FACTORIZATION IN FINITE-CODIMENSIONAL IDEALS OF GROUP ALGEBRAS

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ABSTRACT. Let G be a σ-compact, locally compact group and $\mathcal{I}$ be a closed 2-sided ideal with finite codimension in $L^1(G)$. It is shown that there are a closed left ideal $\mathcal{L}$ having a right bounded approximate identity and a closed right ideal $\mathcal{R}$ having a left bounded approximate identity such that $\mathcal{I} = \mathcal{L} + \mathcal{R}$. The proof uses ideas from the theory of boundaries of random walks on groups.

1. INTRODUCTION

A Banach algebra $\mathcal{A}$ is said to be *idempotent* if $\mathcal{A}^2 = \mathcal{A}$ where $\mathcal{A}^2$ is defined to be span$\{ab : a, b \in \mathcal{A}\}$. Of course $\mathcal{A}$ is idempotent if it has an identity, but it can be a difficult and fruitful question to decide whether a Banach algebra which does not have an identity is idempotent. This question was tackled by W. Rudin, see [27], in the case when $\mathcal{A}$ is the group algebra $L^1(G)$ of a non-discrete, locally compact group G and he showed that $L^1(G)$ is idempotent for many groups G. Shortly afterwards P.J. Cohen showed that if a Banach algebra $\mathcal{A}$ has a bounded approximate identity then every element in $\mathcal{A}$ is a product, see [2], [1], Theorem 11.10 or [10], Theorem 32.22. Since, by [10] Theorem 20.27, every group algebra has a bounded approximate identity, it follows that group algebras are idempotent.

The question of whether an ideal in a group algebra is idempotent generally depends on properties of the ideal and of the group for its answer. Ideals in $L^1(G)$ do not, in general, have a bounded approximate identity, see [26] and [14], and may fail to be idempotent, see [29], Theorem 7.6.3 and [10], Theorem 42.16. In all these examples G is abelian and the ideals have infinite codimension. However it is shown here that for an arbitrary locally compact group G, if $\mathcal{I}$ is an ideal with finite codimension in $L^1(G)$, then $\mathcal{I}$ is idempotent. This will be an immediate consequence of the following

Theorem 1.1. *Let G be a σ-compact, locally compact group and $\mathcal{I}$ be a closed 2-sided ideal with finite codimension in $L^1(G)$. Then*

$$\mathcal{I} = \mathcal{L} + \mathcal{R}\,,$$

where $\mathcal{L}$ is a closed left ideal in $L^1(G)$ with a right bounded approximate identity and $\mathcal{R}$ is a closed right ideal in $L^1(G)$ with a left bounded approximate identity.

For σ-compact G the theorem implies directly that finite-codimensional ideals in $L^1(G)$ are idempotent, and indeed that every element in the ideal is a sum of 2 products. The same conclusion holds if G is not σ-compact because each

element f of the ideal is an L^1-function on G and hence is supported on some open, σ-compact subgroup of G. Applying the theorem to the subalgebra of $L^1(G)$ consisting of functions supported on this open subgroup shows that f is the sum of 2 products in $\mathcal{I}$.

Idempotence of proper ideals in group algebras has been established previously in a number of cases, which usually require that the group be amenable. Once again, one of the earliest results is due to Rudin, who showed in [28] that, if G is compact and $\mathcal{I}$ is an ideal in $L^1(G)$ which has a Banach space complement, then $\mathcal{I}$ has a bounded approximate identity. It was shown by H. Reiter in [21] that the codimension one ideal

$$L_0^1(G) = \left\{ f \in L^1(G) : \int_G f \, dm \right\},$$

where m denotes the left invariant Haar measure on G, has a bounded approximate identity if and only if G is amenable. This theorem of Reiter was extended by Liu, van Rooij and Wang who showed in [17], Theorem 2 that, if G is amenable, then every complemented ideal in $L^1(G)$ has a bounded approximate identity. Their argument developed further the method used by Rudin in [28]. As a final result in this direction, cohomological techniques show that, when G is an amenable group, an ideal $\mathcal{I}$ in $L^1(G)$ has a bounded approximate identity if and only if it is weakly complemented, see [9], Proposition VII.2.37 or [3]. As observed above, even in the case when G is abelian there are many ideals which do not satisfy this condition.

Since finite-codimensional ideals are complemented, the theorem proved here says nothing new about amenable groups; indeed its conclusion is not as strong as known results. For non-amenable groups though it is just about as strong as can be expected (but see the discussion of connected groups below). In the case when $\mathcal{I} = L_0^1(G)$, this is shown by the previously cited theorem of Reiter, [21]. More generally, it is shown in [36] that, if G is not amenable, then no finite-codimensional ideal in $L^1(G)$ has a bounded approximate identity.

Even when G is not amenable, idempotence of certain ideals in $L^1(G)$ can be shown by considering amenable subgroups of G. If G is locally compact and H is a closed subgroup of G, then

$$\mathcal{J}_H = \operatorname{span} \left\{ f - f * \delta_h : h \in H \right\}^-,$$

is a closed left ideal in $L^1(G)$. When H is a normal subgroup $\mathcal{J}_H$ is a 2-sided ideal. Then $\mathcal{J}_H$ has a right bounded approximate identity if and only if H is amenable, see [22], Theorems 10.1 and 10.2. This fact may be used, in conjunction with a Lie groups argument, to show that, if G is connected, then finite-codimensional ideals in $L^1(G)$ are idempotent, see [38]. These methods in fact show that $L_0^1(G)$ is a sum of two left ideals each having a right bounded approximate identity, i.e., that the approximate identities are on the same side.

Free groups with 2 or more generators are neither amenable nor connected and so the results cited above say nothing about idempotence of finite-codimensional ideals in their group algebras. There are few factorization theorems applying to general group algebras. In [34] it was shown that, if $\mathcal{I}$ has codimension 1 in $L^1(G)$, then every element of $\mathcal{I}$ is a sum of 4 products and in [37] the number of products required was reduced to 2. The proof in [37] corresponds to part of the argument used here and we recall the main points of that proof in section 3 below. In [35] it was shown that, if G is discrete and finitely generated, then each ideal having

codimension 2 in $L^1(G)$ is idempotent, the number of products required increasing with the number of generators of G. The theorem proved here improves these results by being applicable to all finite-codimensional ideals, all locally compact groups and by showing that only 2 products are required in all cases.

The rest of the paper gives the proof of the theorem, which is set out in sections as follows. In the next section it is shown that to each finite-codimensional ideal $\mathcal{I}$ in $L^1(G)$ there corresponds a finite-dimensional, bounded representation $\rho_\mathcal{I}$ of G. The closure of the range of $\rho_\mathcal{I}$ is a compact group, K, and harmonic analysis on K plays an essential role in the proof. In the case when $\mathcal{I} = L_0^1(G)$, K becomes trivial and the proof reduces to the argument given in [37]. This argument uses ideas connected with random walks on G which must be incorporated in the general case and the requisite facts are recalled in section 3. Section 4 uses the random walk techniques to produce functions in $L^1(G)$ which convolve approximately like matrix coefficients of $\rho_\mathcal{I}$. In the case when G is amenable these approximations are related to the weak containment of the irreducible representations of K in the regular representation of G but are, of course, much weaker. The functions produced in section 4 are used in section 5 to define $\mathcal{L}$, $\mathcal{R}$ and their approximate identities and to show that $\mathcal{I} = \mathcal{L} + \mathcal{R}$. The final section discusses some further consequences of the theorem and some open problems.

2. The representation $\rho_\mathcal{I}$ and compact group K

Let $\mathcal{I}$ be a 2-sided, closed ideal with finite codimension in $L^1(G)$ and define $\mathcal{V} = L^1(G)/\mathcal{I}$. Then $\mathcal{V}$ is a finite-dimensional Banach algebra.

Since $\mathcal{I}$ is an ideal, it is closed under translation and so the regular representation of G on $L^1(G)$ induces a finite-dimensional representation of G by isometries on $\mathcal{V}$. Call this representation $\rho_\mathcal{I}$. Thus

$$\rho_\mathcal{I}(x)(f + \mathcal{I}) = \delta_x * f + \mathcal{I} \qquad (x \in G, \ f + \mathcal{I} \in \mathcal{V}).$$

Since $\rho_\mathcal{I} : G \to \text{Isometries}(\mathcal{V})$, the closure of $\rho_\mathcal{I}(G)$ in the operator-norm topology on $\mathcal{V}$ is a compact group of isometries on $\mathcal{V}$. Define $K = \rho_\mathcal{I}(G)^-$ to be this compact group, so that $\rho_\mathcal{I} : G \to K$.

Now the identity representation of K on $\mathcal{V}$ is finite-dimensional, continuous and bounded and so it induces a finite-dimensional representation Θ of $L^1(K)$ on $\mathcal{V}$ given by

$$\Theta(\phi)v = \int_K \phi(k)kv\,dm_K(k) \qquad (v \in \mathcal{V}, \ \phi \in L^1(K)).$$

Define $\tilde{\mathcal{I}} = \text{kernel}(\Theta)$. Then $\tilde{\mathcal{I}}$ is a finite-codimensional, closed, 2-sided ideal in $L^1(K)$ and so has a bounded approximate identity, by [28] or [17]. This bounded approximate identity can be described explicitly in terms of the representation theory of K.

It follows from [10], Theorem 38.7 that there is a finite set, S, of irreducible unitary representations, σ, of K such that

$$\tilde{\mathcal{I}} = \{f \in L^1(K) : \hat{f}(\sigma) = 0, \ \sigma \in S\},$$

where $\hat{f}(\sigma)$ denotes the Fourier transform of f at σ, [10], 28.34.

Let $\mathfrak{H}_\sigma$ denote the Hilbert space on which σ represents K and suppose that the dimension of $\mathfrak{H}_\sigma$ is d_σ. Choose an orthonormal basis $\{h_\alpha\}$ for $\mathfrak{H}_\sigma$ and define

$$c_{\alpha\beta}^\sigma(k) = \langle \sigma(k)h_\beta, \ h_\alpha \rangle$$

to be the matrix coefficient of $\sigma(k)$ with respect to this basis, see [10], 27.5. Then the functions $c_{\alpha\beta}^\sigma$ are continuous on K and satisfy the orthogonality relations

$$(1) \qquad \int_K c_{\alpha\beta}^\sigma \bar{c}_{\gamma\delta}^\tau \, dm_K = \begin{cases} \frac{1}{d_\sigma}, & \text{if } \sigma = \tau, \, \alpha = \gamma \text{ and } \beta = \delta \\ 0, & \text{otherwise}, \end{cases}$$

where $\bar{c}_{\gamma\delta}^\tau$ denotes the complex conjugate of $c_{\gamma\delta}^\tau$, see [10], 27.15 and 27.19. They also satisfy the translation relations

$$(2) \qquad \delta_k * \bar{c}_{\alpha\beta}^\sigma = \sum_{\gamma=1}^{d_\sigma} c_{\gamma\alpha}^\sigma(k) \bar{c}_{\gamma\beta}^\sigma,$$

see [10], Theorem 27.20(i) and convolve as scaled matrix units

$$\bar{c}_{\alpha\beta}^\sigma * \bar{c}_{\gamma\delta}^\tau = \begin{cases} \frac{1}{d_\sigma} \bar{c}_{\alpha\delta}^\sigma, & \text{if } \sigma = \tau \text{ and } \beta = \gamma \\ 0, & \text{otherwise}, \end{cases}$$

[10], Theorem 27.20(iii).

