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CUNTZ-KRIEGER ALGEBRAS OF INFINITE GRAPHS AND MATRICES

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ABSTRACT. We show that the Cuntz-Krieger algebras of infinite graphs and infinite $\{0, 1\}$-matrices can be approximated by those of finite graphs. We then use these approximations to deduce the main uniqueness theorems for Cuntz-Krieger algebras and to compute their K-theory. Since the finite approximating graphs have sinks, we have to calculate the K-theory of Cuntz-Krieger algebras of graphs with sinks, and the direct methods we use to do this should be of independent interest.

The Cuntz-Krieger algebras $\mathcal{O}_A$ were introduced by Cuntz and Krieger in 1980, and have been prominent in operator algebras ever since. At first the algebras $\mathcal{O}_A$ were associated to a finite matrix A with entries in $\{0, 1\}$, but it was quickly realised that they could also be viewed as the C^*-algebras of a finite directed graph [33]. Over the past few years, originally motivated by their appearance in the duality theory of compact groups [22], authors have considered analogues of the Cuntz-Krieger algebras for infinite graphs and matrices (see [21], [27], [10], [13], [1] and the survey articles [18], [24]). The class of Cuntz-Krieger algebras now encompasses a vast array of important C^*-algebras, including Toeplitz algebras, $\mathcal{O}_\infty$ and AF-algebras, as well as those arising in duality theory.

A directed graph E consists of a vertex set E^0, an edge set E^1, and range and source maps $r, s : E^1 \to E^0$. A Cuntz-Krieger E-family consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying

$$(0.1) \qquad S_e^* S_e = P_{r(e)} \text{ and } P_v = \sum_{\{e\,:\,s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink;}$$

the graph algebra $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$. The equations (0.1) make sense as they stand for row-finite graphs, in which the index set $\{e \in E^1 : s(e) = v\}$ for the sum is always finite. If a vertex v emits infinitely many edges, the sum does not make sense in a C^*-algebra, because infinite sums of projections cannot converge in norm. However, it was observed in [13] that the general theory of Cuntz-Krieger algebras carries over to arbitrary countable graphs if one simply removes the relations involving infinite sums from (0.1), and requires instead that the range projections $S_e S_e^*$ are mutually orthogonal and dominated by $P_{s(e)}$.

Exel and Laca have described a different generalisation of the Cuntz-Krieger algebras for infinite matrices A [10]. Their defining relations are complicated: loosely speaking, one has to include a Cuntz-Krieger relation whenever a row-operation

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on A yields a finitely non-zero vector. The resulting *Exel-Laca algebras* include the graph algebras of [13] (and the results of [10] are used in [13]), but there exist matrices A for which $\mathcal{O}_A$ is not a graph algebra (see Remark 4.4). The analysis of [10] is deep, and depends on the machinery of partial actions [12]; Szymański has shown that Exel-Laca algebras can also be analysed using Pimsner's results on the Cuntz-Pimsner algebras of Hilbert bimodules [30].

We show here that the Exel-Laca algebras are direct limits of C^*-algebras of finite graphs, and that this approximation process can be used to derive the main theorems about them. We hope that, since the theory of algebras of finite graphs is by now relatively elementary (see [1] and §1 below), this provides a more friendly route to the Exel-Laca theory. We also believe that the approximation process we describe is itself a powerful tool, which will be helpful, for example, when working with K-theory.

It is an intrinsic feature of our construction that the approximating graphs have sinks. (In the language of $\{0,1\}$-matrices, we need to allow rows of zeros.) It is understood in principle how to adapt the general theory to cover graphs with sinks [1, §1], but calculating the K-theory requires some work. Our approach to the computation of K-theory is new: we use the skew-product graphs of Kumjian and Pask [19] to avoid much of the usual chasing through stable isomorphisms and duality [8], [25], [23]. Thus this approach should be of interest even to those who only encounter graphs without sinks.

We begin in Section 1 by describing our approximation procedure for the graph algebras of infinite graphs, which is based on the isomorphism between the C^*-algebra of a graph and that of its dual. In Section 2, we describe the analogous approximation of Exel-Laca algebras, which is based on the usual isomorphism of a Cuntz-Krieger algebra with a graph algebra. Once we have the approximation, we can easily deduce the uniqueness theorems for Exel-Laca algebras; our gauge-invariant uniqueness theorem is apparently new, even for the algebras of infinite graphs. In Section 3, we calculate the K-theory of $C^*(E)$ when E is a row-finite graph with sinks, and in Section 4 we show how to apply this to Exel-Laca algebras. We close with a section of concluding remarks, in which we consider questions of finiteness, stable rank and approximate finite-dimensionality.

1. The C^*-algebras of infinite graphs

Let $E = (E^0, E^1, r, s)$ be a (countable) directed graph. A Cuntz-Krieger E-family consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

(G1) $S_e^* S_e = P_{r(e)}$,
(G2) $S_e S_e^* \leq P_{s(e)}$,
(G3) $P_v = \sum_{s(e)=v} S_e S_e^*$ if $s^{-1}(v)$ is finite and non-empty.

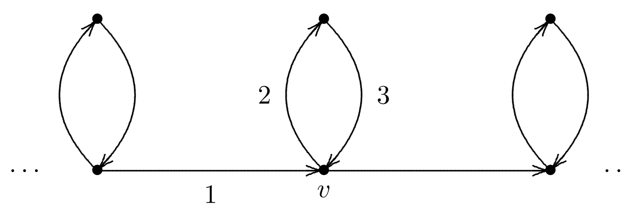
The C^*-algebra $C^*(E)$ of E is the universal C^*-algebra generated by a Cuntz-Krieger family $\{s_e, p_v\}$; there are various ways of showing that there is such a C^*-algebra, either by direct arguments [15], [20] or by appealing to general machines [2], [13]. If $\{S_e, P_v\}$ is a Cuntz-Krieger family, we denote by $\pi_{S,P}$ the representation of $C^*(E)$ such that $\pi_{S,P}(p_v) = P_v$ and $\pi_{S,P}(s_e) = S_e$. The universality of $C^*(E)$ implies that there is a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ such that $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$.

The key idea in what follows is to approximate the dual of a graph rather than the graph itself. The dual is by definition the graph $\hat{E}$ with $\hat{E}^0 = E^1$,

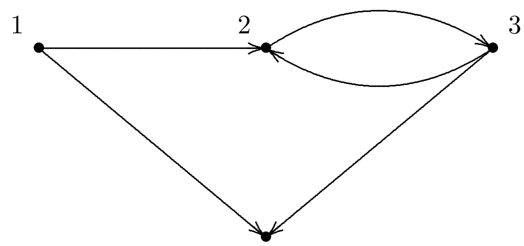
$$\hat{E}^1 = \{(e, f) \in E^1 \times E^1 : r(e) = s(f)\},$$

$\hat{r}(e, f) = f$ and $\hat{s}(e, f) = e$. The embeddings of the approximating graph algebras which we describe in Lemma 1.2 below are modelled on the isomorphism of $C^*(\hat{E})$ onto $C^*(E)$ constructed in [1, Corollary 2.5].

To construct our approximations, we start with a finite subgraph F of E, and form its dual $\hat{F}$. We then look at the vertices in $r(F^1)$ which emit in E both edges in F and edges which are not in F: for each such vertex v we add a sink to $\hat{F}$, and for each edge e in F ending at v we add a new edge to $\hat{F}$ going from the vertex e in $\hat{F}$ to the sink. For example, if E is the following infinite graph and F is its subgraph consisting of the edges labelled 1, 2, and 3,



then our procedure yields the following finite graph:



More formally, we make the following definition.

Definition 1.1. Let E be a directed graph without sinks and let $F \subset E^1$ be a finite set. Let E_F denote the finite directed graph in which

$$E_F^0 = F \cup (r(F) \cap s(F) \cap s(E^1 \setminus F)),$$
$$E_F^1 = \{(e, f) \in F \times E_F^0 : r(e) = s(f)\},$$

$s(e, f) = e$, and $r(e, f) = f$.

When E is a finite directed graph without sinks and $F = E^1$, we have $E_F = \hat{E}$, but in general E_F may have many sinks even if E has none. We think of the vertices in E_F as representing projections in the C^*-subalgebra of $C^*(E)$ generated by $\{s_e : e \in F\}$: the projection corresponding to a vertex $e \in F$ is the range projection $s_e s_e^*$, and the one corresponding to a vertex $v \in r(F) \cap s(F) \cap s(E^1 \setminus F)$ is $p_v - \sum\{s_f s_f^* : f \in F, s(f) = v\}$ (which is a minimal projection in $C^*(s_e : e \in F)$). The next lemma makes this precise.

Lemma 1.2. *Let E be a directed graph without sinks and let $F \subset E^1$ be a finite set of edges. Then $C^*(E_F)$ is naturally isomorphic to the C^*-subalgebra of $C^*(E)$ generated by $\{s_e : e \in F\}$.*

Proof. Let A denote the C^*-subalgebra of $C^*(E)$ generated by $\{s_e : e \in F\}$. The projections

$$\{s_e s_e^* : e \in F\} \cup \Big\{p_v - \sum_{f \in F,\ s(f)=v} s_f s_f^* : v \in E_F^0 \cap E^0\Big\}$$

and the partial isometries

$$\{s_e s_f s_f^* : e, f \in F,\ r(e) = s(f)\} \\ \cup \Big\{s_e\Big(p_{r(e)} - \sum_{f \in F,\ s(f)=r(e)} s_f s_f^*\Big) : e \in F,\ r(e) \in E_F^0 \setminus F\Big\}$$

form a Cuntz-Krieger E_F-family in A. Since every s_e is a finite sum of elements of this family, the family generates A. Thus the universal property of $C^*(E_F)$ gives a surjective homomorphism $\phi : C^*(E_F) \to A$ which carries generators to generators. Let α, γ be the gauge actions of $\mathbb{T}$ on $C^*(E_F)$ and $C^*(E)$, respectively. Since A is γ-invariant and $\phi \circ \alpha_z = \gamma_z \circ \phi$ for $z \in \mathbb{T}$, the gauge-invariant uniqueness theorem [1, Theorem 2.1] implies that ϕ is an isomorphism. $\square$

When E is the graph with one vertex and infinitely many edges, $C^*(E)$ is the Cuntz algebra $\mathcal{O}_\infty$, and Lemma 1.2 yields the description of $\mathcal{O}_\infty$ as a direct limit of Toeplitz-Cuntz algebras $\mathcal{TO}_n$ used in [7]. However, the approximation technique based on Lemma 1.2 is substantially different from the one used by Pask and Raeburn in [25].

