t \to \infty} \|x(t)\| = 0$*

Definition 2.3 *A hybrid system (1),(2) controlled by a switching controller $\Gamma(\xi, x)$ is said to be stable in the class of exogenous disturbances $d(t) \in \mathcal{G}$ if the following conditions are satisfied: (1) $x(t) \in \Omega$ for all $t \geq t_0$, $t_0 \in \mathbf{N}_+$; (2) There exists a finite $t^* \in \mathbf{N}$ s.t. $x(t) \in \Omega_1$ $\forall t \geq t^*$.*

Definition 2.4 *A hybrid system (1),(2) with $d(t) = 0$ ($d(t) \in \mathcal{G}$, resp.) is stabilizable in the class $\mathcal{L}$ if the exists a controller $\Gamma(\xi, x)$ s.t. the controlled system is asymptotically stable (stable, resp.).*

The two major steps essential in the proposed method, namely, *discrete event composition of the system* and *cell trajectories analysis* are presented below.

3 Discrete Event Composition

The main idea behind the *discrete event composition* is superposition of the cell partitions associated with each discrete mode upon one

another. This allows us to obtain a cell partition based description of the original system where the evolution of the continuous dynamics within each cell explicitly depends on the chosen switching sequence. Before proceed-
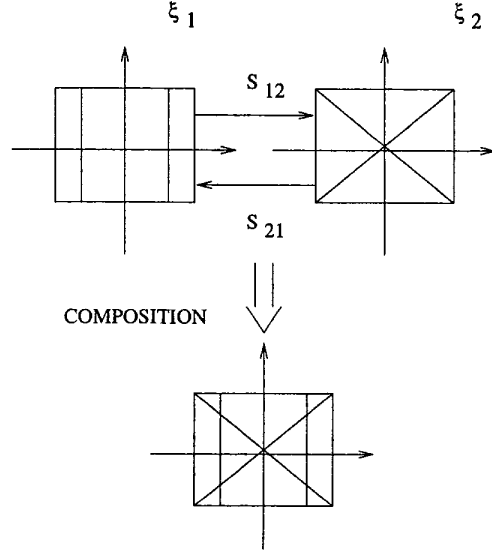

Figure 2: Discrete Event Composition

ing further we recall that a convex polytope $\Omega_j(\xi)$ can be constructed as the intersection of $p(\xi)$ linear half-spaces, each given by an affine inequality $c_{jl}(\xi)x \geq z_{jl}(\xi)$, $l = 1, 2, \cdots p(\xi)$. Given decompositions of $\Omega = \bigcup_{i=1}^{L(\xi)} \Omega_i(\xi)$, $\forall \xi \in Q$ we define the discrete event composition as $\Omega = \bigcup_{i=1}^{J} \Omega_i$ with each cell Ω_i, satisfying the conditions: **(C1)** Each Ω_i, is a convex polyhedron; **(C2)** For every hyperplane $H \in \{H_{jl\xi}\}$, $H_{jl\xi} = \{x : c_{jl}(\xi)x = z_{jl}(\xi), 1 \leq l < p(\xi), \xi \in Q\}$ and every Ω_j $H \bigcap \Omega_j \neq \emptyset \Leftrightarrow H \bigcap \Omega_j \subset \partial \{\Omega_j\}$; **(C3)** For each face, $\mathcal{F}_i$, $i = 1, 2, ..., s$ of Ω_j there exists an $H \in \{H_{jl\xi}\}$ s.t. $\mathcal{F}_i \in H$.

An example of discrete event composition satisfying conditions (C1)-(C3) is given in Figure 2.

4 Cell Trajectories Analysis

Having performed the discrete event composition we can start analysing stability/stabilizability properties of the system. The following assumption is used in the further development.

(A8) $|\lambda_{max}(A(1,\xi))| < 1$ for at least one switching index $\xi \in Q$ defined on Ω_1.