The subspace of $L^1(K)$ spanned by $\{\bar{c}_{\alpha\beta}^\sigma\}$ is an ideal in $L^1(K)$ which is isomorphic to $\mathcal{V}$ and also to $\bigoplus_{\sigma \in S} \mathcal{B}(\mathfrak{H}_\sigma)$, where $\mathcal{B}(\mathfrak{H}_\sigma) \cong M_{d_\sigma}$. In the following, $\mathcal{V}$ **will be identified with this ideal**. Thus $\{\bar{c}_{\alpha\beta}^\sigma\}$ is a basis for $\mathcal{V}$. The identity element of $\mathcal{V}$ is then the function

$$(3) \qquad z = \sum_{\sigma \in S} d_\sigma \left(\sum_{\alpha=1}^{d_\sigma} \bar{c}_{\alpha\alpha}^\sigma \right).$$

The function z: (i) belongs to the centre of $L^1(K)$; (ii) is idempotent, i.e., $z*z = z$; and   (iii) $L^1(K) = \tilde{\mathcal{I}} \oplus \mathcal{V}$, where $\mathcal{V} = L^1(K) * z$ and $\tilde{\mathcal{I}} = L^1(K) * (\delta_e - z)$. These properties of z are shown in [10], Theorem 27.24.

The bounded approximate identity for $\tilde{\mathcal{I}}$ has the form $u_\lambda * (\delta_e - z)$, where (u_λ) is a bounded approximate identity for $L^1(K)$. The functions u_λ may also be chosen to be central, see [10], Theorem 28.53.

Unless G itself is compact, there is no function in $L^1(G)$ with properties (i)–(iii). For instance, if $\mathcal{I} = L_0^1(G)$, then K is the trivial group and z is the constant function 1. However functions can be found in $L^1(G)$ which satisfy analogous properties in an approximate way. In the case when $\mathcal{I} = L_0^1(G)$ these functions are probability measures.

3. Probability measures on G and the case when $\mathcal{I} = L_0^1(G)$

The case of the theorem when $\mathcal{I} = L_0^1(G)$ has been proved in [37], where the ideals $\mathcal{L}$ and $\mathcal{R}$ are described in terms of a random walk on G.

For each probability measure μ on G define

$$\mathcal{J}_\mu = \left\{ f - f * \mu : f \in L^1(G) \right\}^-.$$

Then $\mathcal{J}_\mu$ is a closed left ideal in $L^1(G)$, $\mathcal{J}_\mu \subset L_0^1(G)$, and $\mathcal{J}_\mu$ has a right bounded approximate identity $\left(u_{\lambda,n} = u_\lambda * (\delta_e - \frac{1}{n} \sum_{k=1}^n \mu^{*k}) \right)$, where (u_λ) is a bounded approximate identity for $L^1(G)$. Similarly, define

$$_\mu\mathcal{J} = \left\{ f - \mu * f : f \in L^1(G) \right\}^-.$$

Then $_\mu\mathcal{J}$ is a closed, right ideal in $L^1(G)$, $_\mu\mathcal{J} \subset L_0^1(G)$ and $_\mu\mathcal{J}$ has a left bounded approximate identity.

Theorem 3.1 ([37], 3.8)**.** *Let μ be a probability measure on G such that μ is absolutely continuous with respect to Haar measure and such that $\mathrm{supp}(\mu) = G$. Then*

$$L_0^1(G) = {}_\mu\mathcal{J} + \mathcal{J}_\mu.$$

Any choice of probability measure satisfying the hypotheses gives such a decomposition of $L_0^1(G)$ and so it is far from being unique.

The argument from [37] is to be incorporated in the current proof and needed facts and notation from that paper are recalled in this section. An important idea is the notion of the boundary of the random walk with transition probability μ. This is defined in probabilistic terms in [7], [11] and [18], VI.2 but the following functional analytic definition is more appropriate here.

It may be shown, see [37], that the quotient space $L^1(G)/\mathcal{J}_\mu$ becomes an abstract L^1-space if we define its positive cone to be the closure of $L^1(G)^+/\mathcal{J}_\mu$, where $L^1(G)^+$ denotes the set of positive functions in $L^1(G)$. By the theorem of Kakutani, see [12] or [16], Theorem 1.1.2, there are a measure space (Ω, ν) and an isometric isomorphism $B : L^1(G)/\mathcal{J}_\mu \to L^1(\Omega, \nu)$ such that $B\big((L^1(G)^+/\mathcal{J}_\mu)^-\big) = L^1(\Omega, \nu)^+$.

An action of G on Ω may be defined so that Ω is a measurable G-space, ν is quasi-invariant and that under the induced action of $L^1(G)$ on $L^1(\Omega, \nu)$, B is a left $L^1(G)$-module isomorphism, see section 2 of [37]. The left action of $L^1(G)$ on $L^1(\Omega, \nu)$ will be denoted $(f, \xi) \mapsto f.\xi$ $\big(f \in L^1(G), \ \xi \in L^1(\Omega, \nu)\big)$. Thus, if $\xi = B(g + \mathcal{J}_\mu) \in L^1(\Omega, \nu)$, then

$$f.\xi = f.B(g + \mathcal{J}_\mu) = B(f * g + \mathcal{J}_\mu).$$

A consequence of this construction of (Ω, ν) is that

$$(4) \qquad \int_G f \, dm = \int_\Omega B(f + \mathcal{J}_\mu) \, d\nu \qquad \big(f \in L^1(G)\big).$$

When μ is absolutely continuous with respect to m and $\mathrm{supp}(\mu) = G$, the measure ν may be chosen such that $B(\mu + \mathcal{J}_\mu) = \mathbf{1}_\Omega$. (In probability theory, μ is said to be *non-degenerate* and ν to be μ-*stationary* in this case. In algebraic terms, μ is a right modular unit for $\mathcal{J}_\mu$.) Then $\mu.\mathbf{1}_\Omega = \mathbf{1}_\Omega$ and B is given by

$$B(f + \mathcal{J}_\mu) = f.\mathbf{1}_\Omega \qquad \big(f \in L^1(G)\big).$$

If $A \subset \Omega$ is measurable, then $\mathbf{1}_A$, the characteristic function of A, lies in the positive cone of $L^1(\Omega, \nu)$, and so the construction of $L^1(\Omega, \nu)$ implies that for every $\epsilon > 0$ there is a positive $f_A \in L^1(G)$ such that

$$(5) \qquad \qquad \|\mathbf{1}_A - f_A.\mathbf{1}_\Omega\|_{L^1(\Omega,\nu)} < \epsilon.$$

If $\{A_1, A_2, \ldots, A_p\}$ is a measurable partition of Ω, we may choose positive functions f_{A_i} $(i = 1, 2, \ldots, p)$, such that $\|\mathbf{1}_{A_i} - f_{A_i}.\mathbf{1}_\Omega\|_{L^1(\Omega,\nu)} < \epsilon$ for each i. Moreover, these functions may be chosen so that

$$\left\| \mathbf{1}_\Omega - \left(\sum_{i=1}^p f_{A_i} \right).\mathbf{1}_\Omega \right\|_{L^1(\Omega,\nu)} < \epsilon.$$

It follows, by replacing each f_{A_i} by $f_{A_i} * \big(\frac{1}{n}\sum_{k=1}^n \mu^{*k}\big)$ if necessary, that the functions f_{A_i} may be chosen such that

$$(6) \qquad \qquad \left\| \mu' - \sum_{i=1}^p f_{A_i} \right\|_{L^1(G)} < \epsilon,$$

for some μ' in the convex hull of the convolution powers of μ.

The decomposition of $L_0^1(G)$ may be deduced from properties of the operator T in $\mathcal{B}\big(L^1(\Omega, \nu)\big)$ defined by

$$T\xi = \mu.\xi \qquad \big(\xi \in L^1(\Omega, \nu)\big).$$

This T is a positive, norm 1 operator on $L^1(\Omega, \nu)$. Furthermore, T maps bounded functions to bounded functions and its restriction to $L^\infty(\Omega, \nu)$ also has norm 1. The ergodic theorem for operators, [6], Corollary VIII.5.5, implies then that $\left(\sum_{k=1}^n T^k\right)/n$ converges in the strong operator topology to a projection P such that

$$\text{range}(P) = \big\{\xi \in L^1(\Omega, \nu) : T\xi = \xi\big\} \qquad \text{and } \text{kernel}(P) = \big[(I - T)L^1(\Omega, \nu)\big]^-.$$

It is shown in [37] that $T\xi = \xi$ only if ξ is constant, whence $\text{kernel}(T)$ has codimension 1 in $L^1(\Omega, \nu)$, i.e.,

$$(7) \qquad\qquad L_0^1(\Omega, \nu) = \big\{\xi - \mu.\xi : \xi \in L^1(\Omega, \nu)\big\}^-.$$

Since $B(L_0^1(G) + \mathcal{J}_\mu) = L_0^1(\Omega, \nu)$, see (4), we have

$$L_0^1(G) = {}_\mu\mathcal{J} + \mathcal{J}_\mu.$$

What is required in the following is the fact that

$$(8) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \mu^{*k}.\xi \xrightarrow{\|\cdot\|} P\xi = \left(\int_\Omega \xi \, d\nu\right) \mathbf{1}_\Omega \qquad \big(\xi \in L^1(\Omega, \nu)\big).$$

4. Some Approximation Results

In this section G will be a σ-compact, locally compact group, K an arbitrary compact group, and $\rho : G \to K$ a continuous group homomorphism with range dense in K. It is shown how harmonic analysis on K may be pulled back, in an approximate sense, to G via ρ. Should the irreducible representations of G which factor through ρ happen to be weakly contained in the regular representation of G, which occurs only if G is amenable, see [19], Theorem 4.21 or [8], Theorem 3.5.2, these approximations could be made with the aid of weak containment. They continue to hold when G is not amenable however. The approximations will be used in section 4 with K and ρ as defined in section 1.

Let ϕ be a Borel measurable function on K. Then $\phi \circ \rho$ is a Borel function on G which will be denoted $\tilde{\phi}$. For each measure, λ, on G define the measure $\tilde{\phi}\lambda$ by

$$\left(\tilde{\phi}\lambda\right)(E) = \int_E \tilde{\phi}(x) \, d\lambda(x) = \int_E (\phi \circ \rho)(x) \, d\lambda(x) \qquad \text{for all Borel } E \subset G.$$

Let λ be a bounded measure on G. Then λ has a direct image under ρ which is the measure $\tilde{\rho}(\lambda)$ defined on K by

$$\tilde{\rho}(\lambda)(E) = \lambda\left(\rho^{-1}(E)\right) \qquad \text{for all Borel } E \subset K.$$

The map $\tilde{\rho}$ is a homomorphism from $M(G)$, the convolution algebra of bounded measures on G, to $M(K)$. In the following $\tilde{\rho}(\lambda)$ will be denoted simply by $\tilde{\lambda}$. Then

$$(\tilde{\phi}\lambda)^{\sim} = \phi\tilde{\lambda} \qquad (\phi \in \mathcal{B}(K), \ \lambda \in M(G))$$

and for positive λ

$$\|\tilde{\phi}\lambda\|_{M(G)} = \|\phi\tilde{\lambda}\|_{M(K)} \qquad (\phi \in \mathcal{B}(K), \ \lambda \in M(G)),$$

where $\mathcal{B}(K)$ denotes the space of bounded Borel measurable functions on K.

