Special Section Technical Notes and Correspondence

Performance Degradation in Feedback Control Due to Constraints

Tristan Perez, Graham C. Goodwin, and Maria M. Serón

***Abstract*—In this note, we present a method to characterize the degradation in performance that arises in linear systems due to constraints imposed on the magnitude of the control signal to avoid saturation effects. We do this in the context of cheap control for tracking step signals.**

***Index Terms*—Antiwindup, cheap control, constrained performance.**

I. Introduction

The presence of constraints on inputs, in general, produces a degradation in the achievable closed-loop performance. Therefore, it is of practical importance to quantify and understand the fundamental aspects of this performance degradation. In this note, we take an initial step by proposing a way to characterize the closed-loop performance degradation that arises in single-input–single-output (SISO) linear feedback systems due to constraints on the magnitude of the control signal.

We analyze a suboptimal cheap control strategy that simply saturates the unconstrained cheap controller. We propose as a measure of performance the $\mathcal{L}_2$-norm ("energy") of the tracking error when a unitary reference is applied with the system initially at rest. The results also apply for output rejection if we measure the performance by the $\mathcal{L}_2$-norm of the output when a step output disturbance is applied since the two problems are analogous. The analysis yields analytical expressions for the cost that describes the degradation in performance and provides a benchmark against which the performance of other control strategies can be assessed. A preliminary version of these results has been presented in [8].

II. Preliminaries: Linear Limiting Optimal Control

In this section, we review some results of linear quadratic optimal control. These results will be used as a basis for the subsequent analysis of the performance of constrained systems.

Consider a linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu, \qquad x \in \mathbb{R}^n; \quad u \in \mathbb{R}\\ y &= Cx, \qquad y \in \mathbb{R}; \quad x(0) = x_0\end{aligned} \tag{1}$$

which is assumed to be stabilizable and detectable. Consider also the problem of regulating the output to a constant set-point r starting from zero initial state. Let us define the error variables

$$e = y - r \quad \tilde{x} = x - \bar{x} \quad \tilde{u} = u - \bar{u} \tag{2}$$

where the "bar"-variables denote the steady state values corresponding to the set-point r. Then, the optimal constant set-point tracking problem consists of finding the stabilizing feedback control law that minimizes the following cost:

$$J_{\mathrm{TRACK}}(\varepsilon) = \int_0^\infty (e^2(t) + \varepsilon^2 \tilde{u}^2(t))\, dt. \tag{3}$$

The control law that achieves zero steady-state error in the original variables is [6]

$$u = -K_\varepsilon x + \bar{K}_\varepsilon r \tag{4}$$

where

$$K_\varepsilon = \frac{1}{\varepsilon^2} B^{\mathrm{T}} P(\varepsilon) \quad \bar{K}_\varepsilon = C(BK_\varepsilon - A)^{-1} B \tag{5}$$

and $P(\varepsilon)$ satisfies

$$A^{\mathrm{T}} P(\varepsilon) + P(\varepsilon) A + C^{\mathrm{T}} C - \frac{1}{\varepsilon^2} P(\varepsilon) B B^{\mathrm{T}} P(\varepsilon) = 0. \tag{6}$$

The cheap control framework assumes $\varepsilon \to 0$. The optimal value of J_{TRACK} for this case can be obtained by transforming the system into its normal form or zero dynamics form [5], i.e., $\tilde{x}$ is transformed to $[\eta^{\mathrm{T}} \ z^{\mathrm{T}}]^{\mathrm{T}}$ by a linear transformation that takes the system into the following form:

$$\begin{aligned}\dot{\eta} &= A_0 \eta + B_0 z_1\\ \dot{z} &= A_1 \eta + A_2 z + B_1 u\\ y &= z_1.\end{aligned} \tag{7}$$