In view of **(A7)-(A8)** we assume that Ω_1 is a positive invariant set for the dynamics $x(t+1) = A(\xi_i)x(t)$, otherwise it is always possible to construct a subset $\hat{\Omega}_1 \subset \Omega_1$ satisfying Assumption (A8).

Definition 4.1 *The mapping $\Omega_j \times \Gamma(\xi,x) \to \text{Tr}(\Omega_j)$ where $\text{Tr}(\Omega_j) = \{\Omega_k : \Omega_k \cap \Omega_j A(j,\xi)) \neq \varnothing\}$ is said to be a cell mapping of the cell Ω_j with respect to the controller $\Gamma(\xi,x)$.*

Definition 4.2 *The transition $\Omega_j \to \Omega_k$ is said to be a cell transition if $\Omega_k \in \text{Tr}(\Omega_j)$.*

Definition 4.3 *A sequence $\mathbf{P}_n(\Omega_j) = \Omega_j \to \Omega_k \to \Omega_l \to \dots \to \Omega_s$ composed of n transitions is said to be a cell trajectory of the hybrid system (1), (2) of length $n \in \mathbf{N}$. We say that a given cell trajectory is non-cyclic iff $\Omega_i \neq \Omega_j$ for all possible i, j, otherwise the cell trajectory is cyclic.*

An illustrative example of the notions introduced above is given by Figure 5. Here $\Omega_1 \to \{\Omega_2, \Omega_3, \Omega_4, \Omega_5\}$ is a cell mapping, $\{\Omega_1 \to \Omega_2\}$, $\{\Omega_1 \to \Omega_3\}$, $\{\Omega_1 \to \Omega_4\}$ and $\{\Omega_1 \to \Omega_5\}$ are cell transitions and $\Omega_1 \to \Omega_5 \to \Omega_7$ being a cell trajectory. In the sequel we will use the
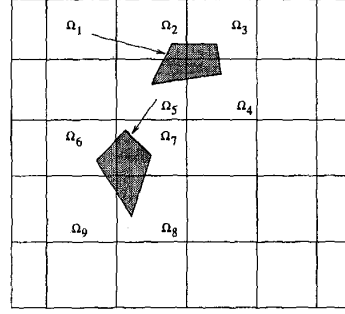

Figure 3: Cell Transitions

notation $\mathcal{P}_n(\Omega_j)$ to refer to the set of all possible cell trajectories originating from the cell Ω_j. We note that it follows trivially that every non-cyclic cell trajectory is finite.

4.1 Stability Results

First we present two stability results applicable to the hybrid system (1), (2) controlled by a fixed controller $\Gamma(\xi,x)$. This will enable us to proceed with the switching controller design.

Theorem 4.1 *The hybrid system (1), (2) controlled by a fixed switching controller $\Gamma(\xi,x)$ is stable with $d(t) = 0$ if every cell trajectory generated by the system and defined on $\Omega - \Omega_1$ is non-cyclic.*

Theorem 4.2 *The hybrid system (1), (2) controlled by $\Gamma(\xi,x)$ is unstable in the presence of $d(t) \in \mathcal{G}$ if the closed loop system generates at least one cyclic cell trajectory defined on $\Omega - \Omega_1$.*

The theorems above provide sufficient and necessary stability conditions, respectively, for two slightly different classes of hybrid systems. To be precise, the difference is in the presence/absence of the exogenous disturbance $d(t) \in \mathcal{G}$. Making the cell partition of Ω denser

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is expected to narrow the gap between the sufficient and necessary conditions above. This, however, can only be achieved at a higher computational price. The following algorithms can be used for stability analysis.

Algorithm A *(Forward Propagation Algorithm)*

STEP 1. Initialise $j = 1$ and $n = 2$.

STEP 2. Compute $\{\mathbf{P}_n(\Omega_j) \; : \; \mathbf{P}_n(\Omega_j) \in \mathcal{P}_n(\Omega_j)\}$.