It will be important later that the process of passing from E to the finite approximation E_F preserves the loop structure. As in [20], an *exit* from a loop L is an edge $e \in E^1 \setminus L$ whose source $s(e)$ is also the source of an edge in the loop.

Lemma 1.3. *Let E be a directed graph without sinks and let $F \subset E^1$ be a finite set. If $L = (x_1, \dots, x_n)$ is a loop in E_F, then there exists a unique loop $L' = (e_1, \dots, e_n)$ in E such that $\{e_i\}_{i=1}^n \subset F$, $x_i = (e_i, e_{i+1})$ for $i = 1, \dots, n-1$ and $x_n = (e_n, e_1)$. Furthermore, L has an exit if and only if L' does.*

Proof. Since different edges in E_F come from different edges in E, any exit for L in E_F comes from an exit for L'. On the other hand, if L' has an exit, then there is a vertex in F which emits two edges e, f in E, at least one of which, say f, is in F. If $e \in F$ too, then e is an exit in F; if $e \notin F$, then there is an edge from $s(e)$ to a sink which leaves L. $\square$

We need to know how to relate the C^*-algebras of arbitrary graphs to those of graphs with neither sinks nor sources. As was shown in [1, Lemma 1.2] for row-finite graphs, we can add tails at sinks without substantially changing $C^*(E)$. An obvious analogue of this procedure for dealing with sources was described in [13]: by *adding a head at a vertex w* we mean extending E to a graph F, in which

$$F^0 = E^0 \cup \{v_i : -\infty < i \le -1\}, \quad F^1 = E^1 \cup \{e_i : -\infty < i \le -1\},$$

and r, s are extended to F^1 by $r(e_i) = v_{i+1}$ (and $r(e_{-1}) = w$) and $s(e_i) = v_i$. The proof of the following lemma is almost identical to that of [1, Lemma 1.2], and hence is omitted.

Lemma 1.4. *Let E be a directed graph and let F be the graph obtained by adding a tail at each sink in E and a head at each source in E. Then:*

(1) *for each Cuntz-Krieger E-family $\{S_e, P_v\}$ on a Hilbert space $\mathcal{H}_E$, there is a Hilbert space $\mathcal{H}_F = \mathcal{H}_E \oplus \mathcal{H}_T$ and a Cuntz-Krieger F-family $\{T_e, Q_v\}$ such that $T_e = S_e$ for $e \in E^1$, $Q_v = P_v$ for $v \in E^0$, and $\sum_{v \notin E^0} Q_v$ is the projection on $\mathcal{H}_T$;*

(2) *if $\{T_e, Q_v\}$ is a Cuntz-Krieger F-family, then $\{T_e, Q_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family. If w is a sink (source) in E such that $Q_w \neq 0$, then $Q_v \neq 0$ for every vertex v on the tail (head) attached to w;*

(3) *if $\{t_e, q_v\}$ are the canonical generators of $C^*(F)$, then the homomorphism $\pi_{t,q}$ corresponding to the Cuntz-Krieger E-family $\{t_e, q_v : e \in E^1, v \in E^0\}$ is an isomorphism of $C^*(E)$ onto the C^*-subalgebra of $C^*(F)$ generated by $\{t_e, q_v : e \in E^1, v \in E^0\}$, which is the full corner in $C^*(F)$ determined by the projection $p := \sum_{v \in E^0} q_v$.*

Exactly how the sum $\sum_{v \in E^0} q_v$ defines a projection p in $M(C^*(F))$ is explained in [1, Lemma 1.1]. We are now ready to give our proof of the Cuntz-Krieger uniqueness theorem for C^*-algebras of arbitrary graphs. This result was originally proved in [13, Theorem 2] using the machinery of [10].

Theorem 1.5. *Suppose that E is a directed graph in which every loop has an exit, and that $\{S_e, P_v\}$, $\{T_e, Q_v\}$ are two Cuntz-Krieger E-families in which all the projections P_v and Q_v are non-zero. Then there is an isomorphism ϕ of $C^*(S_e, P_v)$ onto $C^*(T_e, Q_v)$ such that $\phi(S_e) = T_e$ and $\phi(P_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$.*

Proof. We first claim that it suffices to prove the theorem for graphs without sinks or sources. Indeed, given this, the general case follows from Lemma 1.4, as in the first paragaph of the proof of [1, Theorem 3.1]. So assume that E has no sinks or sources. We shall prove the theorem by showing that the representations $\pi_{S,P}$ and $\pi_{T,Q}$ of $C^*(E)$ are faithful; then $\phi := \pi_{T,Q} \circ \pi_{S,P}^{-1}$ is the required isomorphism. By symmetry, it is enough to show that $\pi_{S,P}$ is faithful.

Write $E^1 = \bigcup_{n=1}^{\infty} F_n$ as the increasing union of finite subsets F_n, and let B_n be the C^*-subalgebra of $C^*(E)$ generated by $\{s_e : e \in F_n\}$. By Lemma 1.2 there are isomorphisms $\phi_n : C^*(E_{F_n}) \to B_n$ which respect the generators. Since all loops in F_n have exits by Lemma 1.3, [1, Theorem 3.1] implies that $\pi_{S,P} \circ \phi_n$ is an isomorphism, and hence is isometric. Thus $\pi_{S,P}$ is isometric on the dense $*$-subalgebra $\bigcup_n B_n$ of $C^*(E)$, and hence on all of $C^*(E)$; in particular, it is an isomorphism. □

2. The Cuntz-Krieger algebras of infinite matrices

Let I be a countable set and let $A = \big(A(i,j)\big)$ be an $I \times I$ matrix with entries in $\{0, 1\}$, in which no row is identically zero. The *Exel-Laca algebra* $\mathcal{O}_A$ is by definition the universal C^*-algebra generated by a family of partial isometries $\{s_i : i \in I\}$ satisfying the following relations:

(EL1) $s_i^* s_i$ and $s_j^* s_j$ commute for all $i, j \in I$;
(EL2) $s_i^* s_j = 0$ whenever $i \neq j$;
(EL3) $(s_i^* s_i) s_j = A(i,j) s_j$ for all $i, j \in I$; and
(EL4) for every pair X, Y of finite subsets of I such that

$$S(X,Y) := \{k \in I : A(i,k) = 1 \text{ for all } i \in X \text{ and } A(j,k) = 0 \text{ for all } j \in Y\}$$

has at most finitely many elements,

$$\Big(\prod_{i\in X} s_i^*s_i\Big)\Big(\prod_{j\in Y}(1-s_j^*s_j)\Big) = \sum_{k\in S(X,Y)} s_ks_k^*.$$

(It is easiest to think of the 1 in (EL4) as the identity of the multiplier algebra of $\mathcal{O}_A$.) As in [15] and [1, §1], the uniqueness of the universal object $\mathcal{O}_A$ implies the existence of a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut}\mathcal{O}_A$ such that $\gamma_z(s_i) = zs_i$ for $z \in \mathbb{T}$ and $i \in I$.

For finite matrices, the Exel-Laca algebras are the usual Cuntz-Krieger algebras, and hence are precisely the C^*-algebras of finite directed graphs without sinks. While we cannot always realise the Cuntz-Krieger algebra $\mathcal{O}_A$ of an infinite matrix as a graph algebra (see Remark 4.4), we can always construct a directed graph E_A from a $\{0,1\}$-matrix A by taking $E_A^0 := I$,

$$E_A^1 = \{(i,j) \in I\times I : A(i,j) = 1\},$$

and defining $s(i,j) = i$ and $r(i,j) = j$; this graph played an important role in Exel and Laca's analysis of $\mathcal{O}_A$ [10]. For finite A, the identification of $\mathcal{O}_A$ with $C^*(E_A)$ takes a generating family $\{s_i\}$ to the Cuntz-Krieger E_A-family $\{t_{(i,j)}, q_i\} := \{s_is_js_j^*, s_is_i^*\}$. Our approximation of $\mathcal{O}_A$ by graph algebras uses the same idea: we start with a finite subset F of the index set, and aim to view $\{s_is_js_j^*, s_is_i^* : i,j \in F\}$ as a Cuntz-Krieger family of a finite graph. In general s_i could be strictly larger than $\sum_{j\in F} s_is_js_j^*$; to recover s_i, we need to include other summands of the form s_ip. From our point of view, the sets $S(X,Y)$ of (EL4) arise because we have to include such a term whenever p is a minimal projection in the C^*-subalgebra of $\mathcal{O}_A$ generated by $\{s_i : i \in F\}$, and these minimal projections turn out to be

$$\Big\{\Big(\prod_{i\in X} s_i^*s_i\Big)\Big(\prod_{j\in F\setminus X}(1-s_j^*s_j)\Big)\Big(1-\sum_{k\in F} s_ks_k^*\Big) : \\ \emptyset \neq X \subset F \text{ satisfies } S(X, F\setminus X) \not\subset F\Big\}.$$

The extra vertices in our approximating graph are in one-to-one correspondence with these minimal projections. It is important to note that the extra vertices X are all sinks, and that if $A(i,j) = 1$ only for $j \in F$, then there are no edges of the form (i,X).