Since G is σ-compact, there is a probability measure μ on G which is absolutely continuous with respect to Haar measure and such that $\operatorname{supp}(\mu) = G$. Choose a fixed probability measure with these properties. (This is the only point at which it is necessary that G be σ-compact.) The closed, convex hull of the set of convolution powers of μ will be denoted $\operatorname{co}(\mu)$.

Since $\operatorname{supp}(\mu) = G$ and the range of ρ is dense in K, the support of $\tilde{\mu}$ is dense in K. Then, since K is compact, the convolution powers of $\tilde{\mu}$ converge to m_K, the Haar measure on K, in the weak-$*$ topology and also in the sense that, for every open subset $U \subset K$,

$$(9) \qquad \lim_{n \to \infty} \tilde{\mu}^{*n}(U) = m_K(U),$$

see [13] or [25], Theorem VI.3.2.

Theorem 4.1. (i) *Let $U \subset K$ be open and let $\xi \in L^1(\Omega, \nu)$. Then $m_K(U) \left(\int_\Omega \xi \, d\nu \right) \mathbf{1}_\Omega$ belongs to the norm closure of $\left\{ (\tilde{\mathbf{1}}_U \mu').\xi : \mu' \in \operatorname{co}(\mu) \right\}$.*
(ii) *Let $\phi \in C(K)$ and $\xi \in L^1(\Omega, \nu)$. Then $\left(\int_K \phi \, dm_K \right) \left(\int_\Omega \xi \, d\nu \right) \mathbf{1}_\Omega$ belongs to the norm closure of $\left\{ (\tilde{\phi}\mu').\xi : \mu' \in \operatorname{co}(\mu) \right\}$.*

Proof. (i) Suppose for the time being that ξ is non-negative. We establish first that $m_K(U) \left(\int_\Omega \xi \, d\nu \right) \mathbf{1}_\Omega$ belongs to the weak closure in $L^1(\Omega, \nu)$ of

$$\left\{ (\tilde{\mathbf{1}}_U \mu').\xi : \mu' \in \operatorname{co}(\mu) \right\}.$$

For this, let $\{A_1, A_2, \ldots, A_p\}$ be a measurable partition of Ω. Then by (5) and (6) there are positive functions $f_{A_i} \in L^1(G)$ such that:

$$(10) \qquad f_{A_i}.\mathbf{1}_\Omega - \mathbf{1}_{A_i} \text{ is arbitrarily small ;}$$

$$(11) \qquad \int_\Omega f_{A_i}.\mathbf{1}_\Omega \, d\nu = \int_G f_{A_i} \, dm = \nu(A_i) \qquad (i = 1, 2, \ldots, p) ; \quad \text{and}$$

$$(12) \qquad f_{A_1} + f_{A_2} + \cdots + f_{A_p} - \mu' \text{ is arbitrarily small for some } \mu' \in \operatorname{co}(\mu).$$

Combining with (8) and replacing each f_{A_i} by $f_{A_i} * \left(\frac{1}{n} \sum_{k=1}^n \mu^{*k} \right)$ for some n, which does not affect (10)–(12), these functions may be chosen to also satisfy:

$$(13) \qquad f_{A_i}.\xi - \left(\int_\Omega \xi \, d\nu \right) \mathbf{1}_{A_i} \text{ is arbitrarily small for } i = 1, 2, \ldots, p.$$

With the aid of (9), these functions may be chosen to satisfy one further condition. Since m_K is translation invariant, (9) implies that for every $x \in G$

$$(\delta_x * \mu^{*n})^{\sim}(U) \to m_K(U) \qquad \text{as } n \to \infty.$$

Since μ is absolutely continuous with respect to Haar measure on G, the map $x \mapsto (\delta_x * \mu^{*n})^{\sim}(U) : G \to [0, 1]$ is continuous for each n. Hence, by the dominated convergence theorem, we have for every $f \in L^1(G)$ that

$$\begin{aligned} (f * \mu^{*n})^{\sim}(U) \quad &= \quad \int_G f(x) (\delta_x * \mu^{*n})^{\sim}(U) \, dm(x) \\ &\longrightarrow \quad m_K(U) \int_G f(x) \, dm(x) \qquad \text{as } n \to \infty. \end{aligned}$$

In particular,

$$(f_{A_i} * \mu^{*n})^\sim(U) \xrightarrow{\ n\ } \left(\int_G f_{A_i}\, dm \right) m_K(U) = \nu(A_i) m_K(U).$$

Pulling back to a statement about functions on G yields

$$\int_G \tilde{\mathbf{1}}_U(f_{A_i} * \mu^{*n})\, dm = \left\| \tilde{\mathbf{1}}_U(f_{A_i} * \mu^{*n}) \right\|_{L^1(G)} \xrightarrow{\ n\ } \nu(A_i) m_K(U).$$

Hence, once again replacing f_{A_i} by $f_{A_i} * \mu^{*n}$ if necessary, it may be supposed that:

(14) $\qquad\qquad \left\| \tilde{\mathbf{1}}_U f_{A_i} \right\|_{L^1(G)} - \nu(A_i) m_K(U)$ is arbitrarily small.

Clearly $\tilde{\mathbf{1}}_U f_{A_i} \leq f_{A_i}$ and so $(\tilde{\mathbf{1}}_U f_{A_i}).\xi \leq f_{A_i}.\xi$. (Here we use that $\xi \geq 0$.) Hence, because $A_i \cap A_j = \emptyset$, (13) implies that

$$\int_{A_j} (\tilde{\mathbf{1}}_U f_{A_i}).\xi\, d\nu \approx \begin{cases} \int_\Omega (\tilde{\mathbf{1}}_U f_{A_i}).\xi\, d\nu, & \text{for } i = j \\ 0, & \text{for } i \neq j. \end{cases}$$

Since $\tilde{\mathbf{1}}_U f_{A_i}$ and ξ are positive, $\int_\Omega (\tilde{\mathbf{1}}_U f_{A_i}).\xi\, d\nu = \left\| \tilde{\mathbf{1}}_U f_{A_i} \right\|_{L^1(G)} \int_\Omega \xi\, d\nu$. With the aid of (14) this yields that

$$\int_{A_j} (\tilde{\mathbf{1}}_U f_{A_i}).\xi\, d\nu \approx \begin{cases} \nu(A_i) m_K(U) \int_\Omega \xi\, d\nu, & \text{if } i = j \\ 0, & \text{for } i \neq j. \end{cases}$$

Now let $F = \sum_{j=1}^p \lambda_j \mathbf{1}_{A_j}$ be a simple function in $L^\infty(\Omega, \nu)$. Then we may choose functions f_{A_i} which satisfy the above approximations and such that $f_{A_1} + f_{A_2} + \cdots + f_{A_p} \approx \mu'$ for some μ' in $\mathrm{co}(\mu)$. Then

$$\begin{aligned}
\int_\Omega (\tilde{\mathbf{1}}_U \mu').\xi F\, d\nu &\approx \sum_{i,j=1}^p \lambda_j \int_{A_j} (\tilde{\mathbf{1}}_U f_{A_i}).\xi\, d\nu \\
&\approx \sum_{i=1}^p \lambda_i \nu(A_i) m_K(U) \left(\int_\Omega \xi\, d\nu \right) \\
&= \left(\int_\Omega F\, d\nu \right) m_K(U) \left(\int_\Omega \xi\, d\nu \right).
\end{aligned}$$

This approximation can be made for any finite number of simple functions simultaneously by using a partition $\{A_1, \ldots, A_p\}$ subordinate to all of them.

Since the simple functions $F = \sum_{j=1}^p \lambda_j \mathbf{1}_{A_j}$, ranging over all measurable partitions $\{A_j\}$ of Ω and all λ_j, form a dense subset of $L^\infty(\Omega, \nu)$, it follows as claimed that $(\int_\Omega \xi\, d\nu) m_K(U) \mathbf{1}_\Omega$ belongs to the weak closure of $\{(\tilde{\mathbf{1}}_U \mu').\xi : \mu' \in \mathrm{co}(\mu)\}$.

Now $\{(\tilde{\mathbf{1}}_U \mu').\xi : \mu' \in \mathrm{co}(\mu)\}$ is convex and so, by [6], Corollary V.2.14, its weak closure and norm closure are the same. Hence $(\int_\Omega \xi\, d\nu) m_K(U) \mathbf{1}_\Omega$ belongs to the norm closure of this set. Moreover, if $\xi_1, \ldots, \xi_s$ are non-negative functions in $L^1(\Omega, \nu)$ and $\epsilon > 0$, there is $\mu' \in \mathrm{co}(\mu)$ such that

$$\left\| (\tilde{\mathbf{1}}_U \mu')\, .\xi_r - \left(\int_\Omega \xi_r\, d\nu \right) m_K(U) \mathbf{1}_\Omega \right\|_{L^1(\Omega, \nu)} < \epsilon \qquad (1 \leq r \leq s).$$

To see this, note that the above arguments will show that $\left(\left(\int_\Omega \xi_r\, d\nu \right) m_K(U) \mathbf{1}_\Omega \right)_{r=1}^s$ belongs to the weak closure in $L^1(\Omega, \nu)^s$ of $\left\{ \left((\tilde{\mathbf{1}}_U \mu').\xi_r \right)_{r=1}^s : \mu' \in \mathrm{co}(\mu) \right\}$ and then apply [6], Corollary V.2.14 to $L^1(\Omega, \nu)^s$.

Finally an arbitrary $\xi \in L^1(\Omega, \nu)$ may be written as a linear combination of 4 non-negative functions. Hence, since $\xi \mapsto (\tilde{\mathbf{1}}_U \mu').\xi$ and $\xi \mapsto \int_\Omega \xi \, d\nu$ are linear, $\left(\int_\Omega \xi \, d\nu\right) m_K(U) \mathbf{1}_\Omega$ is the norm limit of a sequence of functions of the form $(\tilde{\mathbf{1}}_U \mu').\xi$ ($\mu' \in \text{co}(\mu)$) for each $\xi \in L^1(\Omega, \nu)$.

(ii) A similar argument, using the fact that

$$\int_K \phi \, d\mu^{*n} \to \int_K \phi \, dm_K \qquad \text{as } n \to \infty$$

proves the claim when ϕ is positive, whence it follows for all $\phi \in C(K)$. Alternatively, the claim may be deduced from (i) by approximating ϕ m_K-almost everywhere by a linear combination of characteristic functions of open sets. $\square$

The proof does not show which μ' in $\text{co}(\mu)$ give the approximations because of the appeal to [6], Corollary V.2.14 in passing from the weak closure to the norm closure. For this reason the lemma cannot be stated in the limit form

$$\left\| \left(\tilde{\mathbf{1}}_U \frac{1}{n} \sum_{k=1}^n \mu^{*k}\right).\xi - m_K(U) \left(\int_\Omega \xi \, d\nu\right) \mathbf{1}_\Omega \right\|_{L^1(\Omega, \nu)} \longrightarrow 0 \qquad \text{as } n \to \infty$$

despite having

$$\frac{1}{n} \sum_{k=1}^n \tilde{\mu}^{*k}(U) \to m_K(U), \qquad \frac{1}{n} \sum_{k=1}^n \mu^{*k}.\xi \to \left(\int_\Omega \xi \, d\nu\right) \mathbf{1}_\Omega$$

$$\text{and } \left\| f * \left(\frac{1}{n} \sum_{k=1}^n \mu^{*k}\right) \right\|_{L^1(G)} \to \|B(f + \mathcal{J}_\mu)\|_{L^1(\Omega, \nu)} \text{ as } n \to \infty.$$

The results below are most conveniently stated in a limit form and to do this it is necessary to adjoin an ideal point ∞ to $\text{co}(\mu)$ and to define a topology $\mathcal{T}$ of neighbourhoods of ∞.