STEP 3. If every cell trajectory $\mathbf{P}_n(\Omega_j) \in \mathcal{P}_n(\Omega_j)$ is non-cyclic then GO TO STEP 4 otherwise the system is unstable, STOP.

STEP 4. If all the cell trajectories enter Ω_1 then GO TO STEP 6 otherwise GO TO STEP 5.

STEP 5. Set $n = n + 1$. GO TO STEP 3.

STEP 6. If $j < L$ then set $j = j + 1$ and $n = 2$ and GO TO STEP 2 otherwise STOP.

The main idea behind this algorithm is directly computing all valid cell trajectories and analysing their properties. A different solution to the problem at hand can be found by recursively extending domains of attraction starting with the cell Ω_1 being a positive invariant set by definition.

Algorithm B *(Backward Propagation Algorithm)*

STEP 1. Set $W_1 = \Omega_1$, $i = 1$.

STEP 2. Compute recursively $W_{i+1} = W_i \bigcup \{\Omega_j \; : \; \hat{\mathrm{Tr}}(\Omega_j) \subseteq W_i\}$, with $\hat{\mathrm{Tr}}(\Omega_j)$ defined as $\hat{\mathrm{Tr}}(\Omega_j) = \bigcup_{\Omega_k \in \mathrm{Tr}(\Omega_j)} \Omega_k$.

STEP 3. If $W_i \subset W_{i+1}$ then set $i = i + 1$ and GO TO STEP 2 otherwise $W_i = W_{atr}$ and STOP.

It is easy to show that the condition $W_{atr} = \Omega$ is equivalent to the absence of cyclic cell trajectories. If, however, $W_{atr} \subset \Omega$ then the set W_{atr} is exactly the domain of attraction for the hybrid system.

5 Switching Controller Design

By definition the existence of a stabilizing controller is equivalent to the existence of a proper distribution of the switching indices over the cell partition of the system which would guarantee the absence in the closed loop system of cyclic cell trajectories. First, we modify Algorithm B taking into account the multiple choice of the switching indices available for each cell.

Algorithm C *(Switching Controller Design)*

STEP 1. Set $W_1 = \Omega_1$, $i = 1$.

STEP 2. Compute recursively $W_{i+1} = W_i \bigcup \{\Omega_j \; : \; \exists \; \xi \; \text{s.t.} \; \hat{\mathrm{Tr}}(\Omega_j, \xi) \subseteq W_i\}$, with $\hat{\mathrm{Tr}}(\Omega_j, \xi)$ defined as $\hat{\mathrm{Tr}}(\Omega_j, \xi) = \bigcup_{\Omega_k \in \mathrm{Tr}(\Omega_j, \xi)} \Omega_k$.

STEP 3. For each cell Ω_j define a set of valid switching indices $\Xi_j = \{\xi \; : \; \hat{\mathrm{Tr}}(\Omega_j, \xi) \subseteq W_i\}$.

STEP 4. If $W_i \subset W_{i+1}$ then $i = i + 1$ and GO TO STEP 2 otherwise $W_i = W_{contr}$ and STOP.

Relying on the algorithm above we are now in the position to formulate the following results.

Theorem 5.1 *Let $W_{contr} = \Omega$ then the hybrid system (1), (2) with $d(t) = 0$ is asymptotically stabilizable to the the origin by the switching controller $\Gamma(\xi, x)$.*

Theorem 5.2 *Let $W_{contr} \subset \Omega$ then there exists no switching controller of the form $\Gamma(\xi, x)$*

such that the closed loop system with the exogenous disturbance $d(t) \in \mathcal{G}$ is stable.

The interpretation of the results above is similar to that we made in Section 4.1.

6 Conclusions

In this paper we describe a method of switching control design applicable to a class of piecewise linear discrete time hybrid systems. The method is based on the notions of a cell transition and cell trajectory and allows for automated systematic treatment of a class of hybrid systems.

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