Definition 2.1. For each non-empty finite subset F of the index set I we define a finite directed graph $E(A,F)$ by

$$E(A,F)^0 = F \cup \{X : \emptyset \neq X \subset F \text{ satisfies } S(X, F\setminus X) \not\subset F\},$$
$$E(A,F)^1 = \{(i,j) \in F\times F : A(i,j) = 1\} \cup \{(i,X) : i \in X\}.$$

Proposition 2.2. *If $\{S_i\}$ is a family of partial isometries satisfying the relations* (EL1–4)*, then*

$$Q_i := S_iS_i^*, \quad Q_X := \Big(\prod_{i\in X} S_i^*S_i\Big)\Big(\prod_{j\in F\setminus X}(1-S_j^*S_j)\Big)\Big(1-\sum_{k\in F} S_kS_k^*\Big),$$

$T_{(i,j)} = S_iQ_j$ and $T_{(i,X)} = S_iQ_X$ form a Cuntz-Krieger $E(A,F)$-family which generates $C^*(S_i : i \in F)$. If every S_i is non-zero, then every projection Q_v is non-zero.

The proof of this proposition requires a simple lemma.

Lemma 2.3. *Suppose $P_1, \cdots, P_n$ are commuting projections on a Hilbert space $\mathcal{H}$. Then*

$$1 = \sum_{Y \subset \{1,\cdots,n\}} \Big(\prod_{i\in Y} P_i\Big)\Big(\prod_{i\notin Y}(1-P_i)\Big).$$

Proof. By induction on n: multiply the formula for $n = k$ by $P_{k+1} + (1 - P_{k+1})$. □

Proof of Proposition 2.2. The projections Q_i are mutually orthogonal by (EL2), and are orthogonal to Q_X because of the factor $1 - \sum_{k\in F} S_k S_k^*$. The other factors in the Q_X ensure that they are mutually orthogonal. Since $A(i,j) = 1$ implies $S_i^* S_i \geq Q_j$, we have $T_{(i,j)}^* T_{(i,j)} = Q_j = Q_{r(i,j)}$, and since $S_i^* S_i \geq Q_X$ whenever $i \in X$, we have $T_{(i,X)}^* T_{(i,X)} = Q_X = Q_{r(i,X)}$. If $A(i,j) = 1$ only for $j \in F$, then there are no edges of the form (i,X), and

$$\sum_{(i,j)} T_{(i,j)} T_{(i,j)}^* = \sum_{\{j : A(i,j)=1\}} S_i S_j S_j^* S_i^* = S_i(S_i^* S_i) S_i^* = S_i S_i^*$$

follows from the usual Cuntz-Krieger relation (which is (EL4) for the combination $X = \{i\}$ and $Y = \emptyset$). When there do exist edges of the form (i,X), we use Lemma 2.3 and (EL3) to compute

(2.1)
$$\begin{aligned}\sum_{\{X:\, i\in X\}} Q_X &= S_i^* S_i \Big(\sum_{Y\subset F\setminus\{i\}} \Big(\prod_{j\in Y} S_j^* S_j\Big)\Big(\prod_{j\in (F\setminus\{i\})\setminus Y} (1 - S_j^* S_j)\Big)\Big(1 - \sum_{k\in F} S_k S_k^*\Big)\Big)\\ &= S_i^* S_i \Big(1 - \sum_{k\in F} S_k S_k^*\Big)\\ &= S_i^* S_i \Big(1 - \sum_{\{k\in F: A(i,k)=1\}} S_k S_k^*\Big).\end{aligned}$$

Now we have

$$\sum_{(i,j)} T_{(i,j)} T_{(i,j)}^* + \sum_X T_{(i,X)} T_{(i,X)}^* = \sum_{\{j\in F: A(i,j)=1\}} S_i S_j S_j^* S_i^* + \sum_{\{X\subset F:\, i\in X\}} S_i Q_X S_i^*,$$

which equals $S_i S_i^*$ by (2.1). Thus $\{T_e, Q_v\}$ is a Cuntz-Krieger $E(A,F)$-family.

Equation (2.1) also implies that we can recover S_i as

$$S_i = \sum_{(i,j)} T_{(i,j)} + \sum_{\{X:\, i\in X\}} T_{(i,X)} = S_i\Big(\sum_{\{j\in F: A(i,j)=1\}} S_j S_j^* + \sum_{\{X:\, i\in X\}} Q_X\Big), \tag{2.2}$$

so the operators T_e and Q_v generate $C^*(S_i)$. For the last comment, note that $S(X, F\setminus X) \not\subset F$ implies $Q_X \geq S_k S_k^*$ for some $k \notin F$, and hence that $Q_X \neq 0$. □

Corollary 2.4. *Let A be an $I \times I$ matrix with entries in $\{0,1\}$ and no zero rows. Then for every non-empty finite subset F of I, the graph algebra $C^*(E(A,F))$ is naturally isomorphic to the C^*-subalgebra of $\mathcal{O}_A$ generated by $\{s_i : i \in F\}$.*

Proof. Applying the proposition to the canonical generating family $\{s_i\}$ of $\mathcal{O}_A$ gives a Cuntz-Krieger $E(A,F)$-family $\{T_e, Q_v\}$ which generates $C^*(s_i : i \in F)$, and in which each of the projections Q_v is non-zero. Since the gauge action on $\mathcal{O}_A$ leaves $C^*(s_i : i \in F)$ invariant, it follows from the gauge-invariant uniqueness theorem of [1, Theorem 2.1] that $\pi_{T,Q}$ is an isomorphism of $C^*(E(A,F))$ onto $C^*(s_i : i \in F)$. □

Remark 2.5. This corollary allows us to realise $\mathcal{O}_A$ as a direct limit of graph algebras, and hence to replace the axioms of Exel and Laca by a sequence of Cuntz-Krieger relations for finite graphs. In the proof of Proposition 2.2, however, we made no obvious use of the relations (EL4) except in the special case $X = \{i\}$ and $Y = \emptyset$. So it is reassuring to observe that we can recover the full strength of (EL4) from the graph relations.

To see this, suppose Z and Y are finite subsets of I such that $S(Z, Y)$ is finite, and choose a finite subset F of I which contains Z, Y and $S(Z, Y)$. Let $\{T_e, Q_v\}$ be a Cuntz-Krieger $E(A, F)$-family; we want to show that the partial isometries S_i defined by (2.2) satisfy (EL4) for the pair Z, Y. Since

$$S_i^* S_i = \sum_{k\in F} Q_k + \sum_{\{X\subset F:\, i\in X,\ S(X,\ F\setminus X)\not\subset F\}} Q_X,$$

and since $S(Z, Y) \subset F$, we have

(2.3)

$$\Big(\prod_{i\in Z} S_i^* S_i\Big)\Big(\prod_{j\in F\setminus Z}(1 - S_j^* S_j)\Big) = \sum_{k\in S(Z,Y)} Q_k + \sum_{\{X:\, S(X,F\setminus X)\not\subset F,\ Z\subset X,\ Y\subset F\setminus X\}} Q_X.$$

But $Z \subset X$ and $Y \subset F \setminus X$ imply that $S(X, F \setminus X) \subset S(Z, Y)$; thus there are no subsets X of F which satisfy all the requirements of the second sum in (2.3), and (2.3) gives the required version of (EL4).

Example 2.6. Let $\Gamma = \langle g_1\rangle * \langle g_2\rangle * \dots$ be a countably infinite free product of cyclic groups with generators g_i. The group Γ has a boundary $\partial\Gamma$, which is a compact Hausdorff space on which Γ acts naturally [31]. The crossed product C^*-algebras $C(\partial\Gamma) \times \Gamma$ were investigated in [31], [32], and it was suggested in [31] that they could be viewed as the Cuntz-Krieger algebras of certain infinite $\{0, 1\}$-matrices. The work of Exel and Laca [10] has provided the necessary machinery to formalise that intuitive statement: $C(\partial\Gamma) \times \Gamma$ is $\mathcal{O}_A$, where A has all entries 1 except for square blocks R_i along the diagonal, which are 2×2 identity matrices when g_i has infinite order, and $(m_i - 1) \times (m_i - 1)$ zero matrices when g_i has finite order m_i. The approximations from [31, Propositions 3.2 and 4.4] and [31, Remark 4.7] served as a prototype for our construction.

The following is an analogue of the gauge-invariant uniqueness theorem of [15, Theorem 2.3] and [1, Theorem 2.1].