In (7), $\eta \in \mathbb{R}^m$, $z = [z_1, z_2, \ldots, z_{n-m}]^{\mathrm{T}} \in \mathbb{R}^{n-m}$, and the eigenvalues of A_0 are the m zeros of the system transfer function $G(s) = C(sI - A)^{-1}B$. The η-subsystem is called the zero-dynamics subsystem. Then, the optimal cost is [11]

$$J_{\mathrm{TRACK}}^{\mathrm{OPT}} = \tilde{\eta}(0)^{\mathrm{T}} P_0 \tilde{\eta}(0) \tag{8}$$

where P_0 is the solution of the following Lyapunov equation:

$$A_0^{\mathrm{T}} P_0 + P_0 A_0 = P_0 B_0 B_0^{\mathrm{T}} P_0. \tag{9}$$

Suppose that all the zeros are non minimum phase (NMP). Then, A_0 is nonsingular and the initial condition for the zero-dynamics subsystem is $\tilde{\eta}(0) = A_0^{-1} B_0 r$. Using this value in (8), and assuming $r = 1$, we obtain

$$\begin{aligned}J_{\mathrm{TRACK}}^{\mathrm{OPT}} &= B_0^{\mathrm{T}} \left(A_0^{\mathrm{T}}\right)^{-1} P_0 A_0^{-1} B_0\\ &= 2 \ \mathrm{trace}\ A_0^{-1} = 2 \sum_{i=1}^{m} \frac{1}{q_i}\end{aligned} \tag{10}$$

where $q_i, i = 1, \ldots, n_q$, are the NMP-zeros of the system.

This result was originally obtained in [9] for the case of output disturbances using feed-forward control law.

A. Input–Output Characteristics and Classical Control Loops

An interesting aspect of the cheap control problem is its asymptotic input–output behavior. It was shown in [1] (and analyzed in detail in [6] and [7]) that, as $\varepsilon \to 0$, some closed-loop poles converge to the reflection of the nonminimum phase zeros about the imaginary axis, while

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The authors are with the School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan, NSW 2308, Australia.

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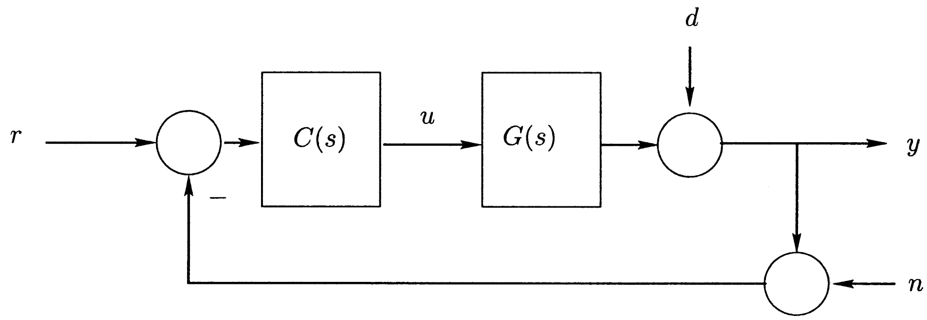

Fig. 1. Standard SISO loop.

the remaining poles tend to infinity in Butterworth patterns. These results are summarized in the following.

Let the transfer function of (1) be

$$G(s) = \alpha \frac{\prod_{i=1}^{m}(s - q_i)}{\prod_{i=1}^{n}(s - p_i)}, \qquad \alpha \neq 0. \tag{11}$$

Using (4) in closed loop with the system (1), and taking $\varepsilon \to 0$, it was shown in [6] that the closed-loop transfer function $T(s)$ from the reference $R(s)$ to the output $Y(s)$ approaches

$$T_{\mathrm{CHEAP}}(s) \to \frac{1}{\chi_{n-m}\left(\frac{s}{\omega_0}\right)} \prod_{i=1}^{m} \frac{-\frac{s}{q_i}+1}{-\frac{s}{\hat{q}_i}+1} \omega_0 = \left(\frac{\alpha^2}{\varepsilon}\right)^{\frac{1}{2(n-m)}} \tag{12}$$

where

$$\hat{q}_i = \begin{cases} q_i, & \text{if } \mathrm{Re}(q_i) \leq 0 \\ -q_i, & \text{if } \mathrm{Re}(q_i) > 0 \end{cases} \tag{13}$$

and χ_{n-m} is a Butterworth polynomial of order $n-m$ and radius 1, ω_0 is the asymptotic radius of the Butterworth configuration of the $n-m$ closed loop poles that tend to infinity, and $q_i, i = 1, \ldots, m$, are the zeros of the open-loop transfer function (11). In (12), we can also see that there is a two-time scale behavior of the closed loop system, since the all-pass factors have slow dynamics and the poles in the Butterworth pattern have fast dynamics. In the limit, as $\varepsilon \to 0$, no matter what the initial condition $(\eta(0), z(0))$ is, the state $(\eta(0^+), z(0^+))$ is on a singular hyperplane given by $y + B_0^{\mathrm{T}} P_0 \eta = 0$, and evolves inside this subspace thereafter. The initial fast response of the state is singular and so is the control that takes the state from the initial condition into the hyperplane. Once the state is on the hyperplane, it presents a slow evolution given by the dynamics according to $\dot{\eta} = P_0^{-1} A_0^{\mathrm{T}} P_0 \eta$ with $y = -B_0^{\mathrm{T}} P_0 \eta$. For systems of higher relative degree, a similar analysis holds [10].