The open neighbourhoods of ∞ include the sets:

$$A(U, \epsilon) = \{\mu' \in \text{co}(\mu) : |\tilde{\mu}'(U) - m_K(U)| < \epsilon\},$$

where $U \subset K$ is open;

$$B(\phi, \epsilon) = \left\{\mu' \in \text{co}(\mu) : \left|\int_K \phi \, d\tilde{\mu}' - \int_K \phi \, dm_K\right| < \epsilon\right\},$$

where $\phi \in C(K)$;

$$C(f, \epsilon) = \left\{\mu' \in \text{co}(\mu) : \left|\|f * \mu'\|_{L^1(G)} - \|B(f + \mathcal{J}_\mu)\|_{L^1(\Omega, \nu)}\right| < \epsilon\right\},$$

where $f \in L^1(G)$;

$$D(U, \xi, \epsilon) = \left\{\mu' \in \text{co}(\mu) : \left\|(\tilde{\mathbf{1}}_U \mu').\xi - m_K(U) \left(\int_\Omega \xi \, d\nu\right) \mathbf{1}_\Omega\right\|_{L^1(\Omega, \nu)} < \epsilon\right\},$$

where $U \subset K$ is open and $\xi \in L^1(\Omega, \nu)$; and

$$E(\phi, \xi, \epsilon) = \left\{\mu' \in \text{co}(\mu) : \left\|(\tilde{\phi}\mu').\xi - \left(\int_K \phi \, dm_K\right)\left(\int_\Omega \xi \, d\nu\right) \mathbf{1}_\Omega\right\|_{L^1(\Omega, \nu)} < \epsilon\right\},$$

where $\phi \in C(K)$ and $\xi \in L^1(\Omega, \nu)$.

It is clear that any intersection of finitely many sets of the forms A, B and C is not empty because $\frac{1}{n} \sum_{k=1}^n \mu^{*k}$ will belong to this intersection for n sufficiently

large. The proof of the Theorem shows that in fact any intersection of finitely many sets of the forms A to E is not empty. We define the topology on $\mathrm{co}(\mu)$ so that these intersections form a filterbase of neighbourhoods of ∞.

Definition 4.2. Let $\mathcal{T}$ be the topology on $\mathrm{co}(\mu)$ with base consisting of all finite intersections of sets of the forms A to E. If for every $\epsilon > 0$ there is an $M \in \mathcal{T}$ such that

$$\|\Phi(\mu') - L\| < \epsilon \text{ whenever } \mu' \in M$$

we shall say that

$$\Phi(\mu') \longrightarrow L \text{ as } \mu' \to \infty.$$

Theorem 4.1 may now be restated as: for every open $U \subset K$, every continuous ϕ on K and every $\xi \in L^1(\Omega, \nu)$,

$$\left(\tilde{\mathbf{1}}_U \mu'\right).\xi \longrightarrow m_K(U)\left(\int_\Omega \xi \, d\nu\right)\mathbf{1}_\Omega \qquad \text{and}$$

$$\left(\tilde{\phi}\mu'\right).\xi \longrightarrow \left(\int_K \phi \, dm_K\right)\left(\int_\Omega \xi \, d\nu\right)\mathbf{1}_\Omega \quad \text{as } \mu' \to \infty.$$

Lemma 4.3. *Let ϕ be a continuous function on K, $M_1 \in \mathcal{T}$ and $\epsilon > 0$. Then there are $\mu_1 \in M_1$ and $M_2 \in \mathcal{T}$ such that*

$$\left\|(\tilde{\phi}\mu_1) * \mu_2 - \left(\int_K \phi \, dm_K\right)\mu_2\right\|_{L^1(G)} < \epsilon$$

for every $\mu_2 \in M_2$.

Proof. Theorem 4.1 shows that there is a $\mu_1 \in M_1$ such that

$$\left\|(\tilde{\phi}\mu_1).\mathbf{1}_\Omega - \left(\int_\Omega \phi \, dm_K\right)\mathbf{1}_\Omega\right\|_{L^1(\Omega,\nu)} < \epsilon.$$

Since $L^1(\Omega, \nu)$ is isometric to $L^1(G)/\mathcal{J}_\mu$, it follows from the definition of the topology $\mathcal{T}$ that there is an $M_2 \in \mathcal{T}$ such that

$$\left\|(\tilde{\phi}\mu_1) * \mu_2 - \left(\int_K \phi \, dm_K\right)\mu_2\right\|_{L^1(G)} < \epsilon$$

for every $\mu_2 \in M_2$. $\qquad\square$

Lemma 4.4. *Let $U \subset K$ be an open set such that $m_K(\overline{U}) = m_K(U)$ and let $\epsilon > 0$ be given. Then there is an open neighbourhood, V, of e in K such that*

$$\|\phi * \mathbf{1}_U - \mathbf{1}_U\|_{L^1(K)} < \epsilon$$

for every $\phi \in C(K)$ satisfying: (i) $\phi \geq 0$; (ii) $\mathrm{supp}(\phi) \subset V$; and (iii) $\int_K \phi \, dm_K = 1$

Proof. Since m_K is a regular measure, there are: a compact set $C \subset U$; and an open set $W \supset \overline{U}$ such that $m_K(W \setminus C) < \epsilon$. Choose V to be a symmetric neighbourhood of e such that $VC \subset U$ and $V\overline{U} \subset W$. Then for every ϕ satisfying (i)–(iii),

$$(\phi * \mathbf{1}_U)(k) = \begin{cases} 1, & \text{if } k \in C \\ 0, & \text{if } k \neq W \\ 0 \leq c \leq 1, & \text{if } k \in W \setminus C. \end{cases}$$

Hence this V satisfies the assertion of the lemma. $\qquad\square$

Definition 4.5. For $\phi \in C(K)$ and $k \in K$ the *left translate* of ϕ by k is the function $\delta_k * \phi$. It will be denoted by ϕ_k.

We shall use the fact that if $\operatorname{supp}(\phi) \subset V$, then $\operatorname{supp}(\phi_k) \subset kV$.

Lemma 4.6. *Let $M_1 \in \mathcal{T}$ and $\epsilon > 0$ be given. Let $U \subset K$ be an open set such that $m_K(\overline{U}) = m_K(U)$ and let $k \in K$. Choose $V \subset K$ as in Lemma 4.4 and let $\phi \in C(K)$ satisfy* (i)–(iii) *of that Lemma. Then there are $\mu_1 \in M_1$ and $M_2 \in \mathcal{T}$ such that*

$$\left\| (\tilde{\phi}_k \mu_1) * (\tilde{\mathbf{1}}_U \mu_2) - (\phi_k * \mathbf{1}_U)^{\sim} \mu_2 \right\|_{L^1(G)} < 5\epsilon$$

for every $\mu_2 \in M_2$.

Proof. Let W and C be as chosen in the proof of Lemma 4.4. Then, by translation invariance of m_K, $m_K(kW \setminus kC) < \epsilon$. By Lemma 4.3 and the definition of $\mathcal{T}$ there are $\mu_1 \in M_1$ and $M_2 \in \mathcal{T}$ such that for every $\mu_2 \in M_2$

$$(15) \qquad \left\| (\tilde{\phi}_k \mu_1) * \mu_2 - \mu_2 \right\|_{L^1(G)} < \epsilon,$$

$$(16) \qquad \tilde{\mu}_2(kW \setminus kC) < \epsilon,$$

$$(17) \qquad \left(\int_K \phi_k \, d\tilde{\mu}_1 \right) \tilde{\mu}_2(U) < m_K(U) + \epsilon$$

$$(18) \qquad \text{and} \qquad \tilde{\mu}_2(kC) > m_K(kC) - \epsilon.$$

The convolution of ϕ_k with $\mathbf{1}_K$ is again the constant function and so we have

$$\mu_2 = \tilde{\mathbf{1}}_U \mu_2 + \tilde{\mathbf{1}}_{K \setminus U} \mu_2 = (\phi_k * \mathbf{1}_U)^{\sim} \mu_2 + (\phi_k * \mathbf{1}_{K \setminus U})^{\sim} \mu_2.$$

Hence (15) may be written

$$\left\| \left[(\tilde{\phi}_k \mu_1) * (\tilde{\mathbf{1}}_U \mu_2) - (\phi_k * \mathbf{1}_U)^{\sim} \mu_2 \right] + \left[(\tilde{\phi}_k \mu_1) * (\tilde{\mathbf{1}}_{K \setminus U} \mu_2) - (\phi_k * \mathbf{1}_{K \setminus U})^{\sim} \mu_2 \right] \right\|_{L^1(G)}$$
$$< \epsilon.$$

We abbreviate the first term inside the norm on the left of this inequality as F_U and the second as $F_{K \setminus U}$ so that the inequality becomes

$$(19) \qquad \left\| F_U + F_{K \setminus U} \right\|_{L^1(G)} < \epsilon.$$

The idea of the proof is to show that F_U and $F_{K \setminus U}$ have small mass where their supports overlap.

Now $\operatorname{supp}(\mathbf{1}_U \tilde{\mu}_2) \subset U$ and $\operatorname{supp}(\phi_k \tilde{\mu}_1) \subset kV$ so that

$$\operatorname{supp}\left((\phi_k \tilde{\mu}_1) * (\mathbf{1}_U \tilde{\mu}_2) \right) \subset kW.$$

It follows that $\mathbf{1}_{kW} \left((\phi_k \tilde{\mu}_1) * (\mathbf{1}_U \tilde{\mu}_2) \right) = (\phi_k \tilde{\mu}_1) * (\mathbf{1}_U \tilde{\mu}_2)$, whence

$$\tilde{\mathbf{1}}_{kW} \left((\tilde{\phi}_k \mu_1) * (\tilde{\mathbf{1}}_U \mu_2) \right) = (\tilde{\phi}_k \mu_1) * (\tilde{\mathbf{1}}_U \mu_2)$$

because each of these measures is positive. Similarly, $\operatorname{supp}\left((\phi_k * \mathbf{1}_U) \tilde{\mu}_2 \right) \subset kW$ and it follows that

$$(20) \qquad F_U = \tilde{\mathbf{1}}_{kW} F_U = \tilde{\mathbf{1}}_{kC} F_U + \tilde{\mathbf{1}}_{kW \setminus kC} F_U.$$

It may be seen in a similar way that $\operatorname{supp}(\tilde{F}_{K \setminus U}) \subset K \setminus kC$. Hence $\tilde{\mathbf{1}}_{kC} F_{K \setminus U} = 0$ and the intersection of the supports of F_U and $F_{K \setminus U}$ is contained in $\operatorname{supp}(\tilde{\mathbf{1}}_{kW \setminus kC})$.