Theorem 2.7. *Let A be an $I \times I$ matrix with entries in $\{0, 1\}$ and no zero rows, let $\{T_i : i \in I\}$ be a family of operators on a Hilbert space $\mathcal{H}$ satisfying* (EL1)–(EL4), *and let π be the representation of $\mathcal{O}_A$ such that $\pi(s_i) = T_i$. Suppose that each T_i is non-zero and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(T_i)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for $z \in \mathbb{T}$. Then π is faithful.*

Proof. Let F be a finite subset of I. Then $C^*(s_i : i \in F)$ is isomorphic to the graph algebra $C^*(E(A, F))$ by Corollary 2.4, and this isomorphism is equivariant for the gauge actions. The projections in $B(\mathcal{H})$ corresponding to vertices of $E(A, F)$ are all non-zero: Q_i because T_i is, and Q_X because (EL4) implies the existence of j such that $T_j T_j^* \le Q_X$. Thus applying [1, Theorem 2.1] to the corresponding representation of the graph algebra $C^*(E(A, F))$ shows that π is faithful on $C^*(s_i : i \in F)$, and hence is isometric there. Thus π is isometric on the dense subalgebra of $\mathcal{O}_A$ generated by $\{s_i\}$, and hence on all of $\mathcal{O}_A$. $\square$

Theorem 2.8 (Exel and Laca [10, Theorem 13.1]). *Suppose that A is an $I \times I$ $\{0,1\}$-matrix in which no row is identically zero, and that all loops in the associated graph E_A have exits. If $\{S_i : i \in I\}$ and $\{T_i : i \in I\}$ are two families of non-zero partial isometries satisfying the relations* (EL1)–(EL4)*, then there is an isomorphism ϕ of $C^*(S_i)$ onto $C^*(T_i)$ such that $\phi(S_i) = T_i$ for all i.*

Proof. For each finite subset F of I, we consider the graph $E(A,F)$. By applying Proposition 2.2 to the families $\{S_i\}$ and $\{T_i\}$, we obtain two Cuntz-Krieger $E(A,F)$-families in which the projections are all non-zero. Since the extra vertices X in $E(A,F)$ are all sinks, every loop in $E(A,F)$ comes from a loop in E_A, and hence has an exit. Thus we can apply Theorem 3.1 of [1] to these Cuntz-Krieger families, and obtain an isomorphism ϕ_F of $C^*(S_i : i \in F)$ onto $C^*(T_i : i \in F)$ such that $\phi_F(S_i) = T_i$ for $i \in F$. These combine to give a $*$-algebra isomorphism ϕ of $\bigcup_F C^*(S_i : i \in F)$ onto $\bigcup_F C^*(T_i : i \in F)$ such that $\phi(S_i) = T_i$ for all i; this isomorphism is isometric because each ϕ_F is, and hence extends to the completion $C^*(S_i : i \in I)$ of $\bigcup_F C^*(S_i : i \in F)$. □

3. K-theory for graphs with sinks

Every graph algebra and Cuntz-Krieger algebra carries a canonical gauge action γ of $\mathbb{T}$. As in [25], we compute K-theory using the dual Pimsner-Voiculescu sequence for γ. In general, if $\alpha : \mathbb{T} \to \operatorname{Aut} A$ is an action of $\mathbb{T}$ on a C^*-algebra A, then the dual Pimsner-Voiculescu sequence is a six-term exact sequence

$$\begin{tikzcd} K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r, "1-\hat{\alpha}_*^{-1}"] & K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r] & K_0(A) \arrow[d] \\ K_1(A) \arrow[u] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l, "1-\hat{\alpha}_*^{-1}"'] \end{tikzcd} \tag{3.1}$$

in which $\hat{\alpha}$ is the generator of the dual action of $\mathbb{Z}$. That there is such a sequence is proved, for example, in [3, Section 10.6]. We shall need to know that this sequence is functorial in the sense that an equivariant homomorphism $\phi : (A, \mathbb{T}, \alpha) \to (B, \mathbb{T}, \beta)$ induces maps $K_i(A) \to K_i(B)$ and $K_i(A \rtimes_\alpha \mathbb{T}) \to K_i(B \rtimes_\beta \mathbb{T})$ which make a commutative cube with the dual Pimsner-Voiculescu sequences of (A,α) and (B,β) on opposite faces. This functoriality is not made explicit in the original papers. However, Connes' treatment of the Thom isomorphism [6] emphasises naturality of the isomorphism, so functoriality of the original Pimsner-Voiculescu sequence follows from the naturality of the various isomorphisms used to deduce it from Connes' theorem (see [6, Section V]). Since the Takesaki-Takai duality isomorphism is also natural, we can deduce the naturality of the dual Pimsner-Voiculescu sequence.

It was pointed out in [19] that the C^*-algebra of a skew-product $E \times_c G$ is a crossed product $C^*(E) \rtimes \hat{G}$ by an action of the dual group $\hat{G}$. We need a converse: we want to realise the crossed product by the gauge action γ as the C^*-algebra of the skew-product $E \times_1 \mathbb{Z}$, in which

$$(E \times_1 \mathbb{Z})^0 = E^0 \times \mathbb{Z}, \quad (E \times_1 \mathbb{Z})^1 = E^1 \times \mathbb{Z},$$

$s(e,n) = (s(e), n-1)$ and $r(e,n) = (r(e), n)$. This skew product carries a canonical action of $\mathbb{Z}$ by translation, which in turn induces an action $\beta : \mathbb{Z} \to \operatorname{Aut} C^*(E \times_1 \mathbb{Z})$ characterised by

$$\beta_m(p_{(v,n)}) = p_{(v,n+m)} \quad \text{and} \quad \beta_m(s_{(e,n)}) = s_{(e,n+m)}. \tag{3.2}$$

To establish the identification of $C^*(E) \rtimes_\gamma \mathbb{T}$ with $C^*(E \times_1 \mathbb{Z})$, we have to find a Cuntz-Krieger $(E \times_1 \mathbb{Z})$-family inside $C^*(E) \rtimes_\gamma \mathbb{T}$. To do this, it is helpful to note that applying the integrated form of the canonical embedding $u : \mathbb{T} \to M(C^*(E) \rtimes_\gamma \mathbb{T})$ to the functions $z \mapsto z^n$ in $L^1(\mathbb{T})$ gives a family $\{\chi_n\}$ of mutually orthogonal projections in $C^*(E) \rtimes_\gamma \mathbb{T}$.

Lemma 3.1. *Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger family in $C^*(E)$. Then $t_{(e,n)} := s_e \chi_n$, $q_{(v,n)} := p_v \chi_n$ form a Cuntz-Krieger $(E \times_1 \mathbb{Z})$-family, and the canonical homomorphism $\pi_{t,q} : C^*(E \times_1 \mathbb{Z}) \to C^*(E) \rtimes_\gamma \mathbb{T}$ is an isomorphism which carries the action β of $\mathbb{Z}$ by translation on $C^*(E \times_1 \mathbb{Z})$ into the dual action $\hat{\gamma}$.*

Proof. The formula $\chi_n = \int z^n u_z \, dz$ and the defining relations $u_z s_e = z s_e u_z$, $u_z p_v = p_v u_z$ imply that $\chi_n s_e = s_e \chi_{n+1}$ and $\chi_n p_v = p_v \chi_n$, and an easy calculation using these relations shows that $\{t_{(e,n)}, q_{(v,n)}\}$ is a Cuntz-Krieger $(E \times_1 \mathbb{Z})$-family. Since the graph $E \times_1 \mathbb{Z}$ has no loops, Theorem 1.5 implies that $\pi_{t,q}$ is injective. An application of the Stone-Weierstrass Theorem shows that the functions $z \mapsto z^n$ span a $\|\cdot\|_1$-dense $*$-subalgebra of $C(\mathbb{T})$, and it follows that the elements $\{s_\mu s_\nu^* \chi_n : \mu, \nu \in E^*, n \in \mathbb{Z}\}$ span a dense subspace of $C^*(E) \rtimes_\gamma \mathbb{T}$. Hence $\pi_{t,q}$ is surjective. The defining relation $\hat{\gamma}_1(u_z) = z u_z$ implies that $\hat{\gamma}_1(\chi_n) = \chi_{n+1}$; since $\hat{\gamma}$ fixes $\{s_e, p_v\}$, the last assertion follows easily. □

Since the skew-product has no loops, its C^*-algebra $C^*(E \times_1 \mathbb{Z})$ is AF by [20, Theorem 2.4] (see §5.4 below). Thus $K_1(C^*(E) \rtimes_\gamma \mathbb{T}) = 0$, and the six-term exact sequence (3.1) collapses to

(3.3)

$$0 \longrightarrow K_1(C^*(E)) \longrightarrow K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \xrightarrow{1-\hat{\gamma}_*^{-1}} K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \longrightarrow K_0(C^*(E)) \longrightarrow 0.$$

We can now formulate the main result of this section.

Theorem 3.2. *Let E be a row-finite graph, let W be the set of sinks in E, and let $V = E^0 \setminus W$. The $E^0 \times E^0$ vertex matrix*

$$M(v, w) := \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}$$

has block form

$$M = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $E^0 = V \cup W$. We define $K : \mathbb{Z}^V \to \mathbb{Z}^V \oplus \mathbb{Z}^W$ by $K(x) = \big((1 - B^t)x, -C^t x\big)$, and $\phi : \mathbb{Z}^V \oplus \mathbb{Z}^W \to K_0(C^*(E) \rtimes_\gamma \mathbb{T})$ in terms of the usual basis by $\phi(e_v) = [p_v \chi_1]$. Then ϕ restricts to an isomorphism $\phi|$ of $\ker K$ onto $K_1(C^*(E))$, and induces an isomorphism $\overline{\phi}$ of $\operatorname{coker} K$ onto $K_0(C^*(E))$ such that the following diagram commutes:

(3.4)

$$\begin{tikzcd}
\ker K \arrow[r] \arrow[d, "\phi|"] & \mathbb{Z}^V \arrow[r, "K"] \arrow[d, "\phi"] & \mathbb{Z}^V \oplus \mathbb{Z}^W \arrow[r] \arrow[d, "\phi"] & \operatorname{coker} K \arrow[d, "\overline{\phi}"] \\
K_1(C^*(E)) \arrow[r] & K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \arrow[r, "1-\hat{\gamma}_*^{-1}"] & K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \arrow[r] & K_0(C^*(E)).
\end{tikzcd}$$

The proof of this theorem will occupy most of this section. We begin by noting that, because $(C^*(E) \rtimes_\gamma \mathbb{T}, \mathbb{Z}, \hat{\gamma}) \cong (C^*(E \times_1 \mathbb{Z}), \mathbb{Z}, \beta)$, it is enough to compute the kernel and cokernel of

$$1 - \beta_*^{-1} : K_0(C^*(E \times_1 \mathbb{Z})) \to K_0(C^*(E \times_1 \mathbb{Z})).$$