The results presented so far deal with the cheap control problem for state feedback; however, for standard SISO control loops (see Fig. 1) similar results hold [2]. One way to obtain such results is by using expression (12) and the affine parametization of all stabilizing controllers, see for example [3] or [4]. Specifically, (12) can be expressed as

$$T_{\mathrm{CHEAP}}(s) = G(s) Q_{\mathrm{CHEAP}}(s) \tag{14}$$

where $Q_{\mathrm{CHEAP}}(s)$ is a stable proper transfer function satisfying

$$Q_{\mathrm{CHEAP}}(s) = G_{MP}^{-1}(s) F(s) \quad F(s) = \frac{1}{\chi_{n-m}\left(\frac{s}{\omega_0}\right)} \tag{15}$$

where χ_{n-m} is as in (12), and $G_{MP}(s)$ has the same poles of $G(s)$ and the reflection of the nonminimum phase zeros of $G(s)$ through the imaginary axis. If the plant $G(s)$ is stable, the controller $C(s)$ in Fig. 1 can be parameterized as

$$C(s) = Q(s)^{-1}(1 - G(s)Q(s)). \tag{16}$$

For the case of unstable plants, an alternative parameterization to (16) can be used; see, for example, [4]. However, for ease of exposition, we consider open-loop stable plants. Under these conditions, we have the following result for the standard single-loop case.

Proposition I.1: Assuming that the standard control loop is internally stable, and that $C(s)$ provides integral action, then for unitary step reference

$$\inf \int_0^\infty e^2(t)\, dt = 2 \sum_{i=1}^{n_q} \frac{1}{q_i} \tag{17}$$

where $\{q_i \colon i = 1, \ldots, n_q\}$ is the set of zeros in the ORHP of $L(s) = C(s)G(s)$.

Proof: By parameterizing $C(s)$ as in (16) with Q satisfying (15), then $T(s) = T_{\mathrm{CHEAP}}(s)$ in the limit as $\varepsilon \to 0$; and therefore $Y(s) = Y_{\mathrm{CHEAP}}(s)$ and the tracking errors of both single loop and state feedback compensator schemes are the same. Finally, it follows that the latter controller provides integral action from (4), and (17) is immediate from (10). ■

The simple proof for Proposition I.1 highlights the link between the single loop structure and the state feedback compensator, and serves as a basis to extend the results to the constrained cases. However, it should be noted, that this Proposition can also be demonstrated directly using quadratic optimal synthesis to obtain $Q(s)$; see, for example, [2].

III. Constrained Cheap Control Performance

In this note, we address the problem of tracking step references. We propose as a measure of the performance limitations in the presence of input constraints the value of the $\mathcal{L}_2$-norm of the tracking error for the cheap antiwindup scheme shown in Fig. 2 [4], where

$$\mathrm{sat}_\Delta(z) = \min\{z, \Delta \mathrm{sign}(z)\}, \qquad \Delta > 0. \tag{18}$$

Antiwindup schemes, such as the one shown in Fig. 2, provide the simplest solution to avoid excessive performance degradation due constrains, and have been thoroughly analyzed in the literature; see, for example, [4] and the references therein. This motivates us to use this scheme to quantify the performance degradation.

The controller $C(s)$ in Fig. 2 is the biproper cheap controller parameterized as in (16) with $Q(s)$ satisfying (15) and c_∞ is its high frequency gain. It is easily seen that when the system is not saturated, the inner loop in Fig. 2 reduces to $C(s)$; and therefore, the closed-loop transfer function without saturation equals that given in (12). It is also worth noting that the antiwindup scheme shown in Fig. 2 is equivalent to saturating the state feedback cheap controller scheme. We will comment on this later.

The performance for SISO systems subject to the constraint $|u(t)| \leq \Delta \forall t$ will then be measured by the value of the cost function given in (3) when a unit step reference is applied with the system initially at rest.