Since $0 \le \phi_k * \mathbf{1}_U \le 1$ it follows from (16) that $((\phi_k * \mathbf{1}_U)\tilde{\mu}_2)(kW \setminus kC) < \epsilon$, which implies that

$$\left\| \tilde{\mathbf{1}}_{kW \setminus kC} \left((\phi_k * \mathbf{1}_U)^{\sim}\mu_2 \right) \right\|_{L^1(G)} < \epsilon.$$

Also, by (17) we have

$$(\phi_k \tilde{\mu}_1) * (\mathbf{1}_U \tilde{\mu}_2)(kW) < m_K(U) + \epsilon$$

and, since $\operatorname{supp}(\phi_k) \subset kV$ and $V^{-1}C \subset U$, we have

$$
\begin{aligned}
(\phi_k \tilde{\mu}_1) * (\mathbf{1}_U \tilde{\mu}_2)(kC) &= ((\phi_k \tilde{\mu}_1) * \tilde{\mu}_2)(kC) \\
&> \tilde{\mu}_2(kC) - \epsilon, \quad \text{by (15),} \\
&> m_K(kC) - 2\epsilon, \quad \text{by (18).}
\end{aligned}
$$

Hence $(\phi_k \tilde{\mu}_2) * (\mathbf{1}_U \tilde{\mu}_2)(kW \setminus kC) < 3\epsilon$ and

$$\left\| \tilde{\mathbf{1}}_{kW \setminus kC} \left((\tilde{\phi}_k \mu_1) * (\tilde{\mathbf{1}}_U \mu_2) \right) \right\|_{L^1(G)} < 3\epsilon.$$

Therefore $\|\tilde{\mathbf{1}}_{kW \setminus kC} F_U\|_{L^1(G)} < 4\epsilon$. Hence, by (20),

$$
\begin{aligned}
\|F_U\|_{L^1(G)} &= \left\| \tilde{\mathbf{1}}_{kC} F_U \right\|_{L^1(G)} + \left\| \tilde{\mathbf{1}}_{kW \setminus kC} F_U \right\|_{L^1(G)} \\
&< \left\| \tilde{\mathbf{1}}_{kC} \left(F_U + F_{K \setminus U} \right) \right\|_{L^1(G)} + 4\epsilon, \quad \text{since } \tilde{\mathbf{1}}_{kC} F_{K \setminus U} = 0, \\
&\le \left\| F_U + F_{K \setminus U} \right\|_{L^1(G)} + 4\epsilon, \\
&< 5\epsilon, \quad \text{by (19).}
\end{aligned}
$$

$\square$

Theorem 4.7. *Let $M_1 \in \mathcal{T}$ and $\epsilon > 0$ be given. Let $\phi_1, \phi_2 \in C(K)$. Then there are $\mu_1 \in M_1$ and $M_2 \in \mathcal{T}$ such that*

$$\left\| (\tilde{\phi}_1 \mu_1) * (\tilde{\phi}_2 \mu_2) - (\phi_1 * \phi_2)^{\sim}\mu_2 \right\|_{L^1(G)} < \epsilon$$

for every $\mu_2 \in M_2$.

Proof. Suppose for now that $\phi_1 \ge 0$ and $\int_K \phi_1 \, dm_K = 1$.

Since $m_K(K) = 1$, there are at most countably many s in $\phi_2(K)$ such that the level set $L_s = \{k \in K : \phi_2(k) = s\}$ has non-zero measure. By covering $\phi_2(K)$ with open disks whose boundaries do not contain any such s, we may find complex values s_i and mutually disjoint open sets $U_i \subset K$, $i = 1, 2, \ldots, p$, such that

$$(21) \qquad\qquad |s_i - \phi_2(k)| < \epsilon \quad \text{for} \quad k \in U_i$$

$$(22) \qquad\qquad \text{and} \qquad K \setminus \left(\bigcup_{i=1}^{p} U_i \right) \quad \text{has measure 0.}$$

Then $|\phi_2(k) - \sum_{i=1}^{p} s_i \mathbf{1}_{U_i}(k)| < \epsilon$ for all $k \in \bigcup_{i=1}^{p} U_i$.

Note that (22) implies that $m_K(\overline{U}_i) = m_K(U_i)$ for each i. Hence a symmetric neighbourhood V of e in K may be chosen as in Lemma 4.4 such that $VC_i \subset U_i$ and $V\overline{U}_i \subset W_i$ where $C_i \subset U_i \subset W_i$ satisfy

$$m_K(W_i \setminus C_i) < \frac{\epsilon}{5\sum_{i=1}^{p} |s_i|}.$$

Since K is compact there are $k_1, k_2, \ldots, k_q$ in K such that $\{k_j V : j = 1, 2, \ldots, q\}$ is an open cover. Let $\{u^{(j)} : j = 1, 2, \ldots, q\}$ be a partition of unity subordinate

to this open cover, so that $u^{(j)} \in C(K)$ and $\text{supp}(u^{(j)}) \subset k_j V$ for each j and $\sum_{j=1}^{q} u^{(j)} = \mathbf{1}_K$. Define

$$t_j = \int_K u^{(j)} \phi_1 \, dm_K$$

and

$$\psi^{(j)} = \begin{cases} t_j^{-1} \delta_{k_j^{-1}} * (u^{(j)}\phi_1), & \text{if } t_j \neq 0 \\ 0, & \text{if } t_j = 0. \end{cases}$$

Then: (i) $\psi^{(j)} \geq 0$ (ii) $\text{supp}(\psi^{(j)}) \subset V$, (iii) $\int_K \psi^{(j)} \, dm_K = 1$ for each j with $t_j \neq 0$ and

$$(23) \qquad \sum_{j=1}^{q} t_j \psi_{k_j}^{(j)} = \phi_1.$$

By Lemma 4.6 there are $\mu_1 \in M_1$ and $M_2 \in \mathcal{T}$ such that

$$\left\| \left(\tilde{\psi}_{k_j}^{(j)} \mu_1 \right) * \left(\tilde{\mathbf{1}}_{U_i} \mu_2 \right) - \left(\psi_{k_j}^{(j)} * \mathbf{1}_{U_i} \right)^{\sim} \mu_2 \right\|_{L^1(G)} < \frac{\epsilon}{\sum_{i=1}^{p} |s_i|}$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$ and for every $\mu_2 \in M_2$. Hence

$$\left\| \left(\tilde{\psi}_{k_j}^{(j)} \mu_1 \right) * \left(\sum_{i=1}^{p} s_i \tilde{\mathbf{1}}_{U_i} \mu_2 \right) - \left(\psi_{k_j}^{(j)} * \sum_{i=1}^{p} s_i \mathbf{1}_{U_i} \right)^{\sim} \mu_2 \right\|_{L^1(G)} < \epsilon$$

for $1 \leq j \leq q$ and for every $\mu_2 \in M_2$. Now (23) implies that

$$\left\| \left(\tilde{\phi}_1 \mu_1 \right) * \left(\sum_{i=1}^{p} s_i \tilde{\mathbf{1}}_{U_i} \mu_2 \right) - \left(\phi_1 * \sum_{i=1}^{p} s_i \mathbf{1}_{U_i} \right)^{\sim} \mu_2 \right\|_{L^1(G)} < \epsilon$$

for every $\mu_2 \in M_2$, where we have used that $\sum_{j=1}^{q} t_j = \int_K \phi_1 \, dm_K = 1$.

Since $\left| \sum_{i=1}^{p} s_i \mathbf{1}_{U_i}(k) - \phi_2(k) \right| < \epsilon$ for every k not in a set of measure 0,

$$\left\| \phi_1 * \phi_2 - \phi_1 * \left(\sum_{i=1}^{p} s_i \mathbf{1}_{U_i} \right) \right\|_{\infty} < \epsilon.$$

Furthermore, by choosing a smaller M_2 if necessary, it may be supposed that

$$\| \phi_1 \tilde{\mu}_1 \|_{M(K)} \left\| \left(\sum_{i=1}^{p} s_i \mathbf{1}_{U_i} \right) \tilde{\mu}_2 - \phi_2 \tilde{\mu}_2 \right\|_{M(K)} < \epsilon.$$

for every $\mu_2 \in M_2$. Therefore

$$\left\| \left(\tilde{\phi}_1 \mu_1 \right) * \left(\tilde{\phi}_2 \mu_2 \right) - (\phi_1 * \phi_2)^{\sim} \mu_2 \right\|_{L^1(G)} < 3\epsilon$$

for every $\mu_2 \in M_2$.

The desired estimate may be obtained for arbitrary $\phi_1 \in C(K)$ by writing ϕ_1 as a linear combination of four positive functions whose integral over K is 1 and by scaling ϵ. $\qquad\square$

The final approximation theorem in this section relies on the following lemma, which is an immediate consequence of Theorem 4.1.

Lemma 4.8. *Let $F \subset C(K)$ be compact. Then*

$$(\tilde{\phi}\mu').\mathbf{1}_\Omega \to \left(\int_K \phi\, dm_K\right)\mathbf{1}_\Omega$$

uniformly for ϕ in F as $\mu' \to \infty$.

Note that, since K is compact, the set $\phi_K = \{\phi_k = \delta_k * \phi : k \in K\}$ is compact for every $\phi \in C(K)$.

Theorem 4.9. *Let $\phi \in C(K)$ and $f \in L^1(G)$. Then*

$$\left(\tilde{\phi}(f * \mu')\right).\mathbf{1}_\Omega \to \left(\int_K \phi\, dm_K\right)f.\mathbf{1}_\Omega \ as\ \mu' \to \infty.$$

Proof. Note first of all that $f * \mu' = \int_G f(x)\delta_x * \mu'\, dm(x)$ and $\tilde{\phi}(\delta_x * \mu') = \delta_x * (\tilde{\phi}_{x^{-1}}\mu')$, whence

$$\tilde{\phi}(f * \mu') = \int_G f(x)\delta_x * (\tilde{\phi}_{x^{-1}}\mu')\, dm(x).$$

Also $f.\mathbf{1}_\Omega = \int_G f(x)\delta_x.\mathbf{1}_\Omega\, dm(x)$. Hence

$$\left\|\left(\tilde{\phi}(f * \mu')\right).\mathbf{1}_\Omega - \left(\int_K \phi\, dm_K\right)f.\mathbf{1}_\Omega\right\|_{L^1(\Omega,\nu)}$$

$$= \left\|\int_G f(x)\delta_x.\left(\left(\tilde{\phi}_{x^{-1}}\mu'\right).\mathbf{1}_\Omega - \left(\int_K \phi\, dm_K\right)\mathbf{1}_\Omega\right)dm(x)\right\|_{L^1(\Omega,\nu)}$$

$$\leq \int_G |f(x)|\left\|\left(\tilde{\phi}_{x^{-1}}\mu'\right).\mathbf{1}_\Omega - \left(\int_K \phi\, dm_K\right)\mathbf{1}_\Omega\right\|_{L^1(\Omega,\nu)} dm(x).$$

Now $\{\tilde{\phi}_{x^{-1}} : x \in G\} \subset \tilde{\phi}_K$, which is compact, and so

$$\left(\tilde{\phi}_{x^{-1}}\mu'\right).\mathbf{1}_\Omega \to \left(\int_K \phi_{x^{-1}}\, dm_K\right)\mathbf{1}_\Omega$$

uniformly for x in G as $\mu' \to \infty$ and $\int_K \phi_{x^{-1}}\, dm_K = \int_K \phi\, dm_K$ for every $x \in G$ because m_K is translation invariant. Hence

$$\left\|\left(\tilde{\phi}_{x^{-1}}\mu'\right).\mathbf{1}_\Omega - \left(\int_K \phi\, dm_K\right)\mathbf{1}_\Omega\right\|_{L^1(\Omega,\nu)} \to 0$$

uniformly for $x \in G$ as $\mu' \to \infty$. Therefore

$$\left\|\left(\tilde{\phi}(f * \mu')\right).\mathbf{1}_\Omega - \left(\int_K \phi\, dm_K\right)f.\mathbf{1}_\Omega\right\|_{L^1(\Omega,\nu)} \to 0 \ as\ \mu' \to \infty.$$