For integers $m \le n$ we denote by $E \times_1 [m,n]$ the subgraph of $E \times_1 \mathbb{Z}$ with vertices $\{(v,k) : m \le k \le n,\ v \in E^0\}$ and edges $\{(e,k) : m < k \le n,\ e \in E^1\}$. We allow $m = -\infty$, with the obvious modification of the definition. Since any path in $E \times_1 [m,n]$ has length at most $n-m$, we can use the arguments in the proofs of Proposition 2.1, Corollary 2.2 and Corollary 2.3 of [20] to deduce that $C^*(E \times_1 [m,n])$ is a direct sum of copies of the compact operators (on spaces of varying dimensions), indexed by the set of sinks in $E \times_1 [m,n]$, and that each direct summand contains precisely one projection $p_{(v,k)}$ associated to a sink as a minimal projection. Thus $K_0(C^*(E \times_1 [m,n]))$ is the free abelian group with generators

$$\{[p_{(v,n)}] : v \in V\} \cup \{[p_{(v,k)}] : v \in W,\ m \le k \le n\}.$$

By continuity of K-theory we can let $m \to -\infty$ and deduce that

$$\begin{aligned} K_0(C^*(E \times_1 (-\infty,n])) &= \Big(\bigoplus_{v \in V} \mathbb{Z}[p_{(v,n)}]\Big) \bigoplus \Big(\bigoplus_{k=0}^{\infty} \bigoplus_{v \in W} \mathbb{Z}[p_{(v,n-k)}]\Big) \\ &\cong \mathbb{Z}^V \oplus \mathbb{Z}^{W_n} \oplus \mathbb{Z}^{W_{n-1}} \oplus \dots, \end{aligned}$$

where each copy W_j of W is labelled to indicate its place in the direct sum.

Next we need to see how the inclusions $\imath_n$ and $\imath^n$ of $C^*(E \times_1 (-\infty,n])$ in $C^*(E \times_1 (-\infty,n+1])$ and $C^*(E \times_1 \mathbb{Z})$ behave at the level of K_0. If $v \in V$, then in $K_0(C^*(E \times_1 (-\infty,n+1]))$ we have

$$\begin{aligned} [p_{(v,n)}] &= \sum_{e \in E^1 :\, s(e)=v} [s_{(e,n+1)} s_{(e,n+1)}^*] = \sum_{e \in E^1 :\, s(e)=v} [s_{(e,n+1)}^* s_{(e,n+1)}] \\ &= \sum_{e \in E^1 :\, s(e)=v} [p_{(r(e),n+1)}] = \sum_{w \in E^0} M(v,w)[p_{(w,n+1)}]. \end{aligned}$$

For $k \le n$ and $v \in W$, $[p_{(v,k)}]$ is still a generator in $K_0(C^*(E \times_1 (-\infty,n+1]))$. Thus the induced map from $\mathbb{Z}^V \oplus \mathbb{Z}^{W_n} \oplus \mathbb{Z}^{W_{n-1}} \oplus \cdots$ to $\mathbb{Z}^V \oplus \mathbb{Z}^{W_{n+1}} \oplus \mathbb{Z}^{W_n} \oplus \cdots$ is given by the matrix

$$D = \begin{pmatrix} B^t & 0 & 0 & 0 & \cdot \\ C^t & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ddots \end{pmatrix},$$

and $K_0(C^*(E \times_1 \mathbb{Z}))$ is the direct limit of the system

$$\mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \mathbb{Z}^W \oplus \cdots \xrightarrow{D} \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \mathbb{Z}^W \oplus \cdots \xrightarrow{D} \cdots.$$

From the formulas (3.2) characterising $\beta = \beta_1$ we can deduce that

$$\beta^{-1} : C^*(E \times_1 (-\infty,n]) \to C^*(E \times_1 (-\infty,n]),$$

and that the restriction of β_*^{-1} to $K_0(C^*(E \times_1 (-\infty, n]))$, viewed as a map on $\mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots$, is just multiplication by D. Since the diagram

$$\begin{tikzcd}
C^*(E \times_1 (-\infty, n]) \arrow[r, "\iota_n"] \arrow[d, "1-\beta^{-1}"'] & C^*(E \times_1 (-\infty, n+1]) \arrow[r, "\iota^{n+1}"] \arrow[d, "1-\beta^{-1}"'] & C^*(E \times_1 \mathbb{Z}) \arrow[d, "1-\beta^{-1}"'] \\
C^*(E \times_1 (-\infty, n]) \arrow[r, "\iota_n"] & C^*(E \times_1 (-\infty, n+1]) \arrow[r, "\iota^{n+1}"] & C^*(e \times_1 \mathbb{Z})
\end{tikzcd}$$

commutes, we have the following commutative diagram:
(3.5)

$$\begin{tikzcd}
\mathbb{Z}^V \otimes \mathbb{Z}^W \oplus \mathbb{Z}^W \oplus \cdots \arrow[r, "D"] \arrow[d, "1-D"'] & \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \mathbb{Z}^W \oplus \cdots \arrow[r, "\iota_*^{n+1}"] \arrow[d, "1-D"'] & K_0(C^*(E \times_1 \mathbb{Z})) \arrow[d, "1-\beta_*^{-1}"'] \\
\mathbb{Z}^V \oplus \mathbb{Z}^W \oplus /F^W \oplus \cdots \arrow[r, "D"] & Z^V \oplus \mathbb{Z}^W \oplus \mathbb{Z}^W \oplus \cdots \arrow[r, "t_*^{n+1}"] & K_0(C^*(E \times_1 \mathbb{Z}))
\end{tikzcd}$$

We can therefore realise $K_0(C^*(E \times_1 \mathbb{Z}))$ as the group of equivalence classes $[(x_i)]$ of sequences in $\prod_{i=1}^{\infty}(\mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots)$ which eventually satisfy $x_{i+1} = Dx_i$, where two sequences are equivalent if they eventually coincide. The natural map $\imath_*^n$ takes $x \in \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots$ to the class of the sequence (x_i), where

$$x_i = \begin{cases} 0 & \text{if } i < n, \text{ and} \\ D^{i-n}x & \text{if } i \geq n. \end{cases}$$

Lemma 3.3. *The homomorphism $\imath_*^1$ restricts to an isomorphism of* $\ker(1-D)$ *onto* $\ker(1-\beta_*^{-1})$*, and induces an isomorphism $\bar{\imath}_*^1$ of* $\operatorname{coker}(1-D)$ *onto* $\operatorname{coker}(1-\beta_*^{-1})$.

Proof. Since $x \in \ker(1-D)$ if and only if $x = Dx$, $\imath_*^1$ maps each $x \in \ker(1-D)$ to the class of the constant sequence (x). In particular, $\imath_*^1$ is injective on $\ker(1-D)$, and maps it into $\ker(1-\beta_*^{-1})$. If $z = \imath_*^n(y) \in \ker(1-\beta_*^{-1})$, then we can assume by increasing n that $y = Dy$. But then $z = \imath_*^1(y)$, and we have proved the first claim.

The commutativity of (3.5) implies that $\imath_*^1$ maps the image of $1-D$ into the image of $1-\beta_*^{-1}$, so $\imath_*^1$ induces a map $\bar{\imath}_*^1$ on cokernels. To see that $\bar{\imath}_*^1$ is injective, suppose $(z_i) = \imath_*^1(x) \in \operatorname{im}(1-\beta_*^{-1})$, say $[(z_i)] = [(y_i - Dy_i)]$ for some $(y_i) \in K_0(C^*(E \times_1 \mathbb{Z}))$. Then for large k we have $D^{k-1}x = z_k = y_k - Dy_k$. But then

$$x = x - D^{k-1}x + D^{k-1}x = (1-D)(1 + D + D^2 + \cdots D^{k-2})x + (1-D)y_k$$

belongs to $\operatorname{im}(1-D)$. To show that $\bar{\imath}_*^1$ is surjective, let $\imath_*^n(y) \in K_0(C^*(E \times_1 \mathbb{Z}))$. By commutativity of (3.5), we have

$$\imath_*^n(y) - \imath_*^n(Dy) = \imath_*^n((1-D)y) = (1-\beta_*^{-1})(\imath_*^n(y)),$$

so $\imath_*^n(y)$ and $\imath_*^n(Dy)$ define the same class in $\operatorname{coker}(1-\beta_*^{-1})$. But this implies that $\imath_*^1(y) = \imath_*^n(D^{n-1}y)$ defines the same class as $\imath_*^n(y)$, and hence that $\bar{\imath}_*^1$ is surjective. □

Lemma 3.4. *Let i and j be the inclusions of $\mathbb{Z}^V$ and $\mathbb{Z}^V \oplus \mathbb{Z}^W$ as the first coordinates of $\mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots$. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{Z}^V \arrow[r, "K"] \arrow[d, "i"'] & \mathbb{Z}^V \oplus \mathbb{Z}^W \arrow[d, "j"] \\
\mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots \arrow[r, "1-D"] & \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots,
\end{tikzcd}$$

i is an isomorphism of $\ker K$ *onto* $\ker(1-D)$*, and j induces an isomorphism $\overline{j}$ of* $\operatorname{coker} K$ *onto* $\operatorname{coker}(1-D)$.

Proof. It is straightforward to check that the diagram commutes. In particular, i maps $\ker K$ into $\ker(1-D)$, and it is trivially injective. To see that i maps $\ker K$ onto $\ker(1-D)$, let $(n, m_1, m_2, \ldots) \in \ker(1-D)$. Then

$$(1-B^t)n = 0, \tag{3.6}$$

$$-C^t n + m_1 = 0, \tag{3.7}$$

$$-m_k + m_{k+1} = 0 \quad \text{for } k \geq 1,$$

and we have $m_k = m_1$ for all k. Since $(n, m_1, m_2, \ldots)$ belongs to the direct sum, m_k is eventually 0, and hence $m_k = 0$ for all k. Thus (3.6) and (3.7) imply that $n \in \ker K$ and $(n, m_1, m_2, \ldots) = i(n)$.