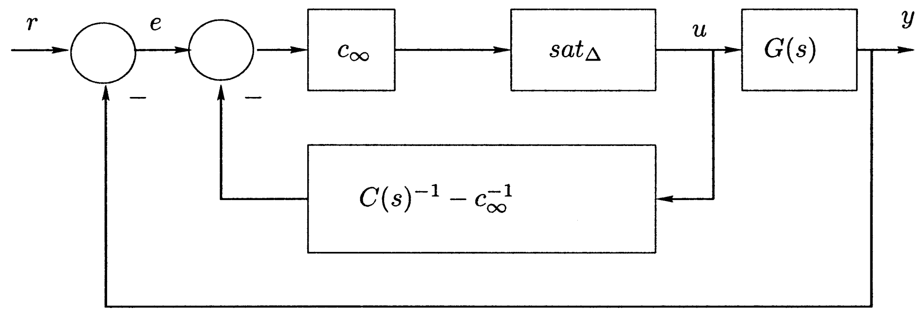

Fig. 2. Simplified antiwindup control loop.

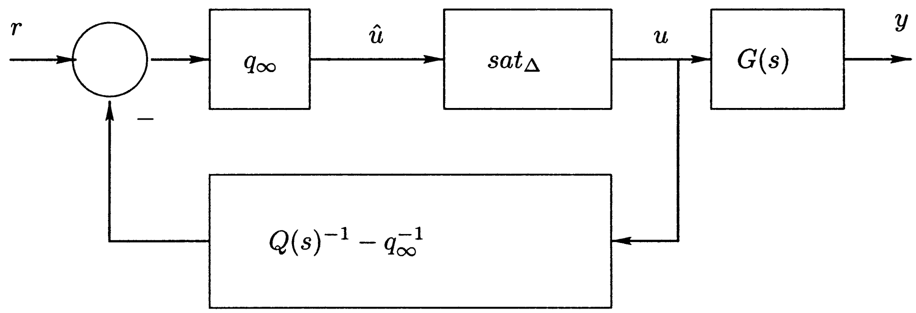

Fig. 3. Antiwindup open-loop equivalent control.

To evaluate the cost (3), we assume that input saturation only occurs in the first part of the evolution of the system.

Assumption 1 (A.1): There exists time $t_{\rm sat} \in [0^+, \infty)$, such that the control of the closed-loop system switches between the saturation levels for $t < t_{\rm sat}$, and never thereafter.

Note that this is a reasonable assumption given the high gain nature of the cheap controller. Under Assumption (A.1), the cost (3) can be separated into two components; one corresponding to the period of time where the control saturates and the other corresponding to the period of time starting when the control leaves saturation until it reaches the final steady-state value $u(\infty)$, i.e.,

$$J = \lim_{\varepsilon \to 0} \int_0^{t_{\rm sat}} (e^2(t) + \varepsilon^2 u^2(t))\, dt + \lim_{\varepsilon \to 0} \int_{t_{\rm sat}}^{\infty} (e^2(t) + \varepsilon^2 u^2(t))\, dt. \quad (19)$$

1) Cost During Saturation: In order to evaluate the first term of the cost (19), we will use the affine parameterization of the controller [cf., (14) and (16)]. For stable plants, the cheap antiwindup scheme in Fig. 2 can be analyzed using the equivalent open-loop scheme shown in Fig. 3, i.e., the latter scheme reproduces the control and output signals of the scheme in Fig. 2. In Fig. 3, $Q(s)$ is the biproper transfer function given by (15), and q_∞ is its high frequency gain. It is easily seen that when the system is not saturated, the loop in Fig. 3 reduces to $Q(s)$.

From Fig. 3, we see that after applying a unit step signal in $r(t)$, the control $u(t)$ typically saturates since the gain q_∞ is usually large for small values of ε. We thus assume, without loss of generality, that $u(0^+) = \Delta$. The control will then switch between Δ and $-\Delta$ until $t = t_{\rm sat}$ when it leaves saturation to continue with a linear evolution. The crucial step in this analysis is then to evaluate the switching times during the saturated regime. We will illustrate the ideas by taking the cases with, at most, one switch in the saturation regime. In these cases, during the saturation period $[0^+, t_{\rm sat})$, $\hat{u}(t)$ (i.e., the signal at the input of the saturation function; see Fig. 3) is given by