$\square$

4.1. **Comparison with weak containment.** In the case when G is amenable, the probability measure μ may be chosen so that Ω is a one point set, see [11], [24] or [37], Theorem 1.2. Then $\mathcal{J}_\mu = L_0^1(G)$ and

$$f.\mathbf{1}_\Omega = \left(\int_G f\, dm\right)\mathbf{1}_\Omega \qquad (f \in L^1(G)).$$

Theorem 4.9 then reduces to the statement that

$$\lim_{\mu' \to \infty}\int_G \tilde{\phi}(f * \mu')\, dm = \left(\int_K \phi\, dm_K\right)\left(\int_G f\, dm\right) \qquad (\phi \in C(K),\ f \in L^1(G)).$$

This restricted case may be seen directly because $\mathcal{J}_\mu$ being equal to $L_0^1(G)$ means that

$$(24) \qquad \lim_{n \to \infty} \left\| f * \left(\frac{1}{n} \sum_{k=1}^{n} \mu^{*k} \right) - \left(\int_G f \, dm \right) \left(\frac{1}{n} \sum_{k=1}^{n} \mu^{*k} \right) \right\|_{L^1(G)} = 0$$

for $f \in L^1(G)$, which implies that the trivial representation of G is weakly contained in the regular representation and also implies the Property P_1, see [20], Section 8.4 and [8], Sections 3.2, 3.5, 3.7. Then the special case of Theorem 4.9 follows because

$$\lim_{n \to \infty} \int_G \tilde{\phi} \mu^{*n} \, dm = \lim_{n \to \infty} \int_K \phi \, d\tilde{\mu}^{*n} \qquad (\phi \in C(K))$$

and, since $\mu^{*n} \overset{\text{w-*}}{\to} m_K$, this limit equals $\int_K \phi \, dm_K$.

That the approximation theorems in this section are much weaker than Property P_1 or weak containment of the trivial representation may be seen by considering the case when $K = \{e\}$. In that case Theorem 4.7 reduces in effect to the familiar estimate

$$\left\| \mu * \left(\frac{1}{n} \sum_{k=1}^{n} \mu^{*k} \right) - \frac{1}{n} \sum_{k=1}^{n} \mu^{*k} \right\|_{L^1(G)} < \frac{2}{n},$$

which is of course much weaker than (24). This latter estimate is essential for the definitions of the bounded approximate identities in $\mathcal{J}_\mu$ and $_\mu\mathcal{J}$ and also for the application of the ergodic theorem described in section 3. The estimates in Theorems 4.1, 4.7 and 4.9 are used in an analogous way in the next section.

5. Proof of the Main Theorem

The proof of the theorem follows the same steps as the case when $\mathcal{I}$ is $L_0^1(G)$. First the ideals $\mathcal{L}$ and $\mathcal{R}$ are defined and it is shown that they have bounded approximate identities. Next the quotient $L^1(G)/\mathcal{L}$ and its subspace $\mathcal{I}/L$ are described and it is shown that the left bounded approximate identity for $\mathcal{R}$ is a bounded approximate identity for $\mathcal{I}/\mathcal{L}$. Finally it is deduced that $\mathcal{I} = \mathcal{L} + \mathcal{R}$.

Let $\rho_\mathcal{I}$, K and z be as defined in section 2. As in section 4, let μ be a fixed probability measure on G which is absolutely continuous with respect to Haar measure and satisfies $\text{supp}(\mu) = G$. We now impose one further condition on μ.

Since G is σ-compact, it has a compact normal subgroup N such that G/N is separable, see [10], Theorem 8.7. Then m_N, the normalized Haar measure on N, belongs to the centre of $M(G)$. We require that $\mu = m_N * \mu$, which may be achieved by replacing μ by $m_N * \mu$ if necessary. The quotient map $L^1(G) \to L^1(G)/\mathcal{J}_\mu$ then factors through the map $T_N : L^1(G) \to L^1(G/N)$, see [20], 3.4.4 for the definition of T_N. Since $L^1(G/N)$ is separable, it follows that $L^1(\Omega, \nu)$ is separable. (It may be shown that in fact the quotient map factors through T_N even without supposing that $\mu = m_n * \mu$, see [37], Proposition 5.2 for a related result.)

5.1. Definition of $\mathcal{L}$ and $\mathcal{R}$.
For the definition of $\mathcal{L}$, choose probability measures μ_1, μ_2, $\ldots$, from $\text{co}(\mu)$ satisfying

$$(25) \qquad \|(\tilde{z}\mu_n) * (\tilde{z}\mu_{n+1}) - \tilde{z}\mu_{n+1}\|_{L^1(G)} < \frac{1}{2^n} \qquad (n = 1, \, 2, \, \ldots).$$

These may be chosen by applying Theorem 4.7 repeatedly as follows. Since $z*z = z$ there are $\mu_1 \in \mathrm{co}(\mu)$ and $M_1 \in \mathcal{T}$ such that

$$\|(\tilde{z}\mu_1) * (\tilde{z}\mu') - \tilde{z}\mu'\|_{L^1(G)} < \frac{1}{2}$$

for every $\mu' \in M_1$. Choose such μ_1 and M_1. Now suppose that $\mu_n \in \mathrm{co}(\mu)$ and $M_n \in \mathcal{T}$ have been chosen such that

$$\|(\tilde{z}\mu_n) * (\tilde{z}\mu') - \tilde{z}\mu'\|_{L^1(G)} < \frac{1}{2^n}$$

for every $\mu' \in M_n$. Then by Theorem 4.7, there are $\mu_{n+1} \in M_n$ and $M_{n+1} \in \mathcal{T}$ such that

$$\|(\tilde{z}\mu_{n+1}) * (\tilde{z}\mu') - \tilde{z}\mu'\|_{L^1(G)} < \frac{1}{2^{n+1}}$$

for every $\mu' \in M_{n+1}$.

It follows from the definition of $\mathcal{T}$ that, if C is a countable subset of $C(K)$, then $\{\mu_n\}$ may also be supposed to satisfy $\int_G \tilde{\phi}\, d\mu_n \to \int_K \phi\, dm_K$ as $n \to \infty$ for every ϕ in C. Now K is a closed subgroup of a finite dimensional matrix group and is therefore separable. Hence $C(K)$ has a dense countable subset and $\{\mu_n\}$ may be chosen to satisfy

$$(26) \qquad \lim_{n\to\infty} \int_G \tilde{\phi}\, d\mu_n = \int_K \phi\, dm_K \text{ for every } \phi \in C(K).$$

The definition of $\mathcal{J}_\mu$ implies immediately that $f * \left(\frac{1}{n}\sum_{k=1}^n \mu^{*k}\right) \to 0$ as $n \to \infty$ for every f in $\mathcal{J}_\mu$ and so it may be further supposed that

$$(27) \qquad \lim_{n\to\infty} f * \mu_n = 0 \text{ for every } f \in \mathcal{J}_\mu.$$

Now (26) and (27), together with the fact that $L^1(\Omega, \nu)$ is separable, imply that the filterbase defining the topology $\mathcal{T}$ can be chosen to be countable. Hence it may be supposed that, in this topology, $\mu_n \to \infty$ as $n \to \infty$.

Define, for positive integers $m \le n$,

$$u_{m,n} = \prod_{r=m}^n (\delta_e - \tilde{z}\mu_r) = (\delta_e - \tilde{z}\mu_m) * (\delta_e - \tilde{z}\mu_{m+1}) * \cdots * (\delta_e - \tilde{z}\mu_n)$$

(The factors in this product do not commute and so it is necessary to specify the order of the product.) Then

$$(28) \qquad u_{m,n+1} - u_{m,n} = -u_{m,n} * (\tilde{z}\mu_{n+1})$$

where $\|u_{m,n} * (\tilde{z}\mu_{n+1})\|_{M(G)} < \left(\frac{1}{2}\right)^n \|u_{m,n-1}\|_{M(G)}$. Hence

$$\|u_{m,n+1}\|_{M(G)} \le \|u_{m,n}\|_{M(G)} + \left(\frac{1}{2}\right)^n \|u_{m,n-1}\|_{M(G)},$$

which implies that

$$\|u_{m,n}\|_{M(G)} \le e \|u_{m,m}\|_{M(G)} \le e(1 + \|z\|_\infty)$$

for all m and n. Then, by (28),

$$\|u_{m,n+1} - u_{m,n}\|_{M(G)} \le \left(\frac{1}{2}\right)^n e(1 + \|z\|_\infty)$$

so that $\{u_{m,n}\}_{n=1}^\infty$ is a Cauchy sequence for each m.

Define
$$u_m = \lim_{n \to \infty} u_{m,n} \qquad (m = 1, \, 2, \, 3, \, \dots).$$

Then for each m

(29) $$\|u_m\|_{M(G)} \le e(1 + \|z\|_\infty).$$

(It is not difficult to see that in fact $\limsup_{m \to \infty} \|u_m\|_{M(G)} \le 1 + \|z\|_{L^1(K)}$.)

When $n \ge m$ we have

(30)
$$
\begin{aligned}
\|u_m * u_n - u_m\|_{M(G)} &= \|(u_m - u_{m,n-1}) * u_n\|_{M(G)} \\
&\le e(1 + \|z\|_\infty) \|u_m - u_{m,n-1}\|_{M(G)} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Also, for $f \in L^1(G)$ and $v \in \mathcal{V} = L^1(G)/\mathcal{I}$ we have
$$f * v = \int_G f(x) \rho_{\mathcal{I}}(x) v \, dm(x).$$

Hence, for each $v \in \mathcal{V}$

$$
\begin{aligned}
\lim_{n \to \infty} (\tilde{z}\mu_n) * v &= \lim_{n \to \infty} \int_G \tilde{z}(x) \rho_{\mathcal{I}}(x) v \, d\mu_n(x) \\
&= \int_K z(k) k v \, dm_K(k), \quad \text{by (26)}, \\
&= v, \quad \text{since } z \text{ is the unit in } \mathcal{V}.
\end{aligned}
$$

Therefore $u_m * v = 0$ for every $v \in \mathcal{V}$, and it follows that

(31) $$f * u_m \in \mathcal{I} \qquad (m = 1, 2, 3, \dots, \ f \in L^1(G)).$$

Define
$$\mathcal{L} = \{f \in L^1(G) : \lim_{m \to \infty} f * u_m = f\}.$$

Then $\mathcal{L}$ is a left ideal in $L^1(G)$ and is closed because $\{u_m\}$ is bounded, see (29).

It follows from (30) that $\mathcal{L} \ne 0$ and, since $\mathcal{I}$ is closed, from (31) that $\mathcal{L} \subset \mathcal{I}$. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be a bounded approximate identity for $L^1(G)$. Then $\{u_\lambda * u_m\}_{(\lambda,m) \in \Lambda \times \mathbb{Z}^+}$ is contained in $\mathcal{L}$ by (30), is a bounded net by (29) and is a right approximate identity for $\mathcal{L}$ by definition. Therefore $\mathcal{L}$ is a closed, left ideal having a right bounded approximate identity.