The commutativity of (3.5) implies that j induces a well-defined map $\overline{j} : \operatorname{coker} K \to \operatorname{coker}(1-D)$. To see that $\overline{j}$ is injective, suppose that $j(n,m) = (n, m, 0, \ldots) \in \operatorname{im}(1-D)$. Then there exists $(n', m_1', \ldots) \in \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots$ such that

$$\begin{aligned}
(1-B^t)n' &= n,\\
-C^t n' + m_1' &= m,\\
-m_k' + m_{k+1}' &= 0 \quad \text{for } k \geq 1.
\end{aligned}$$

Again, because we are working in a direct sum, we must have $m_k' = 0$ for all $k \geq 1$. Thus $(n,m) = K(n')$, and (n,m) defines the zero class in $\operatorname{coker} K$.

To show that $\overline{j}$ is surjective, let $(n, m_1, m_2, \ldots) \in \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots$. We need to find (n', m') and $(n'', m_1'', \ldots)$ such that

$$\begin{pmatrix} n \\ m_1 \\ m_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} n' \\ m' \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} (1-B^t)n'' \\ -C^t n'' + m_1'' \\ -m_1'' + m_2'' \\ \vdots \end{pmatrix}.$$

But we know that $m_k = 0$ for large k, say $k > K$; then we can take $n'' = 0$,

$$m_k'' = \begin{cases} -\sum_{j=k+1}^{K} m_j & \text{for } k < K, \\ 0 & \text{for } k \geq K, \end{cases}$$

$m' = m_1 - m_1''$, and $n' = n$. $\square$

Proof of Theorem 3.2. Consider the following diagram:
(3.8)

$$\begin{tikzcd}
\ker(1-D) \arrow[r] \arrow[d, "\iota_*^1|"'] & \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots \arrow[r, "1-D"] \arrow[d, "\iota_*^1"'] & \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \cdots \arrow[r] \arrow[d, "\iota_*^1"'] & \operatorname{coker} K \arrow[d, "\overline{\iota}_*^1"'] \\
K_1(C^*(E)) \arrow[r] & K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \arrow[r, "1-\hat{\gamma}_*^{-1}"] & K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \arrow[r] & K_0(C^*(E))
\end{tikzcd}$$

This diagram commutes by naturality of K-theory, and the first and fourth vertical arrows are isomorphisms by Lemma 3.3. Now Lemma 3.4 says we can replace the top row by

$$\ker K \to \mathbb{Z}^V \xrightarrow{K} \mathbb{Z}^V \oplus \mathbb{Z}^W \to \operatorname{coker} K,$$

and the result follows. $\square$

4. The K-theory of Exel-Laca algebras

We shall now use Theorem 3.2 to compute the K-theory of an Exel-Laca algebra $\mathcal{O}_A = C^*(s_i)$. For each non-empty subset F of the index set I, let E_F denote the graph $E(A, F)$ of Definition 2.1, so that $C^*(E_F)$ is naturally isomorphic to the C^*-subalgebra $C^*(s_i : i \in F)$ of $\mathcal{O}_A$ by Proposition 2.2. Combining these isomorphisms gives a realisation of $\mathcal{O}_A$ as the direct limit $\varinjlim C^*(E_F)$ over the finite subsets of I directed by inclusion. Since K-theory is continuous, and we have just computed $K_*(C^*(E))$ for a class of graphs which includes every E_F, we have in principle computed

$$K_*(\mathcal{O}_A) = \varinjlim K_*(C^*(E_F)). \tag{4.1}$$

To make this useful, we have to be able to compute the direct limit on the right; we shall explain how the fine print in Theorem 3.2 makes this possible, and illustrate this with some interesting examples.

For each non-empty finite subset F of I, let

$$W_F := \{X \subset F : X \neq \emptyset \text{ and } S(X, F \setminus X) \not\subset F\}$$

be the set of sinks in E_F, and denote by A_F the $F \times (F \cup W_F)$ vertex matrix of E_F. Then Theorem 3.2 gives a commutative diagram

(4.2)

$$\begin{tikzcd} \ker(1 - A_F^t) \arrow[r] \arrow[d, "\phi|"'] & \mathbb{Z}^F \arrow[r, "1-A_F^t"] \arrow[d, "\phi"'] & \mathbb{Z}^F \oplus \mathbb{Z}^{W_F} \arrow[r] \arrow[d, "\phi"'] & \operatorname{coker}(1 - A_F^t) \arrow[d, "\overline{\phi}"'] \\ K_1(C^*(E_F)) \arrow[r] & K_0(C^*(E_F) \rtimes \mathbb{T}) \arrow[r, "1-\hat{\gamma}_*^{-1}"] & K_0(C^*(E_F) \rtimes \mathbb{T}) \arrow[r] & K_0(C^*(E_F)) \end{tikzcd}$$

in which $\phi|$ and $\overline{\phi}$ are isomorphisms, and ϕ sends a basis element e_i or e_X to the class of the corresponding projection $q_i\chi_1$ or $q_X\chi_1$ in $C^*(E_F) \rtimes_\gamma \mathbb{T}$. To compute the direct limit in (4.1), we need to consider a subset G of I such that $F \subset G$. The naturality of the dual Pimsner-Voiculescu sequence gives a commutative diagram

(4.3)

$$\begin{tikzcd} K_1(C^*(E_F)) \arrow[r] \arrow[d] & K_0(C^*(E_F) \rtimes \mathbb{T}) \arrow[r] \arrow[d] & K_0(C^*(E_F) \rtimes \mathbb{T}) \arrow[r] \arrow[d] & K_0(C^*(E_F)) \arrow[d] \\ K_1(C^*(E_G)) \arrow[r] & K_0(C^*(E_G) \rtimes \mathbb{T}) \arrow[r] & K_0(C^*(E_G) \rtimes \mathbb{T}) \arrow[r] & K_0(C^*(E_G)) \end{tikzcd}$$

in which the vertical arrows are induced by the (equivariant) inclusion of $C^*(s_i : i \in F) = C^*(E_F)$ in $C^*(s_i : i \in G) = C^*(E_G)$. The main theorem in the form of (4.2) says that we can replace the middle box by

$$\begin{tikzcd} \mathbb{Z}^F \arrow[r, "1-A_F^t"] \arrow[d, "\psi_{F,G}"'] & \mathbb{Z}^F \oplus \mathbb{Z}^{W_F} \arrow[d, "\phi_{F,G}"] \\ \mathbb{Z}^G \arrow[r, "1-A_G^t"] & \mathbb{Z}^G \oplus \mathbb{Z}^{W_G}, \end{tikzcd} \tag{4.4}$$

provided this diagram commutes, provided

$$\begin{array}{ccc} \mathbb{Z}^F \oplus Z^{W_F} & \xrightarrow{\phi_{F,G}} & \mathbb{Z}^G \oplus \mathbb{Z}^{W_G} \\ \phi \big\downarrow & & \big\downarrow \phi \\ K_0(C^*(E_F) \rtimes_\gamma \mathbb{T}) & \longrightarrow & K_0(C^*(E_G) \rtimes_\gamma \mathbb{T}) \end{array} \tag{4.5}$$

commutes, and provided an analogous diagram for $\psi_{F,G}$ commutes. The projection $q_i\chi_1$ associated to a vertex $i \in F \subset E_F^0$ lies in the subalgebra $C^*(E_F) \rtimes_\gamma \mathbb{T}$ of $C^*(E_G) \rtimes_\gamma \mathbb{T}$, so $\psi_{F,G}$ and $\phi_{F,G}$ both inject $\mathbb{Z}^F$ as a summand of $\mathbb{Z}^G$. To compute $\phi_{F,G}$ on $\mathbb{Z}^{W_F}$, we need to see how the projection q_X associated to a sink X in E_F decomposes in $C^*(E_G)$.

Recall that a sink X in E_F is a non-empty subset of F such that the projection

$$q_X := \Big(\prod_{i\in X} s_i^* s_i\Big)\Big(\prod_{j\in F\setminus X}(1 - s_j^* s_j)\Big)\Big(1 - \sum_{k\in F} s_k s_k^*\Big)$$

is non-zero. Any indices ℓ in G such that $A(i,\ell) = 1$ for $i \in X$ and $A(i,\ell) = 0$ for $i \in F \setminus X$ (in other words, such that $\ell \in S(X, F \setminus X)$) satisfy $q_X \geq s_\ell s_\ell^*$. After removing all such ℓ, the remainder of the set $S(X, F \setminus X)$ may split into several sets $S(Y, G \setminus Y)$ for subsets Y of G with $Y \cap F = X$. In $C^*(s_i : i \in G) = C^*(E_G)$, the projection q_X decomposes as

$$q_X = \sum_{\ell\in S(X,F\setminus X)\cap(G\setminus F)} q_\ell + \sum_{\{Y\subset G\,:\,Y\cap F=X,\ S(Y,G\setminus Y)\not\subset G\}} q_Y.$$

Thus if we write e_i^F, e_X^F for the usual basis elements of $\mathbb{Z}^F \oplus \mathbb{Z}^{W_F}$, the necessary map $\phi_{F,G} : \mathbb{Z}^F \oplus \mathbb{Z}^{W_F} \to \mathbb{Z}^G \oplus \mathbb{Z}^{W_G}$ is defined by

$$\begin{aligned} &\phi_{F,G}(e_i^F) = e_i^G \quad \text{for } i \in F, \text{ and} \\ &\phi_{F,G}(e_X^F) = \sum_{\ell\in S(X,F\setminus X)\cap(G\setminus F)} e_\ell^G + \sum_{\{Y\subset G\,:\,Y\cap F=X,\ S(Y,G\setminus Y)\not\subset G\}} e_Y^G. \end{aligned} \tag{4.6}$$

The description of the inclusion maps ϕ shows that the diagram (4.5) commutes. By recalling that $A_F(i, X) = 1$ precisely when $i \in X$, and chasing a generator e_i for $\mathbb{Z}^F$ through the diagram, we can verify that (4.4) commutes. Thus $\psi_{F,G}$ restricts to a homomorphism

$$\psi_{F,G} : \ker(1 - A_F^t) \cong K_1(C^*(E_F)) \to \ker(1 - A_G^t) \cong K_1(C^*(E_G)),$$

and $\phi_{F,G}$ induces a homomorphism

$$\overline{\phi}_{F,G} : \operatorname{coker}(1 - A_F^t) \cong K_0(C^*(E_F)) \to \operatorname{coker}(1 - A_G^t) \cong K_0(C^*(E_G));$$

the commutativity of (4.5) and its analogue for $\psi_{F,G}$ shows that these homomorphisms agree with the maps induced by the inclusion of $C^*(S_i : i \in F) = C^*(E_F)$ into $C^*(S_i : i \in G) = C^*(E_G)$.