$$\hat{u}(t) = q_\infty \left(1 - \mathrm{L}^{-1}\left\{\left(Q(s)^{-1} - q_\infty^{-1}\right)\frac{\Delta}{s}\right\}\right) \quad (20)$$

where $L^{-1}\{\cdot\}$ denotes the inverse Laplace transform operator. In addition, the control signal leaves saturation when the following condition is satisfied:

$$\hat{u}(t_{\rm sat}) = \Delta. \quad (21)$$

Using (20) and (21), we can determine the time instant $t_{\rm sat}$ at which the control leaves saturation. Also, from Fig. 3, we see that the tracking error during the saturation interval can be calculated as

$$e(t) = y(t) - r = \mathrm{L}^{-1}\left\{G(s)\frac{\Delta}{s}\right\} - 1.$$

With the aforementioned expressions for $e(t)$ and $t_{\rm sat}$, all the ingredients to evaluate the first term of (19) are available. After performing the limits and the integration, we can obtain an analytical expression for the first term of the cost.

2) Cost After Saturation: Once the control signal leaves saturation, the problem reduces to the unconstrained cheap control problem. To find the associated cost we will use the properties of the slow evolution of the state in the singular hyperplane. The first step is to recognize that when the system leaves saturation, the state of the system is on the singular hyperplane. This is easy to show by contradiction: Suppose that the system leaves saturation and never saturates again, and also that the state is not on the singular hyperplane. Then as the control is not saturated the system behaves like the unconstrained problem, and since the state is not on the singular hyperplane there will be a singularity in the control that will make the control saturate. Therefore, once the control leaves saturation, the state must be on the singular hyperplane.

Consequently, once the control leaves saturation at $t = t_{\rm sat}$, the state of the system follows the same trajectories that the unconstrained state would have followed if it had started from the initial condition $[\eta(t_{\rm sat}) z(t_{\rm sat})]'$. Hence, the cost after saturation is

$$\int_{t_{\rm sat}}^{\infty} e^2(t)\, dt = \eta(t_{\rm sat})' P_0 \eta(t_{\rm sat}) \quad (22)$$

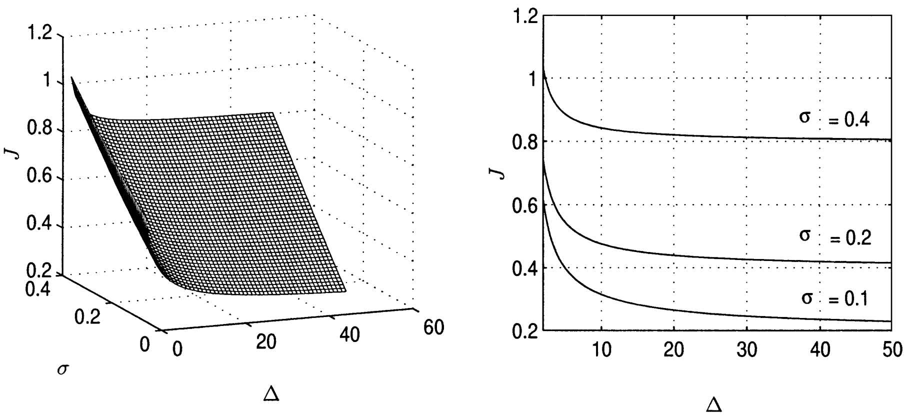

Fig. 4. Cost versus Δ for Example 2.

where $\eta(t_{\mathrm{sat}})$ is determined from

$$\begin{bmatrix}\eta(t_{\mathrm{sat}})\\ z(t_{\mathrm{sat}})\end{bmatrix} = e^{A_n t_{\mathrm{sat}}}\begin{bmatrix}\eta(0)\\ z(0)\end{bmatrix} + \int_0^{t_{\mathrm{sat}}} e^{A_n(t-\tau)} B_n \Delta\, d\tau \qquad (23)$$

where A_n and B_n are given by

$$A_n \triangleq \begin{bmatrix}A_0 & [B_0\ 0\ 0]\\ A_1 & A_2\end{bmatrix} \quad B_n \triangleq \begin{bmatrix}0\\ B_1\end{bmatrix}$$
$$C_n \triangleq [1 \quad 0 \quad \cdots \quad 0]. \qquad (24)$$

Equation (22) allows us to complete the calculation of the cost by evaluating the second term of on the right-hand side of (19).