The right ideal $\mathcal{R}$ is defined in an analogous way. The probability measures μ_n must be chosen to satisfy, in addition to (25)-(27),

(32) $$\|(\tilde{z}\mu_{n+1}) * (\tilde{z}\mu_n) - \tilde{z}\mu_{n+1}\|_{L^1(G)} < \frac{1}{2^n} \qquad (n = 1, 2, \dots).$$

This condition may be imposed on μ_n as it is chosen or, alternatively, if μ is chosen to be a symmetric measure, then it follows automatically from (25). Define, for $m \le n$,
$$w_{m,n} = (\delta_e - \tilde{z}\mu_n) * \cdots * (\delta_e - \tilde{z}\mu_{m+1}) * (\delta_e - \tilde{z}\mu_m)$$

and
$$w_m = \lim_{n \to \infty} w_{m,n}.$$

Next define
$$\mathcal{R} = \{f \in L^1(G) : \lim_{m \to \infty} w_m * f = f\}.$$

Then $\mathcal{R}$ is a closed right ideal in $L^1(G)$ with left bounded approximate identity $\{w_m * u_\lambda\}_{(m,\lambda) \in \mathbb{Z}^+ \times \Lambda}$.

5.2. **Description of $L^1(G)/\mathcal{L}$ and $\mathcal{I}/\mathcal{L}$.** The quotient space $L^1(G)/\mathcal{L}$ may be seen to be isomorphic to $\mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$. For this, define a G-action on $\mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$ by

$$x.(v \otimes \xi) = \rho_\mathcal{I}(x)v \otimes \delta_x.\xi \qquad (v \in \mathcal{V}, \ \xi \in L^1(\Omega, \nu)).$$

Then the map $x \mapsto x.F$ is continuous and bounded for each F in $\mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$ and so we may define an action of $L^1(G)$ on $\mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$ by

$$f.F = \int_G f(x)x.F \, dm(x) \qquad (f \in L^1(G), \ F \in \mathcal{V} \hat{\otimes} L^1(\Omega, \nu))$$

so that $\mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$ becomes a Banach left $L^1(G)$-module.

Define a left $L^1(G)$-module homomorphism $Q : L^1(G) \to \mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$ by

$$Q(f) = f.(z \otimes \mathbf{1}_\Omega) \qquad (f \in L^1(G)).$$

Recall that we have identified $\mathcal{V}$ with an ideal in $L^1(K)$, so that z belongs to $\mathcal{V}$. It will be useful to have the following description of $Q(f)$ in terms of matrix coefficients. We abbreviate the double sum $\sum_{\sigma \in S} \sum_{\alpha,\beta=1}^{d_\sigma}$ as $\sum_{\sigma,\alpha,\beta}$.

Lemma 5.1. $Q(f) = \sum_{\sigma,\alpha,\beta} \bar{c}_{\alpha\beta}^\sigma \otimes (d_\sigma \tilde{c}_{\alpha\beta}^\sigma f).\mathbf{1}_\Omega.$

Proof. By definition
$$
\begin{aligned}
Q(f) &= f.(z \otimes \mathbf{1}_\Omega) \\
&= \int_G f(x)(\rho_\mathcal{I}(x)z \otimes \delta_x.\mathbf{1}_\Omega) \, dm(x) \\
&= \sum_{\sigma,\beta} d_\sigma \int_G f(x)(\rho_\mathcal{I}(x)\bar{c}_{\beta\beta}^\sigma \otimes \delta_x.\mathbf{1}_\Omega) \, dm(x), \text{ by (3),} \\
&= \sum_{\sigma,\alpha,\beta} \int_G d_\sigma f(x)\tilde{c}_{\alpha\beta}^\sigma(x)(\bar{c}_{\alpha\beta}^\sigma \otimes \delta_x.\mathbf{1}_\Omega) \, dm(x), \text{ by (2),} \\
&= \sum_{\sigma,\alpha,\beta} \bar{c}_{\alpha\beta}^\sigma \otimes (d_\sigma \tilde{c}_{\alpha\beta}^\sigma f).\mathbf{1}_\Omega.
\end{aligned}
$$
$\square$

It may now be shown that Q induces an isomorphism between $L^1(G)/\mathcal{L}$ and $\mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$.

Lemma 5.2. *Q is surjective and the kernel of Q equals $\mathcal{L}$.*

Proof. Let F be in $\mathcal{V} \hat{\otimes} L^1(\Omega, \nu)$. Since $\{\bar{c}_{\alpha\beta}^\sigma\}$ is a basis for $\mathcal{V}$, F may be written uniquely as $F = \sum_{\sigma,\alpha,\beta} \bar{c}_{\alpha\beta}^\sigma \otimes \xi_{\alpha\beta}^\sigma$ for some $\{\xi_{\alpha\beta}^\sigma\} \subset L^1(\Omega, \nu)$ and there is a constant $M > 0$ such that

$$\left\| \sum_{\sigma,\alpha,\beta} \bar{c}_{\alpha\beta}^\sigma \otimes \xi_{\alpha\beta}^\sigma \right\| \leq \sum_{\sigma,\alpha,\beta} \|\bar{c}_{\alpha\beta}^\sigma\|_\infty \|\xi_{\alpha\beta}^\sigma\|_{L^1(\Omega,\nu)} \leq M \left\| \sum_{\sigma,\alpha,\beta} \bar{c}_{\alpha\beta}^\sigma \otimes \xi_{\alpha\beta}^\sigma \right\|.$$

The description of the map $B : L^1(G)/\mathcal{J}_\mu \to L^1(\Omega, \nu)$ in section 3 implies that functions $f_{\alpha\beta}^\sigma \in L^1(G)$ may be chosen such that $\|f_{\alpha\beta}^\sigma\|_{L^1(G)} < 2\|\xi_{\alpha\beta}^\sigma\|_{L^1(\Omega,\nu)}$ and with $\xi_{\alpha\beta}^\sigma = f_{\alpha\beta}^\sigma.\mathbf{1}_\Omega$.

Now by Lemma 5.1

$$Q\left(\sum_{\sigma,\alpha,\beta} \tilde{c}^\sigma_{\alpha\beta}(f^\sigma_{\alpha\beta} * \mu')\right) = \sum_{\tau,\gamma,\delta} \bar{c}^\tau_{\gamma\delta} \otimes \left(\sum_{\sigma,\alpha,\beta} d_\tau \tilde{c}^\tau_{\gamma\delta}\tilde{\tilde{c}}^\sigma_{\alpha\beta}(f^\sigma_{\alpha\beta} * \mu')\right).\mathbf{1}_\Omega$$

and, by Theorem 4.9, as $\mu' \to \infty$ the right hand side converges to

$$\sum_{\tau,\gamma,\delta} \bar{c}^\tau_{\gamma\delta} \otimes \left(\sum_{\sigma,\alpha,\beta} d_\tau \int_K c^\tau_{\gamma\delta}\bar{c}^\sigma_{\alpha\beta}\, dm_K \xi^\sigma_{\alpha\beta}\right) \;=\; \sum_{\sigma,\alpha,\beta} \bar{c}^\sigma_{\alpha\beta} \otimes \xi^\sigma_{\alpha\beta}, \qquad \text{by (1),}$$
$$= \; F.$$

Since for every $\mu' \in \mathrm{co}(\mu)$

$$\left\| \sum_{\sigma,\alpha,\beta} \tilde{c}^\sigma_{\alpha\beta}(f^\sigma_{\alpha\beta} * \mu') \right\|_{L^1(G)} \leq 2M \left\| \sum_{\sigma,\alpha,\beta} \bar{c}^\sigma_{\alpha\beta} \otimes \xi^\sigma_{\alpha\beta} \right\|,$$

we have thus shown that for every $\epsilon > 0$ there is

$$g = \sum_{\sigma,\alpha,\beta} \tilde{c}^\sigma_{\alpha\beta}(f^\sigma_{\alpha\beta} * \mu') \text{ in } L^1(G)$$

with $\|g\|_{L^1(G)} \leq 2M\|F\|$ and $\|F - Q(g)\| < \epsilon$. It follows that Q is surjective.

A similar calculation to that in the last paragraph shows that

$$Q(\tilde{z}\mu') \to z \otimes \mathbf{1}_\Omega \text{ as } \mu' \to \infty.$$

Hence $Q(u_m) = 0$ for each m and it follows that $\mathcal{L} \subset \mathrm{kernel}(Q)$.

On the other hand, suppose that $Q(f) = 0$. Then, by Lemma 5.1,

$$\sum_{\sigma,\alpha,\beta} \bar{c}^\sigma_{\alpha\beta} \otimes (d_\sigma \tilde{c}^\sigma_{\alpha\beta}f).\mathbf{1}_\Omega = 0.$$

Since $\{\bar{c}^\sigma_{\alpha\beta}\}$ is a linearly independent set, and by the construction of (Ω, ν), it follows that

$$\tilde{c}^\sigma_{\alpha\beta}f \in \mathcal{J}_\mu \text{ for all } \sigma, \alpha, \beta.$$

Hence

$$\begin{aligned}
f * (\tilde{z}\mu_m) &= \int_G f(x)\delta_x * (\tilde{z}\mu_m)\, dm(x) \\
&= \int_G f(x)\,(\delta_x * \tilde{z})\,(\delta_x * \mu_m)\, dm(x) \\
&= \int_G f(x)\left(\sum_{\sigma,\alpha,\beta} d_\sigma \tilde{c}^\sigma_{\alpha\beta}(x)\tilde{\tilde{c}}^\sigma_{\alpha\beta}\right)(\delta_x * \mu_m)\, dm(x), \quad \text{by (2) and (3),} \\
&= \sum_{\sigma,\alpha,\beta} \tilde{\tilde{c}}^\sigma_{\alpha\beta}\left((d_\sigma \tilde{c}^\sigma_{\alpha\beta}f) * \mu_m\right), \\
&\to \; 0 \text{ as } m \to \infty, \text{ by (27).}
\end{aligned}$$

Therefore $f * u_m \to f$ as $m \to \infty$ and $\mathrm{kernel}(Q) \subset \mathcal{L}$. $\qquad\square$

It may now be seen that $Q(\mathcal{I}) = \mathcal{V} \otimes L_0^1(\Omega, \nu)$. For this, define the operator $R : \mathcal{V} \otimes L^1(\Omega, \nu) \to \mathcal{V}$ on simple tensors by

$$R(v \otimes \xi) = \left(\int_\Omega \xi \, d\nu \right) v$$

and extend to $\mathcal{V} \otimes L^1(\Omega, \nu)$ by linearity. Then it is clear that R is surjective and it is also an $L^1(G)$-module homomorphism because

$$
\begin{aligned}
R\left(f.(v \otimes \xi)\right) &= R\left(\int_G f(x) \left(\rho_\mathcal{I}(x)v \otimes \delta_x.\xi \right) dm(x) \right) \\
&= \int_G f(x) R(\rho_\mathcal{I}(x)v \otimes \delta_x.\xi) \, dm(x) \\
&= \int_G f(x) \left(\int_\Omega \delta_x.\xi \, d\nu \right) \rho_\mathcal{I}(x)v \, dm(x) \\
&= \left(\int_\Omega \xi \, d\nu \right) \int_G f(x)\rho_\mathcal{I}(x)v \, dm(x) \\
&= \left(\int_\Omega \xi \, d\nu \right) f.v \\
&= f.R(v \otimes \xi).
\end{aligned}
$$

Hence $R\left(Q\left(f\right)\right) = R\left(f.(z \otimes \mathbf{1}_\Omega)\right) = f.z = f + \mathcal{I}$ so that $R \circ Q$ is the quotient map $L^1(G) \to L^1(G)/\mathcal{I}$. It follows that $Q(\mathcal{I}) = \text{kernel}(R) = \mathcal{V} \otimes L_0^1(\Omega, \nu)$.