We sum up our calculations:

Theorem 4.1. *Suppose A is an $I \times I$ matrix with entries in $\{0, 1\}$ and with no zero rows. For each finite subset F of I, define A_F and W_F as above. If G is a finite subset of I containing F, define $\phi_{F,G} : \mathbb{Z}^F \oplus \mathbb{Z}^{W_F} \to \mathbb{Z}^G \oplus \mathbb{Z}^{W_G}$ using* (4.6), *and define $\psi_{F,G} : \mathbb{Z}^F \to \mathbb{Z}^G$ by $\psi_{F,G}(e_i^F) = e_i^G$. Then we have*

$$K_1(\mathcal{O}_A) \cong \varinjlim\big(\ker(1 - A_F^t), \psi_{F,G}\big) \quad \textit{and} \quad K_0(\mathcal{O}_A) \cong \varinjlim\big(\operatorname{coker}(1 - A_F^t), \overline{\phi}_{F,G}\big).$$

Example 4.2. Define an $\mathbb{N}\times\mathbb{N}$ matrix A by

$$A(i,j)=\begin{cases}1 & \text{if } i=j, \text{ or } i=j+2, \text{ or } i\in\{1,2\} \text{ and } j\geq 3, \text{ and}\\ 0 & \text{otherwise;}\end{cases}$$

in other words,

$$A=\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & \cdot \\ 0 & 1 & 1 & 1 & 1 & \cdot \\ 1 & 0 & 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 1 & 0 & \cdot \\ 0 & 0 & 1 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ddots \end{pmatrix}.$$

We use the cofinal family of subsets $F_n := \{1,\dots,2n\}$ to compute the direct limits. For each n, the only subset X of F_n with $S(X, F_n\setminus X)\not\subset F_n$ is $X_n=\{1,2\}$, so E_{F_n} has exactly one sink X_n, and

$$A_{F_n}(i,X_n)=\begin{cases}1 & \text{if } i=1 \text{ or } 2, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

When we embed F_n in F_{n+1}, the map $\phi_{F_n,F_{n+1}}$ sends e_{X_n} to $e_{2n+1}+e_{2n+2}+e_{X_{n+1}}$, so

$$\phi_{F_n,F_{n+1}}(\mathbf{m},m_{X_n})=(\mathbf{m},m_{X_n},m_{X_n},m_{X_n}) \ \text{ for } (\mathbf{m},m_{X_n})\in\mathbb{Z}^{2n}\oplus\mathbb{Z}.$$

Since $(1-A_{F_n}^t)(\mathbf{m})$ is

$$-(m_3,m_4,m_5+m_1+m_2,\dots,m_{2n}+m_1+m_2,m_1+m_2,m_1+m_2,m_1+m_2),$$

the kernel of $(1-A_{F_n}^t)$ is spanned by $(1,-1,0,\dots)$ and

$$\operatorname{im}(1-A_{F_n}^t)=\{(\mathbf{k},k_{X_n}) : k_{2n-1}=k_{2n}=k_{X_n}\}.$$

The map $\psi_{F_n,F_{n+1}}$ is therefore an isomorphism of $\ker(1-A_{F_n}^t)\cong\mathbb{Z}$ onto $\ker(1-A_{F_{n+1}}^t)\cong\mathbb{Z}$, and $K_1(\mathcal{O}_A)\cong\mathbb{Z}$; on the other hand, the range of $\phi_{F_n,F_{n+1}}$ is contained in $\operatorname{im}(1-A_{F_{n+1}})$, so the induced map on cokernels is 0, and $K_0(\mathcal{O}_A)$ vanishes.

Remark 4.3. Since we consider only countable graphs and matrices, the algebras $C^*(E)$ and $\mathcal{O}_A$ are all separable. By the Takesaki-Takai duality theorem, every graph algebra $C^*(E)$ is stably isomorphic to a crossed product $(C^*(E)\rtimes_\gamma\mathbb{T})\rtimes_{\hat\gamma}\mathbb{Z}$ of an AF-algebra by $\mathbb{Z}$, and hence is nuclear (see [4, Corollary 3.2] and [5, Proposition 6.8]) and satisfies the Universal Coefficient Theorem (see [29, Theorem 1.17] and [3, Chapter 23]). The same holds for Exel-Laca algebras, since they are direct limits of graph algebras.

For the matrix A in Example 4.2, $\mathcal{O}_A$ is unital: indeed, we have $s_1^*s_1+s_2s_2^*=1$. Since the graph E_A is transitive, $\mathcal{O}_A$ is purely infinite by [10, Theorem 16.2] and simple by [10, Theorem 14.1]. We can therefore deduce from the classification program (see [17, Theorem 9] or [26, Theorem 4.2.4]) that $\mathcal{O}_A$ is the unique pi-sun algebra with $K_0=0$ and $K_1\cong\mathbb{Z}$, which is usually denoted P_∞. In other words, P_∞ can be realised as an Exel-Laca algebra.

Remark 4.4. The algebra $\mathcal{O}_A\cong P_\infty$ of Example 4.2 is not a graph algebra. To see this, suppose E is a graph such that $\mathcal{O}_A\cong C^*(E)$. Because $C^*(E)$ is spanned by elements of the form $s_\mu s_\nu^*$, the sums $p_G:=\sum_{v\in G}p_v$ as G runs through the finite subsets of E^0 form an approximate identity for $C^*(E)$. Since $\mathcal{O}_A$ has an identity, so does $C^*(E)$, and then $\|1 - p_G\| \to 0$; since $\|p - q\| = \sqrt{2}$ whenever p and q are distinct projections, we deduce that $1 = p_G$ for some G, and that E^0 is finite. Now Theorem 3.2 gives us an exact sequence

$$0 \to K_1(C^*(E)) \to \mathbb{Z}^V \to \mathbb{Z}^V \oplus \mathbb{Z}^W \to K_0(C^*(E)) \to 0$$

in which V and W are finite, and it follows that $\operatorname{rank} K_0(C^*(E)) \geq \operatorname{rank} K_1(C^*(E))$. But we just saw that $K_0(\mathcal{O}_A) = 0$ and $K_1(\mathcal{O}_A)$ has rank 1.

Example 4.5. (This is Example 5.3 of [11].) Let A be the chequerboard $\mathbb{N} \times \mathbb{N}$ matrix defined by $A(i,j) = i - j \mod 2$; thus

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ 0 & 1 & 0 & 1 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ddots \end{pmatrix}.$$

Each $F_n := \{1, 2, \ldots, 2n\}$ has two subsets X such that $S(X, F_n \setminus X) \not\subset F_n$, namely the subsets X_1^n of even numbers and X_2^n of odd numbers. So the graph E_{F_n} has two sinks, and the vertex matrix A_{F_n} is the truncation of A: for example

$$A_{F_2} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(we chose the ordering X_1^n, X_2^n to make A_{F_n} look nice). The map $\phi_{F_n,F_{n+1}} : \mathbb{Z}^{2n} \oplus \mathbb{Z}^2 \to \mathbb{Z}^{2(n+1)} \oplus \mathbb{Z}^2$ is given by

$$\phi_{F_n,F_{n+1}}(\mathbf{m}, m_{X_1}, m_{X_2}) = \big((\mathbf{m}, m_{X_1}, m_{X_2}), m_{X_1}, m_{X_2}\big).$$

On the other hand, $(1 - A_{F_n}^t)(\mathbf{m})$ is

$$\Big(m_1 - \sum_{i=1}^n m_{2i}, m_2 - \sum_{i=1}^n m_{2i-1}, m_3 - \sum_{i=1}^n m_{2i}, \ldots,$$
$$m_{2n} - \sum_{i=1}^n m_{2i-1}, -\sum_{i=1}^n m_{2i}, -\sum_{i=1}^n m_{2i-1}\Big),$$

so $\ker(1 - A_{F_n}^t) = \{0\}$ and

$$q_n : (\mathbf{k}, k_{X_1}, k_{X_2}) \mapsto \Big(\sum_{i=1}^n k_{2i-1} - nk_{X_1} + k_{X_2}, \sum_{i=1}^n k_{2i} - nk_{X_2} + k_{X_1}\Big)$$

induces an isomorphism of $\operatorname{coker}(1 - A_{F_n}^t)$ onto $\mathbb{Z}^2$. A calculation shows that we have $q_{n+1} \circ \phi_{F_n,F_{n+1}} = q_n$, and hence $\phi_{F_n,F_{n+1}}$ induces the identity map on $\mathbb{Z}^2$. Thus $K_1(\mathcal{O}_A) = 0$ and $K_0(\mathcal{O}_A) \cong \mathbb{Z}^2$.