For systems with only one nonminimum phase zero, the evaluation of the cost once the system leaves saturation can be considerably simplified since it is not necessary to solve (23). If the system has only one nonminimum phase zero, the singular hyperplane becomes a line in $\mathbb{R}^n$. Since the singular hyperplane is a line, the output $e = \tilde{z}_1$ after saturation has the same evolution as the unconstrained output shifted in time. As a consequence, we can evaluate the cost after saturation of the constrained cheap control problem using a partial cost of the unconstrained cheap control problem as

$$\int_{t_{\mathrm{sat}}}^{\infty} e^2\, dt \approx \int_{t^*}^{\infty}\left(1 - \mathrm{L}^{-1}\left[\left(\frac{-\frac{s}{q}+1}{\frac{s}{q}+1}\right)\frac{1}{s}\right] dt\right)^2 = \frac{2}{q}e^{-2qt^*}. \qquad (25)$$

The approximation in (25) comes from neglecting the high frequency poles in the Butterworth arrangement in (12). The value t^* is determined from the condition

$$\mathrm{L}^{-1}\left[G(s)\frac{\Delta}{s}\right]\Bigg|_{t_{\mathrm{sat}}} = \mathrm{L}^{-1}\left[\left(\frac{-\frac{s}{q}+1}{\frac{s}{q}+1}\right)\frac{1}{s}\right]\Bigg|_{t^*}.$$

Specifically, if we call $k_{t\,\mathrm{sat}} = \mathrm{L}^{-1}[G(s)(\Delta/s)]|_{t_{\mathrm{sat}}}$, then

$$t^* = -\frac{1}{q}\ln\left(\frac{1-k_{t\,\mathrm{sat}}}{2}\right). \qquad (26)$$

Comparing (25) with (10) for the case of a single nonminimum phase zero, we see that the partial cost (25) is smaller than the total unconstrained cost (10). However, whereas the transition to the singular hyperplane is costless in the unconstrained case, it has a nonzero cost in the constrained case since the state cannot "jump" to the singular hyperplane but has a slow evolution to the hyperplane while the control is saturated. The combination of the two partial costs yields a cost larger than the unconstrained cost (10), as we will see in the examples.

A. Examples

Example 1: Consider the following system:

$$G(s) = \frac{1}{\tau s + 1}.$$

Then, the equivalent open-loop cheap controller is given by

$$Q(s) = \frac{\tau s + 1}{\beta s + 1}$$

where $\beta = \sqrt{\epsilon}$. It also follows that, $q_\infty = \tau/\beta$ and

$$Q(s)^{-1} - q_\infty^{-1} = \frac{(\tau - \beta)}{\tau(\tau s + 1)}. \qquad (27)$$

Using (27) in (20), we have

$$\hat{u}(t) = \frac{\tau}{\beta}\left[1 - \frac{(\tau-\beta)}{\tau}\left(1 - e^{-\frac{t}{\tau}}\right)\Delta\right].$$

This last expression is valid until t_{sat}. Using (21), and taking the limit as $\beta \to 0, t_{\mathrm{sat}}$ is found to be

$$t_{\mathrm{sat}} \to -\tau \ln\left[\frac{\Delta - 1}{\Delta}\right]. \qquad (28)$$

Also, until t_{sat} the tracking error is

$$e(t) = \Delta\left(1 - e^{-\frac{t}{\tau}}\right) - 1. \qquad (29)$$

Finally, since the system is minimum phase, and using (29) and (28), we obtain the following expression for the cost (19):

$$J = \tau\left[\frac{3}{2}(1-\Delta)^2 + (\Delta^2 - 2\Delta + 1) \times \log\left(\frac{\Delta}{\Delta - 1}\right) - \frac{3}{2}\Delta^2 + 2\Delta\right]. \qquad (30)$$

It should be noted that for the set point value of the control signal to be feasible, we need $\Delta > 1$. Since this system is minimum phase, the limiting cheap cost for $\Delta \to \infty$ is zero. Also, as expected, the slower the plant (i.e., larger τ), the higher the cost, since slow plants require more control effort that contribute to saturation in this case.

Even though the results are in agreement with intuition, the expression for the cost (30) is far from trivial. This indicates, that even for a simple case, the structural characteristics of the system and its dynamics combine in a rather involved manner to contribute to the degradation of performance.