5.3. **The approximate identity for $Q(\mathcal{I})$.** Next we show that the left bounded approximate identity for $\mathcal{R}$ is also an approximate identity for $\mathcal{V} \otimes L_0^1(\Omega, \nu)$. For this, note that for each $\bar{c}_{\alpha\beta}^\sigma$ in the basis for $\mathcal{V}$ and each ξ in $L_0^1(\Omega, \nu)$

$$
\begin{aligned}
(\tilde{z}\mu_m). \left(\bar{c}_{\alpha\beta}^\sigma \otimes \xi \right) &= \int_G \tilde{z}(x)\mu_m(x)(\rho_\mathcal{I}(x)\bar{c}_{\alpha\beta}^\sigma \otimes \delta_x.\xi) \, dm(x) \\
&= \sum_{\gamma=1}^{d_\sigma} \bar{c}_{\gamma\beta}^\sigma \otimes \left(\tilde{z}\tilde{c}_{\gamma\alpha}^\sigma \mu_m \right).\xi, \quad \text{by (2)}, \\
&\to \sum_{\gamma=1}^{d_\sigma} \bar{c}_{\gamma\beta}^\sigma \otimes \left(\int_K z c_{\gamma\alpha}^\sigma \, dm_K \right) \left(\int_\Omega \xi \, d\nu \right) \mathbf{1}_\Omega, \quad \text{as } m \to \infty \\
&= 0,
\end{aligned}
$$

where we have used Theorem 4.1(ii) and that $\mu_m \to \infty$ as $m \to \infty$. Therefore $w_m.(\bar{c}_{\alpha\beta}^\sigma \otimes \xi) \to \bar{c}_{\alpha\beta}^\sigma \otimes \xi$ as $m \to \infty$ and $\{w_m * u_\lambda\}_{(m,\lambda) \in Z^+ \times \Lambda}$ is a left bounded approximate identity $\mathcal{V} \otimes L_0^1(\Omega, \nu)$ as well as for $\mathcal{R}$.

5.4. **The decomposition of $\mathcal{I}$.** Let f be in $\mathcal{I}$. Then $Q(f)$ belongs to $\mathcal{V} \otimes L_0^1(\Omega, \nu)$. Hence, by Cohen's factorization theorem, in its module version given in [1], Theorem 11.10 and [10], Theorem 32.22, there are w in $\mathcal{R}$ and F in $\mathcal{V} \otimes L_0^1(\Omega, \nu)$ such that $Q(f) = w.F$. There is g_1 in $\mathcal{I}$ such that $F = Q(g_1)$ and so we have $f = w * g_1 + g_2$ for some g_2 in $\mathcal{L}$, showing that f belongs to $\mathcal{R} + \mathcal{L}$ as desired.

6. Corollaries and Further Questions

6.1. **Automatic continuity.** The theorem implies that finite-codimensional ideals in group algebras are idempotent, that every element is the sum of 2 products in fact. It is shown in [5] that various automatic continuity results follow.

Corollary 6.1. *Let G be a locally compact group. Then:*
- (i) *each finite-codimensional ideal in $L^1(G)$ is closed;*
- (ii) *each derivation D from $L^1(G)$ to a finite-dimensional Banach bimodule X is continuous; and*
- (iii) *each homomorphism from $L^1(G)$ to a finite-dimensional Banach algebra is continuous.*

A similar argument to that given in the introduction implies the

Corollary 6.2. *Let G be a locally compact group and $\mathcal{I}$ be a closed, two-sided ideal with finite codimension in $L^1(G)$. Suppose that $\{a_n\}$ is a sequence in $\mathcal{I}$ which converges to 0. Then there are elements u and v in $\mathcal{I}$ and sequences $\{s_n\}$ and $\{t_n\}$ converging to 0 such that*

$$a_n = s_n * u + v * t_n \text{ for every } n.$$

Corollary 6.2 implies that, if $\mathcal{I}$ is an ideal with finite codimension in $L^1(G)$ and $T : \mathcal{I} \to X$ is a bimodule homomorphism, then T is continuous. The question of automatic continuity of derivations from general group algebras is still open. The standard technique for treating this question, described in [4] and [32], uses two steps. Step 1 shows that a certain ideal, called the **continuity ideal**, has finite codimension in the algebra and step 2 uses an approximate identity argument to show that the restriction of the derivation to the continuity ideal is continuous. This approach does succeed in showing that all derivations from C^*-algebras are continuous, [23], and is used in [38] to show that derivations from $L^1(G)$ are continuous for certain groups G. In the case of general locally compact groups, Corollary 6.2 would suffice to carry out step 2 if step 1 could be completed. However our current understanding of the structure of general group algebras seems inadequate for this. If the Continuum Hypothesis is assumed, there are discontinuous homomorphisms from group algebras of infinite abelian and certain other groups, [30].

6.2. **One-sided factorization.** If G is a connected group, then $L_0^1(G)$ is the sum of 2 left ideals each having a right bounded approximate identity, see [38]. Similar arguments to those above then show that left $L_0^1(G)$-module homomorphisms from $L_0^1(G)$ are continuous. It is not known whether, for a general locally compact group G, if $\mathcal{I}$ is a finite-codimensional ideal in $L^1(G)$ and $\{a_n\}$ is a sequence in $\mathcal{I}$ which converges to 0, there are an integer k and sequences $\{c_n^{(i)}\}$ which converge to 0 and elements $b^{(i)}$ in $\mathcal{I}$, $1 \le i \le k$, such that

$$(33) \qquad a_n = \sum_{i=1}^{k} c_n^{(i)} * b^{(i)} \qquad (n = 1, 2, 3, \dots),$$

i.e., whether there is a one sided factorization of null sequences in $\mathcal{I}$. Such a factorization could be shown for $\ell_0^1(G)$, where G is an arbitrary discrete group, if it could be shown for free groups. Let $\mathbb{F}_k$ be the free group on k generators $x_1, x_2, \dots, x_k$. Then the k left ideals $[\ell^1(\mathbb{F}_k) * (\delta_e - \delta_{x_m})]^-$, $m = 1, 2, \dots, k$, each have a right bounded approximate identity and their sum is dense in $\ell_0^1(\mathbb{F}_k)$.

However this does not suffice to prove that (33) holds in $\ell_0^1(\mathbb{F}_k)$. It may be that factorization properties in $\ell_0^1(\mathbb{F}_k)$ distinguish between the groups $\mathbb{F}_k$, $k = 1, 2, 3,$ $\ldots$, see Question 27 on page 467 of the volume in which [5] appears. Note that parts (d) and (e) of this question are answered in the present paper.

6.3. Homological unitality.
The algebra $\mathcal{A}$ is said to be **homologically unital** or **H-unital** if the complex

$$(34) \qquad \{0\} \longleftarrow \mathcal{A} \xleftarrow{\ d\ } \mathcal{A} \otimes \mathcal{A} \xleftarrow{\ d\ } \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xleftarrow{\ d\ } \ldots$$

where

$$d : a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=1}^{n-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,$$

is acyclic, see [40]. An algebra is H-unital if it has a unit and a Banach algebra is H-unital if it has a bounded approximate identity. Suppose that $\mathcal{A}$ is a finite-codimensional ideal in the group algebra $L^1(G)$. Then, as shown in [17], when G is amenable $\mathcal{A}$ has a bounded approximate identity and so in this case $\mathcal{A}$ is H-unital. The Main Theorem implies that the first homology group of (34) vanishes for arbitrary G but it is not known in general whether a finite-codimensional ideal in $L^1(G)$ can be H-unital when G is not amenable. What is known is that $\ell_0^1(\mathbb{F}_k)$ is not H-unital for $k \geq 2$ because the complex (34) is not exact at the second place: this follows from the same calculation which shows that the second bounded cohomology group of $\mathbb{F}_k$ is non-zero, see [33], Chapter 8. Now the existence of a bounded approximate identity in $L_0^1(G)$ characterises amenability of G, [21], and it may be that H-unitality of $L_0^1(G)$, or perhaps the condition that some of the homology groups of (34) after the first vanish, characterises amenability or some interesting larger class of locally compact groups. The one sided factorization which holds when G is connected might help to show that some further homology groups vanish in this case.

6.4. Extension to other ideals.
A natural question to ask is: which ideals in $L^1(G)$ may be decomposed as in the Main Theorem? It is shown in [38] that, if G is discrete or connected and N is a closed normal subgroup, then the ideal

$$(35) \qquad \mathcal{I}_N = \{f \in L^1(G) : \int_N f(xn)\, m_N(n) = 0 \text{ for almost every } x \text{ in } G\}$$

has such a decomposition. It can be shown that a necessary condition for $\mathcal{I}$ to have such a decomposition is that $\mathcal{I}$ should be weakly complemented. Furthermore, a subspace of an L^1-space is weakly complemented if and only if the quotient by that subspace is a local L^1-space in the sense defined in [15] and so an equivalent necessary condition is that $L^1(G)/\mathcal{I}$ should be a local L^1-space. That the quotient space has this form in the cases where such a decomposition is known may be seen directly because finite-dimensional spaces are isomorphic to L^1-spaces and $L^1(G)/\mathcal{I}_N \cong L^1(G/N)$. Recall too that $L^1(G)/\mathcal{J}_\mu \cong L^1(\Omega, \nu)$ and that this isomorphism was used in an essential way in the application of the ergodic theorem in section 2 and in the proof of Theorem 3.1. This suggests the

Conjecture 6.3. *The ideal $\mathcal{I}$ in $L^1(G)$ has a decomposition*

$$\mathcal{I} = \mathcal{L} + \mathcal{R},$$

where $\mathcal{L}$ is a closed left ideal with a right bounded approximate identity and $\mathcal{R}$ is a closed right ideal with a left bounded approximate identity if and only if $\mathcal{I}$ is weakly complemented.

Further evidence for the conjecture is the fact that, when G is amenable, $\mathcal{I}$ has a bounded approximate identity if and only if it is weakly complemented, see [9], Proposition VII.2.37 and [3]. Also, when G is abelian, the ideal $\mathcal{I}$ is weakly complemented if and only if it is the kernel of a Γ closed element of the coset ring of Γ_d, where Γ denotes the dual group of G and Γ_d denotes Γ with the discrete topology, see [17], Theorem 12. A similar description of the weakly complemented ideals in $L^1(G)$ when G is non-abelian, might allow the techniques used here and in [39] to be used to prove the conjecture. However such a characterization of weakly complemented ideals in non-commutative group algebras seems to be beyond current techniques.

6.5. **One last question.** We have seen that each element in $L_0^1(G)$ is a sum of 2 products and is in fact a product when G is amenable. It is not known whether each element in $L_0^1(G)$ is a product when G is not amenable. This question is particularly intriguing in the case when G is a free group.

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