To recover Exel and Laca's description of $K_*(\mathcal{O}_A)$, we need to relate our target spaces $\mathbb{Z}^F \oplus \mathbb{Z}^{W_F}$ to the target space R_A used in [11], which is the subring of $\ell^\infty(I)$ generated by the rows ρ_i of A and the point masses δ_i. From our point of view, the subsets X of F which parametrise the sinks in E_F are precisely the sets X for which

$$\rho_X := \Big(\prod_{i\in X}\rho_i\Big)\Big(\prod_{j\in F\setminus X}(1-\rho_j)\Big)\Big(1-\sum_{k\in F}\delta_k\Big)$$

is non-zero. Thus the map $e_i \mapsto \delta_i$, $e_X \mapsto \rho_X$ extends to a group isomorphism of $\mathbb{Z}^F \oplus \mathbb{Z}^{W_F}$ onto the additive group of the subring $R_A(F)$ of R_A generated by $\{\delta_i, \rho_i : i \in F\}$. These isomorphisms carry the maps $\phi_{F,G}$ into the inclusions of $R_A(F)$ in $R_A(G)$, and thus induce a group isomorphism of $\varinjlim(\mathbb{Z}^F \oplus \mathbb{Z}^{W_F})$ onto the underlying additive group of R_A (which is what appears in the statement of [11, Theorem 4.5]). If $i \in F$, the image of $\delta_i \in R_A(F)$ under transformation with matrix A_F^t is the row vector ρ_i, written as a sum of $\{\delta_j, \rho_X\}$; thus the maps $1-A_F^t : \mathbb{Z}^F \to \mathbb{Z}^F \oplus \mathbb{Z}^{W_F}$ combine to give the map of $\mathbb{Z}^I = \varinjlim \mathbb{Z}^F$ into $\varinjlim(\mathbb{Z}^F \oplus \mathbb{Z}^{W_F})$ which Exel and Laca call $1 - A^t$. Theorem 4.1 therefore gives:

Corollary 4.6 (Exel and Laca [11, Theorem 4.5]). *Suppose A is an $I \times I$ matrix with entries in $\{0, 1\}$, and suppose that A has no zero rows. Then there is an exact sequence*

$$0 \to K_1(\mathcal{O}_A) \to \mathbb{Z}^I \xrightarrow{1-A^t} R_A \to K_0(\mathcal{O}_A) \to 0.$$

5. Concluding remarks

5.1. We assumed in Section 1 that our graphs did not have sinks, but we did so only to make things clearer: with just minor modifications it is possible to consider arbitrary graphs. For each finite subset F of $E^1 \cup E^0$, we define E_F as before, and then enlarge the set of vertices of E_F by adding the sinks in $F \cap E^0$. The constructions of Section 1 then carry over, and in particular there is a version of Lemma 1.2. Thus we can, at least in principle, calculate the K-theory of $C^*(E)$ for an arbitrary countable graph E by writing $C^*(E)$ as a direct limit of C^*-algebras of finite graphs.

5.2. In Remark 4.4, we saw, rather indirectly, that not every Exel-Laca algebra is a graph algebra. It is therefore natural to ask how $\mathcal{O}_A$ is related to the C^*-algebra of the graph E_A with vertex matrix A. While the answer is not fully clear to us, we can say this much:

Proposition 5.1. *Let A be a $\{0, 1\}$-matrix with no zero rows. Then there is an isomorphism of $C^*(E_A)$ onto a C^*-subalgebra of $\mathcal{O}_A$.*

Proof. We verify that $S_{(i,j)} := S_iS_jS_j^*$ for $(i, j) \in E_A^1$ and $P_i := S_iS_i^*$ for $i \in I$ defines a Cuntz-Krieger E_A-family inside $\mathcal{O}_A$. (EL3) implies (G1), and (G2) is obvious. Fix $i \in I$, and suppose there are only finitely many $j \in I$ for which $S_iS_jS_j^* \neq 0$ or, equivalently, for which $A(i, j) = 1$. Then $S_i^*S_i = \sum_{A(i,j)=1} S_jS_j^*$ by (EL4), and (G3) holds. The universality of $C^*(E_A)$ gives a homomorphism $\pi_{S,P} : C^*(E_A) \to \mathcal{O}_A$. To see that $\pi_{S,P}$ is injective, let $\{F_n\}_{n=1}^\infty$ be an increasing family of finite sets such that $\bigcup_{n=1}^\infty F_n = E_A^1 \cup E_A^0$. We denote by ϕ_n the embedding of $C^*(E_{F_n})$ in $C^*(E_A)$ given by Lemma 1.2. The gauge-invariant uniqueness theorem [1, Theorem 2.1] implies that each $\pi_{S,P} \circ \phi_n$ is injective, and it follows that $\pi_{S,P}$ is injective, as required. □

5.3. The condition "every loop has an exit" identifies the graphs whose C^*-algebras satisfy a Cuntz-Krieger uniqueness theorem; all of these algebras are infinite. We can use our approximation technique to sharpen this statement: a graph C^*-algebra is finite if and only if no loop has exits. The proof uses a simple lemma which is essentially contained in [20, Section 2].

Lemma 5.2. *Let E be a finite directed graph in which no loop has exits. Then there are finite-dimensional C^*-algebras D and B such that $C^*(E) \cong D \oplus (B \otimes C(\mathbb{T}))$.*

Proof. Let $v_1, \ldots, v_d$ be the sinks in E, and $L_1, \ldots, L_b$ the distinct loops in E in which no edge is traversed twice (d or b could be 0). Since the loops have no exits, the sets L_i^0 are pairwise disjoint. We set

$$D_i = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ r(\mu) = r(\nu) = v_i\},$$
$$C_j = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ r(\mu) = r(\nu) \in L_j^0\},$$

and note that D_i, C_j are mutually orthogonal ideals in $C^*(E)$. Corollary 2.2 of [20] implies that each D_i is a matrix algebra, and the last part of the proof of [20, Theorem 2.4] shows that each $C_j \cong M_{n_j}(\mathbb{C}) \otimes C(\mathbb{T})$ for some n_j. Thus we take $D = \bigoplus_{i=1}^d D_i$ and $B = \bigoplus_{j=1}^b M_{n_j}(\mathbb{C})$; if one of d or b vanishes, we just take that factor to be $\{0\}$. $\square$

From Lemma 1.2 (as modified in §5.1), Lemma 1.3 and Lemma 5.2 we deduce:

Corollary 5.3. *Suppose E is a directed graph in which no loop has exits. Then there is an increasing sequence of subalgebras A_n of $C^*(E)$ such that $C^*(E) = \overline{\bigcup_{n=1}^\infty A_n}$ and each A_n has the form $D_n \oplus (B_n \otimes C(\mathbb{T}))$ for finite-dimensional C^*-algebras D_n and B_n.*

A C^*-algebra is called infinite if it contains an infinite projection; that is, if it contains a partial isometry s such that ss^* is a proper subprojection of s^*s. It was observed in the proof of [20, Theorem 2.4] that every loop of E which has exits gives rise to an infinite projection in $C^*(E)$. A C^*-algebra is stably finite if it is finite after tensoring with the compacts or, equivalently, with $M_k(\mathbb{C})$ for every k. A C^*-algebra has stable rank 1 if the invertible elements are dense in its minimal unitisation. The property of having stable rank 1 is preserved under tensoring with $M_k(\mathbb{C})$ [28, Theorem 3.3] and passing to direct limits [28, Theorem 5.1]. Algebras of the form $D \oplus (B \otimes C(\mathbb{T}))$ with D, B finite-dimensional have stable rank 1 [28, Section 3]. Any C^*-algebra with stable rank 1 is stably finite [28, Proposition 3.1]. Thus Corollary 5.3 and Lemma 1.4 imply the following.

Proposition 5.4. *Suppose E is a countable directed graph. Then $C^*(E)$ is finite if and only if no loop in E has exits, in which case it is stably finite and has stable rank 1.*

For row-finite graphs this was proved in [16, Theorem 3.3]. Using our approximation technique from Section 2 and similar reasoning gives a parallel result for Exel-Laca algebras:

Proposition 5.5. *Let A be a $\{0,1\}$-matrix with no zero rows. Then $\mathcal{O}_A$ is finite if and only if no loop in E_A has exits, in which case it is stably finite and has stable rank 1.*

5.4. We have just seen that the conditions "every loop has an exit" and "no loop has exits" characterise those graph C^*-algebras for which there is a Cuntz-Krieger uniqueness theorem, and those which are finite, respectively. Graphs which belong to both classes have no loops at all. It was proved in [20, Theorem 2.4] that a row-finite graph E has no loops if and only if $C^*(E)$ is AF, and that argument extends almost unchanged to arbitrary countable graphs. One can also deduce this from Lemma 1.4 and the argument of Lemma 5.2. Using the approximations of Section 2 this result can be extended to Exel-Laca algebras: $\mathcal{O}_A$ is AF if and only if E_A has no loops. (This result has been obtained by Hjelmborg using different methods [14, Theorem 3.6].)

5.5. Yet another way of calculating the K-theory of $\mathcal{O}_A$ might be to imitate the proof of Theorem 3.2, as follows. Define a new $\{0,1\}$-matrix $A \times_1 \mathbb{Z}$ on the index set $I \times \mathbb{Z}$ by $(A \times_1 \mathbb{Z})((i,n),(j,m)) = \delta_{n,m-1} A(i,j)$. As in Lemma 3.1, the crossed product $\mathcal{O}_A \rtimes_\gamma \mathbb{T}$ is canonically isomorphic to $\mathcal{O}_{A\times_1\mathbb{Z}}$, and the latter is an AF-algebra since $E_{A\times_1\mathbb{Z}}$ has no loops (see §5.4 above). Thus the key idea behind the proof of Theorem 3.2 carries over.

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