Example 2: Consider the following system:

$$G(s) = \frac{2(1-\sigma s)}{(s+1)(s+2)}. \qquad (31)$$

In this case, the equivalent open-loop cheap controller is given by

$$Q(s) = \frac{(s+1)(s+2)}{2(1+\sigma s)(1+\beta s)}. \tag{32}$$

It then follows that $q_\infty = 1/2\sigma\beta$, and

$$Q(s)^{-1} - q_\infty^{-1} = \frac{2\left[(\sigma+\beta-3\sigma\beta)s + (1-2\sigma\beta)\right]}{(s+1)(s+2)}. \tag{33}$$

Using (33) in (20) we have that, as $\beta \to 0$

$$t_{\mathrm{sat}} \to \ln\left[\frac{(1-2\sigma)}{-\Delta(1+\sigma) - \sqrt{\Delta^2(\sigma+1)^2 + \Delta(1-2\sigma)(1-\Delta)}}\right]. \tag{34}$$

The tracking error on the interval $[0^+ t_{\mathrm{sat}})$ is

$$e(t) = \Delta\left[1 - 2(1+\sigma)e^{-t} + (1+2\sigma)e^{-2t}\right] - 1. \tag{35}$$

The value of $k_{t\,\mathrm{sat}}$ given by

$$k_{t\,\mathrm{sat}} = \Delta\left.\left[1 - 2(1+\sigma)e^{-t} + (1+2\sigma)e^{-2t}\right]\right|_{t_{\mathrm{sat}}} \tag{36}$$

and t^{star} is given in (26).

Using (34)–(36), (25), and (26), we can evaluate the cost. The results are shown in Fig. 4. Note that as $\Delta \to \infty$ the limiting cost approaches 2σ, which is consistent with the results of unconstrained cheap control. The results shown in Fig. 4 give insight into the effect of the input constrained achievable performance. It is interesting, for instance, to note that a constraint $\Delta = 5$ (which is five times the steady state input necessary in this case) changes the performance limit associated with a non minimum phase zero at 10 $(\sigma = 0.1)$ to be equivalent to the performance limit achieved *without constraints* for a non minimum phase zero at five. This illustrates the fact that, depending on conditions, the effect of input constraints can swamp linear effects due to right-half plane zeros. This is in accord with intuition.

IV. Conclusion

In this note, we have presented a method to evaluate the degradation in performance of the closed-loop system when constraints are added. We have focused our analysis on open-loop stable SISO systems tracking step references, and used as performance index the value of the $\mathcal{L}_2$-norm of the error.

We have obtained analytical expressions for the cost (performance index) that show how system dynamics and constraints interact to deteriorate the performance. Even for simple cases, the obtained expressions indicate that this interaction is far from trivial. In addition, for some cases, tight constraints can swamp limitations associated with system dynamics.

The information provided by the analysis has implications on the choice of actuator authority and on the need for using tactical strategies to address the problem of constraint handling. Indeed, it is a common practice to ignore constraints in previous stages of control design and then evaluate the performance in the presence of constraints. The proposed method not only can help to assess whether this approach leads to good results but also to decide whether to upgrade the actuator or alternatively to consider more sophisticated control strategies.

Although the presented method has only been illustrated for systems with at most one switch, the path to follow for extending the results to multiswitching systems is clear. However, depending on the dynamics of the system, the calculations of the time switchings can be rather involved.

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Performance Limitations of Nonlinear Periodic Sampled-Data Controllers for L_p Disturbance Rejection

Robert Schmid and Cishen Zhang

***Abstract*—This note presents a performance analysis of periodic nonlinear sampled-data controllers for the rejection of L_p specific and uniform disturbances. Earlier results on the performance of linear periodic controllers are extended to nonlinear controllers. For a given periodic controller, a time invariant controller is constructed which in general gives strictly better L_p disturbance rejection performance than the periodic controller.**

***Index Terms*—Disturbance rejection, L_p space, nonlinear systems, periodic systems, sampled-data systems.**

I. Introduction

Time-varying and nonlinear feedback control is often applied to systems for which conventional linear time invariant control cannot achieve the desired system performance. The use of periodic linear and nonlinear control to achieve particular performance specifications has been actively studied for the last two decades. Periodic control has been shown to have advantages over time-invariant control in a number of areas, including simultaneous stabilization of a number of plants

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R. Schmid is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville 3010, Australia.

C. Zhang is with the School of Electrical and Electronic Engineering, Nanyang Technological University, 639798 Singapore (e-mail: ecszhang@ntu.edu.sg).

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